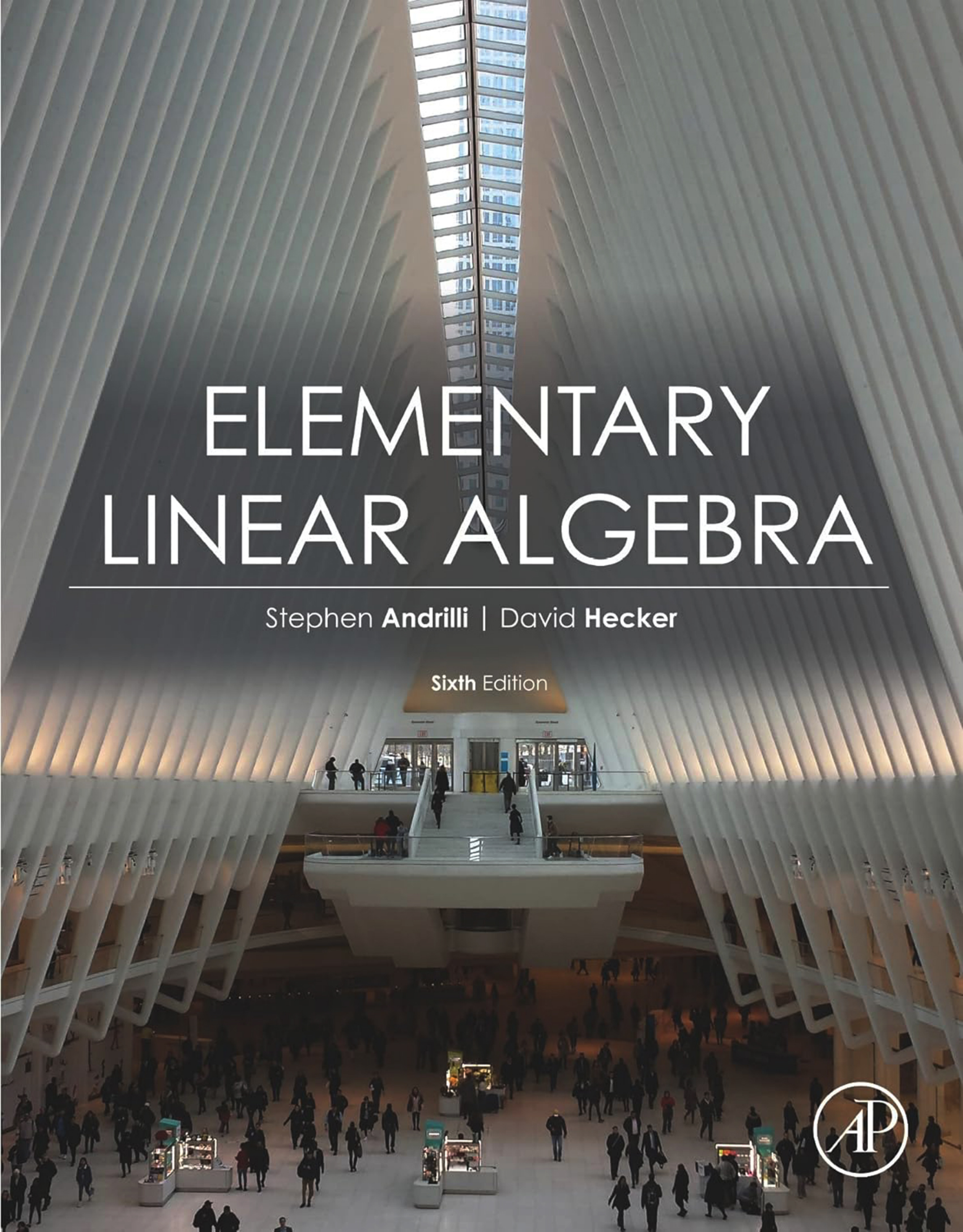

# ELEMENTARY LINEAR ALGEBRA

Stephen **Andrilli** | David **Hecker**

**Sixth** Edition

# Elementary Linear Algebra

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*To our wives, Ene and Lyn, for all their help and encouragement*

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# Preface for the Instructor

This textbook is intended for a sophomore- or junior-level introductory course in linear algebra. We assume the students have had at least one course in calculus.

## Philosophy of the Text

**Helpful Transition From Computation to Theory:** As with all previous editions, our main objective in writing this textbook is to present the basic concepts of linear algebra as clearly as possible. Many concepts such as linear combinations of vectors, the row space of a matrix, and eigenvalues and eigenvectors, are introduced in the earliest chapters in order to facilitate a smoother transition to their roles in later, more theoretical chapters. The "heart" of this text is the material in Chapters 4 and 5 (vector spaces and linear transformations). In particular, we have taken special care to guide students through these chapters as the material more and more heavily emphasizes the abstract theory supporting the computations. Please encourage the students to read the text deeply and thoroughly, taking notes and confirming computations as they progress.

**Applications of Linear Algebra and Numerical Techniques:** This text contains a wide variety of applications of linear algebra, as well as all of the standard numerical techniques typically found in most introductory linear algebra texts. Aside from the many applications and techniques already presented in the first seven chapters, Chapter 8 is devoted entirely to additional applications, while Chapter 9 introduces several other numerical techniques. A summary of these applications and techniques is given in the chart located at the end of the Prefaces.

**Numerous Examples and Exercises:** There are 344 numbered examples in the text, at least one for each major concept or application, as well as for almost every theorem. The text also contains an unusually large number of exercises. There are 1003 numbered exercises, and many of these have multiple parts, for a total of 2988 questions. The exercises within each section are generally ordered by increasing difficulty, beginning with basic computational problems and moving on to more theoretical problems and proofs. Answers are provided at the end of the book for approximately half of the computational exercises; these problems are marked with a star (★). Full solutions to these starred exercises appear in the Student Solutions Manual. The last exercises in each section are True/False questions (there are 510 of these altogether). These are designed to test the students' understanding of fundamental concepts by emphasizing the importance of critical words in definitions or theorems. Finally, there is a set of comprehensive Review Exercises at the end of each of Chapters 1 through 7.

**Assistance in the Reading and Writing of Mathematical Proofs:** To prepare students for a gradual transition to abstract concepts, we introduce them to proof-reading and proof-writing very early in the text, beginning with Section 1.3, which is devoted solely to this topic. For certain long proofs, we present students with an overview so they do not get lost in the details. For every *nontrivial* theorem in Chapters 1 through 6, we have either included a proof, or given detailed hints to enable students to provide a proof on their own. Most of the proofs left as exercises are marked with a wedge symbol (►), and these proofs can be found in the Student Solutions Manual.

**Symbol Table:** Following the Prefaces, for convenience, there is a comprehensive Symbol Table summarizing all of the major symbols for linear algebra that are employed in this text.

**Instructor's Manual:** An Instructor's Manual is available online for all instructors who adopt this text. This manual contains the answers to all exercises, both computational and theoretical. This manual also includes three versions of a Sample Test for each of Chapters 1 through 7, along with corresponding answer keys.

**Student Solutions Manual:** A Student Solutions Manual is freely available at the web site for the textbook (see below) for students using the textbook. This manual contains full solutions for each exercise in the text bearing a star (★) (those whose answers appear in Appendix E). The Student Solutions Manual also contains the proofs of most of the theorems that were left to the exercises. These exercises are marked in the text with a wedge (▶). Because we have compiled this manual ourselves, it utilizes the same styles of proof-writing and solution techniques that appear in the actual text.

**Additional Material on the Web (Including Web Sections):** The web site for this edition of the textbook contains further information about the text. (The web site can be found by going to the Elsevier web site and conducting a search for "Andrilli.") In particular, it contains seven additional web sections of subject content that are available for instructors and students who adopt the text. These web sections range from elementary to advanced topics: Lines and Planes and the Cross Product in $\mathbb{R}^3$, Change of Variables and the Jacobian, Function Spaces, Max-Min Problems in $\mathbb{R}^n$ and the Hessian Matrix, Jordan Canonical Form, Solving First Order Systems of Linear Homogeneous Differential Equations, and Isometries on Inner Product Spaces. These can be covered by instructors in the classroom, or used by the students to explore on their own. These web sections contain 94 additional exercises, many of which have multiple parts, resulting in a total of more than 287 questions altogether.

## Major Changes for the Sixth Edition

- With this edition, the textbook transitions to full-color (replacing the 2-color schema of previous editions). Consequently, improvements that incorporate color have been made to the great majority of the textbook's figures, as well as those in the Instructor's Manual, Student Solutions Manual and Web Sections. Also, color has been added in certain places throughout the text and ancillary materials for emphasis. For example, the pivot entries created during row reduction processes have been colorized in red, eigenvalues in diagonalized matrices have been placed in bold type and colorized in **brown-orange**, and headings for rows and/or columns of matrices have been placed in bold type and colorized in **green**.
- Because various editions of the text have been available for several decades, answers and full solutions for many of the exercises were readily available online. This was problematic for instructors when assigning exercises for graded homework. Therefore, we have changed many of the computational exercises whose answers do not appear in the text (or in the Student Solutions Manual) to provide more options for instructors. Specifically, in Chapters 1 through 6, 365 subportions of exercises (mostly computational in nature) have been replaced, and in addition, 136 completely new subportions have been added.
- (New) **Section 8.6**: A new application section on Linear Recurrence Relations and the Fibonacci Sequence has been added.
- **Section 6.3**: New material is included on the Spectral Theorem, as well as a rearrangement of the material in this section for greater clarity.
- **Section 5.6**: The presentation of the material on eigenvalues, eigenvectors, and diagonalization for linear operators is simplified.

In addition:

- **Section 1.3**: A new subsection is included on the proof technique of reducing a statement to a previously known result.
- **Section 2.2**: A new subsection is included on identifying fundamental solutions for a homogeneous system.
- **Section 2.3**: Two proofs were moved to the exercises (with appropriate hints for the students), and a new example added.
- **Section 3.4**: A new exercise (Exercise 24) has been included to assist students in calculating the characteristic polynomial of a square matrix using only a TI-83/84 calculator.
- **Section 5.1**: A new figure (a commutative diagram) has been included to illustrate one of the properties of a general linear transformation.

## Plans for Coverage

Chapters 1 through 6 have been written in a sequential fashion. Each section is generally needed as a prerequisite for what follows. Therefore, we recommend that these sections be covered in order. However, Section 1.3 (An Introduction to Proofs) can be covered, in whole, or in part, any time after Section 1.2. Also, the material in Section 6.1 (Orthogonal Bases and the Gram-Schmidt Process) can be covered any time after Chapter 4.

The sections in Chapters 7 through 9 as well as the web sections can be covered at any time just as soon as the prerequisites for those sections have been met. (Consult the Prerequisite Chart below for the sections in Chapters 7, 8, and 9.)

The textbook contains much more material than can be covered in a typical 3- or 4-credit course. Some of the material in Chapter 1 could be reviewed quickly if students are already familiar with vector and matrix operations. Two suggested timetables for covering the material in this text are presented below—one for a 3-credit course, and the other for a 4-credit course. A 3-credit course could skip portions of Sections 1.3, 2.3, 3.3, 4.1, 5.5, 5.6, 6.2, and 6.3, and all of Chapter 7. A 4-credit course could cover most of the material of Chapters 1 through 6 (skipping some portions of Sections 1.3, 2.3, and 3.3), and also cover some of Chapter 7.

| | 3-Credit Course | 4-Credit Course |
|---|---|---|
| **Chapter 1** | 5 classes | 5 classes |
| **Chapter 2** | 5 classes | 6 classes |
| **Chapter 3** | 5 classes | 5 classes |
| **Chapter 4** | 11 classes | 13 classes |
| **Chapter 5** | 8 classes | 13 classes |
| **Chapter 6** | 2 classes | 5 classes |
| **Chapter 7** | | 2 classes |
| **Chapters 8, 9 (selections)** | 3 classes | 4 classes |
| **Tests** | 3 classes | 3 classes |
| **Total** | 42 classes | 56 classes |

## Prerequisite Chart for Later Sections

Prerequisites for the material in later sections of the text are listed in the following chart. Each section of Chapter 7 depends on the sections in that chapter that precede it, as well as the prerequisite given in the chart below. In contrast, the sections of Chapters 8 and 9 are generally independent of each other, and they can be covered as soon as their prerequisites from earlier chapters have been met. Also note that the techniques for solving differential equations in Section 8.9 require only Section 3.4 as a prerequisite, although terminology from Chapters 4 and 5 is used throughout Section 8.9.

| Section | Prerequisites |
|---|---|
| Section 7.1 (Complex $n$-Vectors and Matrices) | Section 1.5 |
| Section 7.2 (Complex Eigenvalues and Complex Eigenvectors) | Section 3.4 |
| Section 7.3 (Complex Vector Spaces) | Section 5.2 |
| Section 7.4 (Orthogonality in $\mathbb{C}^n$) | Section 6.3 |
| Section 7.5 (Inner Product Spaces) | Section 6.3 |
| Section 8.1 (Graph Theory) | Section 1.5 |
| Section 8.2 (Ohm's Law) | Section 2.2 |
| Section 8.3 (Least-Squares Polynomials) | Section 2.2 |
| Section 8.4 (Markov Chains) | Section 2.2 |
| Section 8.5 (Hill Substitution: An Intro. to Coding Theory) | Section 2.4 |
| Section 8.6 (Linear Recurrence Relations & Fibonacci Seq.) | Section 3.4 |
| Section 8.7 (Rotation of Axes) | Section 4.7 |
| Section 8.8 (Computer Graphics) | Section 5.2 |
| Section 8.9 (Differential Equations) | Section 5.6 |
| Section 8.10 (Least-Squares Sol'ns for Inconsistent Systems) | Section 6.2 |
| Section 8.11 (Quadratic Forms) | Section 6.3 |
| Section 9.1 (Numerical Methods for Solving Systems) | Section 2.3 |
| Section 9.2 (**LDU** Decomposition) | Section 2.4 |
| Section 9.3 (The Power Method for Finding Eigenvalues) | Section 3.4 |
| Section 9.4 (**QR** Factorization) | Section 6.1 |
| Section 9.5 (Singular Value Decomposition) | Section 6.3 |
| Appendix D (Elementary Matrices) | Section 2.4 |

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We also want to thank those colleagues who have supported our textbook at various stages. We also thank La Salle University and Saint Joseph's University for granting course reductions and sabbaticals to the authors to complete the work on several previous editions.

We especially thank those students and instructors who have reviewed earlier editions of the textbook as well as those who have classroom-tested versions of the earlier editions of the manuscript. Their comments and suggestions have been extremely helpful, and have guided us in shaping the text in many ways.

Last, but most important of all, we want to thank our wives, Ene and Lyn, for bearing extra hardships so that we could work on this text. Their love and support continues to be an inspiration.

# Preface to the Student

**A Quick Overview of the Text:** Chapters 1 to 3 present the basic tools for your study of linear algebra: vectors, matrices, systems of linear equations, inverses, determinants, and eigenvalues. Chapters 4 to 6 then treat these concepts on a higher level: vector spaces, spanning, linear independence, bases, coordinatization, linear transformations, kernel, range, isomorphisms, and orthogonality. Chapter 7 extends the results of earlier chapters to the complex number system. Chapters 8 and 9 present many applications and numerical techniques widely used in linear algebra.

**Strategies for Learning:** Many students find that the transition to abstractness (beginning with general vector spaces in Chapter 4) is challenging. This text was written specifically to help you in this regard. We have tried to present the material in the clearest possible manner with many helpful examples. *Take advantage of this and read each section of the textbook thoroughly and carefully several times over.* Each re-reading will allow you to see connections among the concepts on a deeper level. *You should read the text with pencil, paper, and a calculator at your side.* Reproduce on your own every computation in every example, so that you truly understand what is presented in the text. *Make notes to yourself as you proceed.* Try as many exercises in each section as possible. There are True/False questions to test your knowledge at the end of each section and in the Review Exercises for Chapters 1 to 7. After pondering these first on your own, compare your answers with the detailed solutions given in the Student Solutions Manual. *Ask your instructor questions about anything that you read that you do not comprehend*—as soon as possible, because each new section continually builds on previous material.

**Facility With Proofs:** Linear algebra is considered by many instructors as a transitional course from the freshman computationally-oriented calculus sequence to the junior-senior level courses which put much more emphasis on the reading and writing of mathematical proofs. At first it may seem daunting to write your own proofs. However, most of the proofs that you are asked to write for this text are relatively short. Many useful strategies for proof-writing are discussed in Section 1.3. The proofs that are presented in this text are meant to serve as good examples. *Study them carefully.* Remember that each step of a proof must be validated with a proper reason—a theorem that was proven earlier, a definition, or a principle of logic. Pondering carefully over the definitions and theorems in the text is a very valuable use of your time, for only by fully comprehending these can you fully appreciate how to use them in proofs. Learning how to read and write proofs effectively is an important skill that will serve you well in your upper-division mathematics courses and beyond.

**Student Solutions Manual:** A Student Solutions Manual is available online that contains full solutions for each exercise in the text bearing a star (★) (those whose answers appear in the back of the textbook). Consequently, this manual contains many useful models for solving various types of problems. The Student Solutions Manual also contains proofs of most of the theorems whose proofs were left to the exercises. These exercises are marked in the text by a wedge (▶).

# A Light-Hearted Look at Linear Algebra Terms

As students vector through the space of this text from its initial point to its terminal point, on a one-to-one basis, they will undergo a real transformation from the norm. An induction into the domain of linear algebra is sufficient to produce a pivotal change in their abilities. To transpose students with an empty set of knowledge into higher echelons of understanding, a nontrivial length of time is necessary—one of the prime factorizations to account for in such a system.

One elementary implication is that the students' success is an isomorphic reflection of the homogeneous effort they expend on this complex material. We can trace the rank of their achievement to their resolve to be a scalar of new distances. In a similar manner, there is a symmetric result: their positive definite growth is a function of their overall coordinatization of energy. The matrix of thought behind this parallel assertion is proof that students should avoid the negative consequences of sparse learning. That is, the method of iterative study will lead them in an inverse way to less error, and not rotate them into diagonal tangents of zero worth.

After an interpolation of the kernel of ideas presented here, the students' range of new methods should be graphically augmented in a multiplicity of ways. We extrapolate that one characteristic they will attain is a greater linear independence in problem-solving. An associative feature of this transition is that all these new techniques should become a consistent and normalized part of their identity.

In addition, students will gain a singular appreciation of their mathematical skills, so the resultant skewed change in their self-image should not be of minor magnitude, but complement them fully. Our projection is that the unique dimensions of this text will be a determinant cofactor in enriching the span of their lives, and translate them onto new orthogonal paths of logical truth.

Stephen Andrilli
David Hecker
August, 2021

# Symbol Table

| | |
|---|---|
| $\oplus$ | addition on a vector space (unusual) |
| $\mathcal{A}$ | adjoint (classical) of a matrix **A** |
| $I$ | ampere (unit of current) |
| $\approx$ | approximately equal to |
| $[\mathbf{A} \mid \mathbf{B}]$ | augmented matrix formed from matrices **A** and **B** |
| $\mathbf{p}_L(x)$ | characteristic polynomial of a linear operator $L$ |
| $\mathbf{p}_{\mathbf{A}}(x)$ | characteristic polynomial of a matrix **A** |
| $\mathcal{A}_{ij}$ | cofactor, $(i, j)$, of a matrix **A** |
| $\overline{z}$ | complex conjugate of a complex number $z$ |
| $\overline{\mathbf{z}}$ | complex conjugate of $\mathbf{z} \in \mathbb{C}^n$ |
| $\overline{\mathbf{Z}}$ | complex conjugate of $\mathbf{Z} \in \mathcal{M}_{mn}^{\mathbb{C}}$ |
| $\mathbb{C}$ | complex numbers, set of |
| $\mathbb{C}^n$ | complex $n$-vectors, set of (ordered $n$-tuples of complex numbers) |
| $g \circ f$ | composition of functions $f$ and $g$ |
| $L_2 \circ L_1$ | composition of linear transformations $L_1$ and $L_2$ |
| $\mathbf{Z}^*$ | conjugate transpose of $\mathbf{Z} \in \mathcal{M}_{mn}^{\mathbb{C}}$ |
| $C^0(\mathbb{R})$ | continuous real-valued functions with domain $\mathbb{R}$, set of |
| $C^1(\mathbb{R})$ | continuously differentiable functions with domain $\mathbb{R}$, set of |
| $[\mathbf{w}]_B$ | coordinatization of a vector **w** with respect to a basis $B$ |
| $\mathbf{x} \times \mathbf{y}$ | cross product of vectors **x** and **y** |
| $f^{(n)}$ | derivative, $n$th, of a function $f$ |
| $\lvert\mathbf{A}\rvert$ | determinant of a matrix **A** |
| $\delta$ | determinant of a $2 \times 2$ matrix, $ad - bc$ |
| $\mathcal{D}_n$ | diagonal $n \times n$ matrices, set of |
| $\dim(\mathcal{V})$ | dimension of a vector space $\mathcal{V}$ |
| $\mathbf{x} \cdot \mathbf{y}$ | dot product, or, complex dot product, of vectors **x** and **y** |
| $\lambda$ | eigenvalue of a matrix |
| $E_\lambda$ | eigenspace corresponding to eigenvalue $\lambda$ |
| $\{\ \}$ | empty set |
| $a_{ij}$ | entry, $(i, j)$, of a matrix **A** |
| $F_n$ | Fibonacci sequence, $n$th term |
| $f : X \to Y$ | function $f$ from a set $X$ (domain) to a set $Y$ (codomain) |
| $\phi$ | Golden Ratio $\left(= \frac{1+\sqrt{5}}{2}\right)$ |
| $\mathbf{I}, \mathbf{I}_n$ | identity matrix; $n \times n$ identity matrix |

| | |
|---|---|
| $\Longleftrightarrow$, iff | if and only if |
| $f(S)$ | image of a set $S$ under a function $f$ |
| $f(x)$ | image of an element $x$ under a function $f$ |
| $i$ | imaginary number whose square $= -1$ |
| $\Longrightarrow$ | implies; if...then |
| $< \mathbf{x}, \mathbf{y} >$ | inner product of **x** and **y** |
| $\mathbb{Z}$ | integers, set of |
| $f^{-1}$ | inverse of a function $f$ |
| $L^{-1}$ | inverse of a linear transformation $L$ |
| $\mathbf{A}^{-1}$ | inverse of a matrix **A** |
| $\cong$ | isomorphic |
| $\ker(L)$ | kernel of a linear transformation $L$ |
| $\|\|\mathbf{a}\|\|$ | length, or norm, of a vector **a** |
| $\mathbf{M}_f$ | limit matrix of a Markov chain |
| $\mathbf{p}_f$ | limit vector of a Markov chain |
| $\mathcal{L}_n$ | lower triangular $n \times n$ matrices, set of |
| $L_n$ | Lucas sequence, $n$th term |
| $\|z\|$ | magnitude (absolute value) of a complex number $z$ |
| $\mathcal{M}_{mn}$ | matrices of size $m \times n$, set of |
| $\mathcal{M}_{mn}^{\mathbb{C}}$ | matrices of size $m \times n$ with complex entries, set of |
| $\mathbf{A}_{BC}$ | matrix for a linear transformation with respect to ordered bases $B$, $C$ |
| $\|\mathbf{A}_{ij}\|$ | minor, $(i, j)$, of a matrix **A** |
| $\mathbb{N}$ | natural numbers, set of |
| not $A$ | negation of statement $A$ |
| $\|S\|$ | number of elements in a set $S$ |
| $\Omega$ | ohm (unit of resistance) |
| $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ | ordered basis containing vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ |
| $\mathcal{W}^{\perp}$ | orthogonal complement of a subspace $\mathcal{W}$ |
| $\perp$ | perpendicular to |
| $\mathcal{P}_n$ | polynomials of degree $\leq n$, set of |
| $\mathcal{P}_n^{\mathbb{C}}$ | polynomials of degree $\leq n$ with complex coefficients, set of |
| $\mathcal{P}$ | polynomials, set of all |
| $\mathbb{R}^+$ | positive real numbers, set of |
| $\mathbf{A}^k$ | power, $k$th, of a matrix **A** |
| $f^{-1}(S)$ | pre-image of a set $S$ under a function $f$ |
| $f^{-1}(x)$ | pre-image of an element $x$ under a function $f$ |
| $\mathbf{proj_a b}$ | projection of **b** onto **a** |
| $\mathbf{proj}_{\mathcal{W}}\mathbf{v}$ | projection of **v** onto a subspace $\mathcal{W}$ |
| $\mathbf{A}^+$ | pseudoinverse of a matrix **A** |
| range$(L)$ | range of a linear transformation $L$ |
| rank$(\mathbf{A})$ | rank of a matrix **A** |
| $\mathbb{R}$ | real numbers, set of |
| $\mathbb{R}^n$ | real $n$-vectors, set of (ordered $n$-tuples of real numbers) |

| | |
|---|---|
| $\langle i \rangle \leftarrow c \langle i \rangle$ | row operation of type (I) |
| $\langle i \rangle \leftarrow c \langle j \rangle + \langle i \rangle$ | row operation of type (II) |
| $\langle i \rangle \leftrightarrow \langle j \rangle$ | row operation of type (III) |
| $R(\mathbf{A})$ | row operation $R$ applied to matrix $\mathbf{A}$ |
| $\odot$ | scalar multiplication on a vector space (unusual) |
| $\sigma_k$ | singular value, $k$th, of a matrix |
| $m \times n$ | size of a matrix with $m$ rows and $n$ columns |
| $\text{span}(S)$ | span of a set $S$ |
| $\boldsymbol{\Psi}_{ij}$ | standard basis vector (matrix) in $\mathcal{M}_{mn}$ |
| $\mathbf{i}, \mathbf{j}, \mathbf{k}$ | standard basis vectors in $\mathbb{R}^3$ |
| $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ | standard basis vectors in $\mathbb{R}^n$; standard basis vectors in $\mathbb{C}^n$ |
| $\mathbf{p}_n$ | state vector, $n$th, of a Markov chain |
| $\mathbf{A}_{ij}$ | submatrix, $(i, j)$, of a matrix $\mathbf{A}$ |
| $\sum$ | sum of |
| $\text{trace}(\mathbf{A})$ | trace of a matrix $\mathbf{A}$ |
| $\mathbf{A}^T$ | transpose of a matrix $\mathbf{A}$ |
| $C^2(\mathbb{R})$ | twice continuously differentiable functions with domain $\mathbb{R}$, set of |
| $\mathcal{U}_n$ | upper triangular $n \times n$ matrices, set of |
| $V$ | volt (unit of voltage) |
| $\mathbf{O}$; $\mathbf{O}_n$; $\mathbf{O}_{mn}$ | zero matrix; $n \times n$ zero matrix; $m \times n$ zero matrix |
| $\mathbf{0}$; $\mathbf{0}_{\mathcal{V}}$ | zero vector in a vector space $\mathcal{V}$ |

# Computational & Numerical Techniques, Applications

The following is a list of the most important computational and numerical techniques and applications of linear algebra presented throughout the text.

| Section | Technique/Application |
|---|---|
| **Section 1.1** | Resultant Velocity |
| **Section 1.1** | Newton's Second Law |
| **Section 1.2** | Work (in physics) |
| **Section 1.5** | Shipping Cost and Profit |
| **Section 2.1** | Gaussian Elimination and Back Substitution |
| **Section 2.1** | Curve Fitting |
| **Section 2.2** | Gauss-Jordan Row Reduction Method |
| **Section 2.2** | Balancing of Chemical Equations |
| **Section 2.3** | Determining the Row Space of a Matrix |
| **Section 2.4** | Inverse Method (finding the inverse of a matrix) |
| **Section 2.4** | Solving a System Using the Inverse of the Coefficient Matrix |
| **Section 2.4** | Finding the Determinant of a $2 \times 2$ Matrix ($ad - bc$ formula) |
| **Section 3.1** | Finding the Determinant of a $3 \times 3$ Matrix (Basketweaving) |
| **Section 3.1** | Finding Areas and Volumes Using Determinants |
| **Section 3.2** | Determinant of a Matrix by Row Reduction |
| **Section 3.3** | Determinant of a Matrix by General Cofactor Expansion |
| **Section 3.3** | Cramer's Rule |
| **Section 3.4** | Finding Eigenvalues and Eigenvectors for a Matrix |
| **Section 3.4** | Diagonalization Method (diagonalizing a square matrix) |
| **Section 4.3** | Simplified Span Method (determining span by row reduction) |
| **Section 4.4** | Independence Test Method (linear independence by row reduction) |
| **Section 4.6** | Enlarging Method (enlarging a linear independent set to a basis) |
| **Section 4.7** | Coordinatization Method (coordinatizing with respect to an ordered basis) |
| **Section 4.7** | Transition Matrix Method (transition matrix by row reduction) |
| **Section 5.2** | Determining the Matrix for a Linear Transformation |
| **Section 5.3** | Kernel Method (basis for a kernel of a linear transformation) |
| **Section 5.3** | Range Method (basis for the range of a linear transformation) |
| **Section 5.4** | Determining if a Linear Transformation is One-to-One or Onto |
| **Section 5.5** | Determining if a Linear Transformation is an Isomorphism |
| **Section 5.6** | Generalized Diagonalization Method (diagonalizing a linear operator) |

| Section | Technique/Application |
|---|---|
| **Section 6.1** | Gram-Schmidt Process |
| **Section 6.2** | Finding the Orthogonal Complement of a Subspace |
| **Section 6.2** | Finding the Orthogonal Projection of a Vector Onto a Subspace |
| **Section 6.2** | Finding the Distance From a Point to a Subspace |
| **Section 6.3** | Orthogonal Diagonalization Method |
| **Section 6.3** | Spectral Theorem for an Orthogonally Diagonalizable Matrix |
| **Section 7.2** | Gaussian Elimination for Complex Systems |
| **Section 7.2** | Gauss-Jordan Row Reduction Method for Complex Systems |
| **Section 7.2** | Complex Determinants, Eigenvalues, and Matrix Diagonalization |
| **Section 7.4** | Gram-Schmidt Process for Complex Vectors |
| **Section 7.5** | Finding the Distance Between Vectors in an Inner Product Space |
| **Section 7.5** | Finding the Angle Between Vectors in an Inner Product Space |
| **Section 7.5** | Orthogonal Complement of a Subspace in an Inner Product Space |
| **Section 7.5** | Orthogonal Projection of a Vector Onto an Inner Product Subspace |
| **Section 7.5** | Generalized Gram-Schmidt Process (for inner product spaces) |
| **Section 7.5** | Fourier Series |
| **Section 8.1** | Number of Paths Between Vertices in a Graph/Digraph |
| **Section 8.1** | Determining if a Graph is Connected |
| **Section 8.2** | Finding the Current in a Branch of an Electrical Circuit |
| **Section 8.3** | Finding the Least-Squares Polynomial for a Set of Data |
| **Section 8.4** | Finding the Steady-State Vector for a Markov Chain |
| **Section 8.5** | Encoding/Decoding Messages Using Hill Substitution |
| **Section 8.6** | Determining Solutions for a Linear Recurrence Relation |
| **Section 8.7** | Using Rotation of Axes to Graph a Conic Section |
| **Section 8.8** | Similarity Method (in computer graphics) |
| **Section 8.9** | Solutions of a System of First-Order Differential Equations |
| **Section 8.9** | Solutions to Higher-Order Homogeneous Differential Equations |
| **Section 8.10** | Least-Squares Solutions for Inconsistent Systems |
| **Section 8.10** | Approximate Eigenvalues/Eigenvectors Using Inconsistent Systems |
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Chapter 1

# Vectors and Matrices

**Proof Positive**

In linear algebra, the most fundamental objects of study are vectors and matrices, which have a multitude of practical applications in science and engineering. You are probably already familiar with the use of vectors to describe positions, movements, and forces. The basic properties of matrices parallel those of vectors, but we will find many differences between their more advanced properties, especially with regard to matrix multiplication.

However, linear algebra can also be used to introduce proof-writing skills. The concept of proof is central to higher mathematics, because mathematicians claim no statement as a "fact" until it is proven true using logical deduction. Section 1.3 gives an introductory overview of the basic proof-writing tools that a mathematician uses on a daily basis. Other proofs given throughout the text serve as models for constructing proofs of your own when completing the exercises. With these tools and models, you can begin to develop skills in the reading and writing of proofs that are crucial to your future success in mathematics.

## 1.1 Fundamental Operations With Vectors

In this section, we introduce vectors and consider two operations on vectors: scalar multiplication and addition. We use the symbol $\mathbb{R}$ to represent the set of all **real numbers** (that is, all coordinate values on the real number line).

### Definition of a Vector

**Definition** A **real $n$-vector** is an ordered sequence of $n$ real numbers (sometimes referred to as an **ordered $n$-tuple** of real numbers). The set of all $n$-vectors is represented by the symbol $\mathbb{R}^n$.

For example, $\mathbb{R}^2$ is the set of all 2-vectors (ordered 2-tuples = ordered pairs) of real numbers; it includes $[2, -4]$ and $[-6.2, 3.14]$. $\mathbb{R}^3$ is the set of all 3-vectors (ordered 3-tuples = ordered triples) of real numbers; it includes $[2, -3, 0]$ and $[-\sqrt{2}, 42.7, \pi]$.[1]

The vector in the set $\mathbb{R}^n$ that has all $n$ entries equal to zero is called the **zero $n$-vector**. In the sets $\mathbb{R}^2$ and $\mathbb{R}^3$, the zero vectors are $[0, 0]$ and $[0, 0, 0]$, respectively.

Two vectors in the set $\mathbb{R}^n$ are **equal** if and only if all corresponding entries (called **coordinates**) in their $n$-tuples agree. That is, $[x_1, x_2, \ldots, x_n] = [y_1, y_2, \ldots, y_n]$ if and only if $x_1 = y_1, x_2 = y_2, \ldots,$ and $x_n = y_n$.

A single number (such as $-10$ or $2.6$) is often called a **scalar** to distinguish it from a vector.

### Geometric Interpretation of Vectors

A vector having two coordinates (that is, an element of the set $\mathbb{R}^2$) is frequently used to represent a movement from one point to another in a coordinate plane. From an initial point $(3, 2)$ to a terminal point $(1, 5)$, there is a net decrease of 2 units along the $x$-axis and a net increase of 3 units along the $y$-axis. A vector representing this change would thus be $[-2, 3]$, as indicated by the arrow in Fig. 1.1.

Vectors can be positioned at any desired starting point. For example, $[-2, 3]$ could also represent a movement from an initial point $(9, -6)$ to a terminal point $(7, -3)$.[2]

---

[1] Many texts distinguish between *row* vectors, such as $[2, -3]$, and *column* vectors, such as $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$. However, in this text, we express vectors as row or column vectors as the situation warrants.

[2] We use italicized capital letters and parentheses for the points of a coordinate system, such as $A = (3, 2)$, and boldface lowercase letters and brackets for vectors, such as $\mathbf{x} = [3, 2]$.

**Elementary Linear Algebra.** https://doi.org/10.1016/B978-0-12-822978-1.00011-0

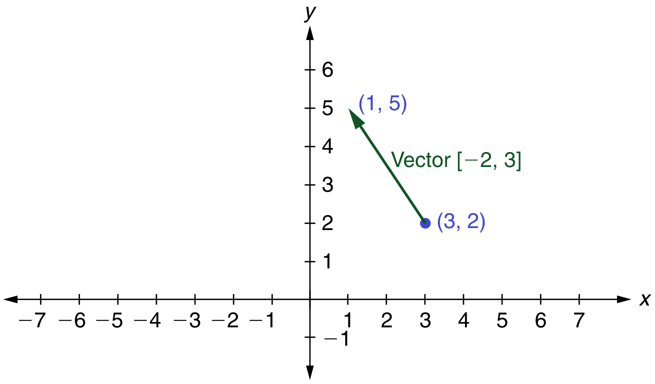


**FIGURE 1.1** Movement represented by the vector $[-2, 3]$

Elements in the set $\mathbb{R}^3$ (that is, vectors having three coordinates) have a similar geometric interpretation: a 3-vector is used to represent movement between points in three-dimensional space. For example, $[2, -2, 6]$ can represent movement from an initial point $(2, 3, -1)$ to a terminal point $(4, 1, 5)$, as shown in Fig. 1.2.

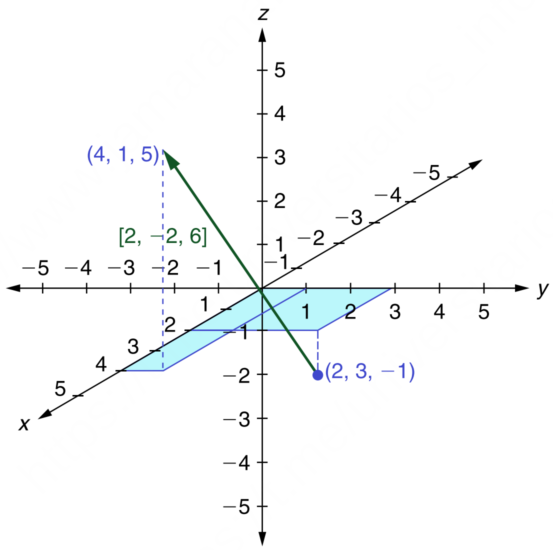


**FIGURE 1.2** The vector $[2, -2, 6]$ with initial point $(2, 3, -1)$

Three-dimensional movements are usually graphed on a two-dimensional page by slanting the $x$-axis at an angle to create the optical illusion of three mutually perpendicular axes. Movements are determined on such a graph by breaking them down into components parallel to each of the coordinate axes.

Visualizing vectors in $\mathbb{R}^4$ and higher dimensions is difficult. However, the same algebraic principles are involved. For example, the vector $\mathbf{x} = [2, 7, -3, 10]$ can represent a movement between points $(5, -6, 2, -1)$ and $(7, 1, -1, 9)$ in a four-dimensional coordinate system.

## Length of a Vector

Recall the **distance formula** in the plane: the distance between two points $(x_1, y_1)$ and $(x_2, y_2)$ is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ (see Fig. 1.3). This formula arises from the Pythagorean Theorem for right triangles. The 2-vector between the points is $[a_1, a_2]$, where $a_1 = x_2 - x_1$ and $a_2 = y_2 - y_1$, so $d = \sqrt{a_1^2 + a_2^2}$. This formula motivates the following definition:

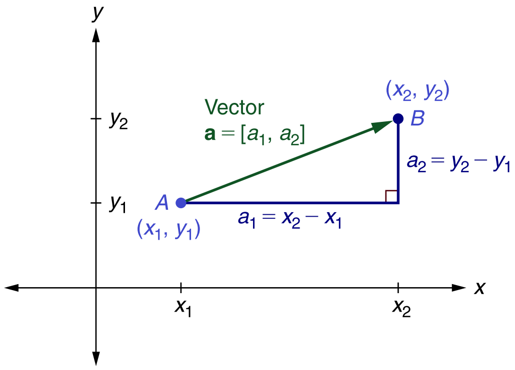


**FIGURE 1.3** The line segment (and vector) connecting points $A$ and $B$, with length $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{a_1^2 + a_2^2}$

**Definition** The **length** (also known as the **norm** or **magnitude**) of a vector $\mathbf{a} = [a_1, a_2, \ldots, a_n]$ in $\mathbb{R}^n$ is $\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$.

**Example 1**

The length of the vector $\mathbf{a} = [4, -3, 0, 2]$ is given by

$$\|\mathbf{a}\| = \sqrt{4^2 + (-3)^2 + 0^2 + 2^2} = \sqrt{16 + 9 + 0 + 4} = \sqrt{29}.$$

Exercise 21 asks you to show that the length of any vector in $\mathbb{R}^n$ is always nonnegative (that is, $\geq 0$), and also that the only vector with length 0 in $\mathbb{R}^n$ is the zero vector $[0, 0, \ldots, 0]$.

Vectors of length 1 play an important role in linear algebra.

**Definition** Any vector of length 1 is called a **unit vector**.

In $\mathbb{R}^2$, the vector $\left[\frac{3}{5}, -\frac{4}{5}\right]$ is a unit vector, because $\sqrt{\left(\frac{3}{5}\right)^2 + \left(-\frac{4}{5}\right)^2} = 1$. Similarly, $\left[-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right]$ is a unit vector in $\mathbb{R}^4$. Certain unit vectors are particularly useful: those with a single coordinate equal to 1 and all other coordinates equal to 0. In $\mathbb{R}^2$ these vectors are represented by $\mathbf{i} = [1, 0]$ and $\mathbf{j} = [0, 1]$; in $\mathbb{R}^3$ they are represented by $\mathbf{i} = [1, 0, 0]$, $\mathbf{j} = [0, 1, 0]$, and $\mathbf{k} = [0, 0, 1]$. In $\mathbb{R}^n$, such vectors, the **standard unit vectors**, are represented by $\mathbf{e}_1 = [1, 0, 0, \ldots, 0]$, $\mathbf{e}_2 = [0, 1, 0, \ldots, 0]$, $\ldots$, $\mathbf{e}_n = [0, 0, 0, \ldots, 1]$. Whenever any of the symbols $\mathbf{i}$, $\mathbf{j}$, $\mathbf{e}_1$, $\mathbf{e}_2$, etc. are used, the actual number of coordinates in the vector is to be understood from context.

## Scalar Multiplication and Parallel Vectors

**Definition** Let $\mathbf{x} = [x_1, x_2, \ldots, x_n]$ be a vector in $\mathbb{R}^n$, and let $c$ be any scalar (real number). Then $c\mathbf{x}$, the **scalar multiple of x by** $c$, is the vector $[cx_1, cx_2, \ldots, cx_n]$.

For example, if $\mathbf{x} = [4, -5]$, then $2\mathbf{x} = [8, -10]$, $-3\mathbf{x} = [-12, 15]$, and $-\frac{1}{2}\mathbf{x} = \left[-2, \frac{5}{2}\right]$. These vectors are graphed in Fig. 1.4. From the graph, you can see that the vector $2\mathbf{x}$ points in the same direction as $\mathbf{x}$ but is twice as long. The vectors $-3\mathbf{x}$ and $-\frac{1}{2}\mathbf{x}$ indicate movements in the direction opposite to $\mathbf{x}$, with $-3\mathbf{x}$ being three times as long as $\mathbf{x}$ and $-\frac{1}{2}\mathbf{x}$ being half as long.

In general, in $\mathbb{R}^n$, multiplication by $c$ **dilates** (expands) the length of the vector when $|c| > 1$ and **contracts** (shrinks) the length when $|c| < 1$. Scalar multiplication by 1 or $-1$ does not affect the length. Scalar multiplication by 0 always yields the zero vector. These properties are all special cases of the following theorem:

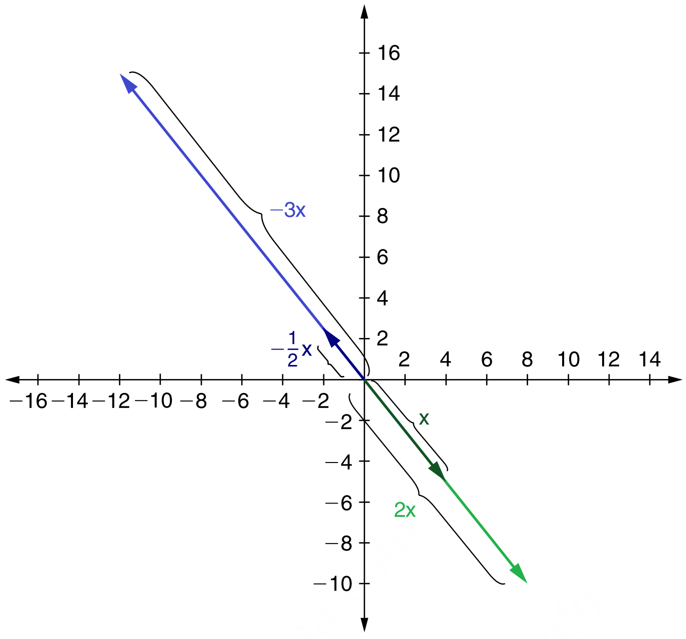


**FIGURE 1.4** Scalar multiples of $\mathbf{x} = [4, -5]$ (all vectors drawn with initial point at origin)

**Theorem 1.1** *Let $\mathbf{x} \in \mathbb{R}^n$, and let $c$ be any real number (scalar). Then*

$$\|c\mathbf{x}\| = |c|\,\|\mathbf{x}\|.$$

*That is, the length of $c\mathbf{x}$ is the absolute value of $c$ times the length of $\mathbf{x}$.*

*Proof.* Suppose $\mathbf{x} = [x_1, x_2, \ldots, x_n]$. Then $c\mathbf{x} = [cx_1, cx_2, \ldots, cx_n]$. Hence, $\|c\mathbf{x}\| = \sqrt{(cx_1)^2 + \cdots + (cx_n)^2} = \sqrt{c^2(x_1^2 + \cdots + x_n^2)} = |c|\sqrt{x_1^2 + \cdots + x_n^2} = |c|\,\|\mathbf{x}\|$. □

We have noted that in $\mathbb{R}^2$, the vector $c\mathbf{x}$ is in the same direction as $\mathbf{x}$ when $c$ is positive and in the direction opposite to $\mathbf{x}$ when $c$ is negative, but have not yet discussed "direction" in higher-dimensional coordinate systems. We use scalar multiplication to give a precise definition for vectors having the same or opposite directions.

**Definition** Two nonzero vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ are **in the same direction** if and only if there is a positive real number $c$ such that $\mathbf{y} = c\mathbf{x}$. Two nonzero vectors $\mathbf{x}$ and $\mathbf{y}$ are **in opposite directions** if and only if there is a negative real number $c$ such that $\mathbf{y} = c\mathbf{x}$. Two nonzero vectors are **parallel** if and only if they are in the same direction or in the opposite direction.

Hence, vectors $[1, -3, 2]$ and $[3, -9, 6]$ are in the same direction, because $[3, -9, 6] = 3[1, -3, 2]$ (or because $[1, -3, 2] = \frac{1}{3}[3, -9, 6]$), as shown in Fig. 1.5. Similarly, vectors $[-3, 6, 0, 15]$ and $[4, -8, 0, -20]$ are in opposite directions, because $[4, -8, 0, -20] = -\frac{4}{3}[-3, 6, 0, 15]$.

The next result follows directly from Theorem 1.1. (A corollary is a theorem that follows immediately from a previous theorem.)

**Corollary 1.2** *If $\mathbf{x}$ is a nonzero vector in $\mathbb{R}^n$, then $\mathbf{u} = \left(\frac{1}{\|\mathbf{x}\|}\right)\mathbf{x}$ is a unit vector in the same direction as $\mathbf{x}$.*

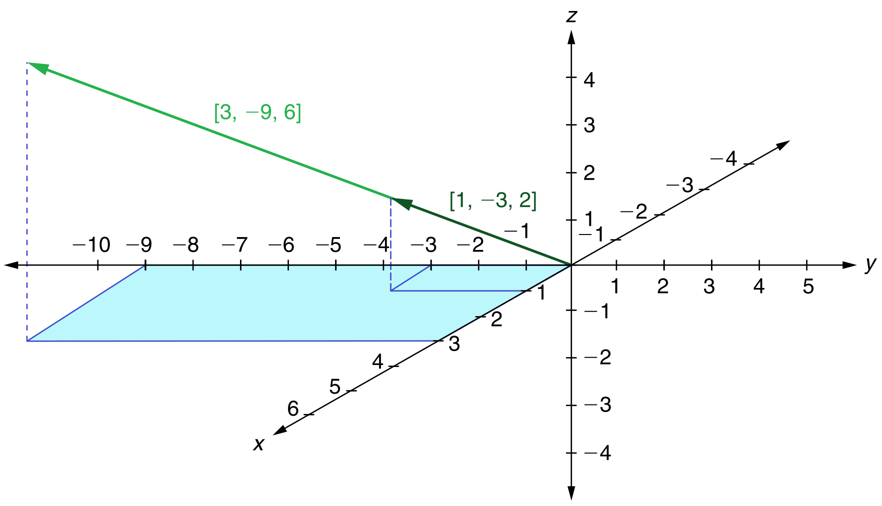


**FIGURE 1.5** The parallel vectors [1, −3, 2] and [3, −9, 6]

*Proof.* The vector $\mathbf{u}$ in Corollary 1.2 is certainly in the same direction as $\mathbf{x}$ because $\mathbf{u}$ is a positive scalar multiple of $\mathbf{x}$ (the scalar is $\frac{1}{\|\mathbf{x}\|}$). Also by Theorem 1.1, $\|\mathbf{u}\| = \left\| \left(\frac{1}{\|\mathbf{x}\|}\right)\mathbf{x} \right\| = \left(\frac{1}{\|\mathbf{x}\|}\right)\|\mathbf{x}\| = 1$, so $\mathbf{u}$ is a unit vector. □

This process of "dividing" a vector by its length to obtain a unit vector in the same direction is called **normalizing** the vector (see Fig. 1.6).

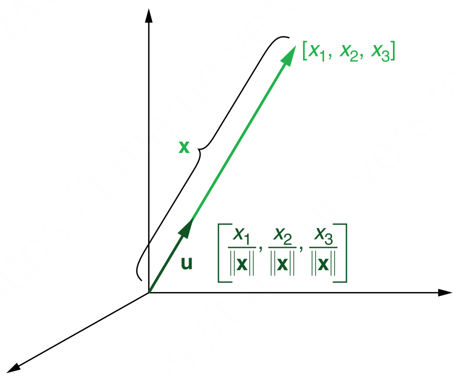


**FIGURE 1.6** Normalizing a vector $\mathbf{x}$ to obtain a unit vector $\mathbf{u}$ in the same direction (with $\|\mathbf{x}\| > 1$)

**Example 2**

Consider the vector $[2, 3, -1, 1]$ in $\mathbb{R}^4$. Because $\|[2, 3, -1, 1]\| = \sqrt{15}$, normalizing $[2, 3, -1, 1]$ gives a unit vector $\mathbf{u}$ in the same direction as $[2, 3, -1, 1]$, which is

$$\mathbf{u} = \left(\frac{1}{\sqrt{15}}\right)[2, 3, -1, 1] = \left[\frac{2}{\sqrt{15}}, \frac{3}{\sqrt{15}}, \frac{-1}{\sqrt{15}}, \frac{1}{\sqrt{15}}\right].$$

## Addition and Subtraction With Vectors

**Definition** Let $\mathbf{x} = [x_1, x_2, \ldots, x_n]$ and $\mathbf{y} = [y_1, y_2, \ldots, y_n]$ be vectors in $\mathbb{R}^n$. Then $\mathbf{x}+\mathbf{y}$, the **sum** of $\mathbf{x}$ and $\mathbf{y}$, is the vector $[x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n]$ in $\mathbb{R}^n$.

Vectors are added by summing their respective coordinates. For example, if $\mathbf{x} = [2, -3, 5]$ and $\mathbf{y} = [-6, 4, -2]$, then $\mathbf{x} + \mathbf{y} = [2 - 6, -3 + 4, 5 - 2] = [-4, 1, 3]$. Vectors cannot be added unless they have the same number of coordinates.

There is a natural geometric interpretation for the sum of vectors in a plane or in space. Draw a vector $\mathbf{x}$. Then draw a vector $\mathbf{y}$ whose initial point is the terminal point of $\mathbf{x}$. The sum of $\mathbf{x}$ and $\mathbf{y}$ is the vector whose *initial* point is the same as that of $\mathbf{x}$ and whose *terminal* point is the same as that of $\mathbf{y}$. The total movement ($\mathbf{x} + \mathbf{y}$) is equivalent to first moving along $\mathbf{x}$ and then along $\mathbf{y}$. Fig. 1.7 illustrates this in $\mathbb{R}^2$.

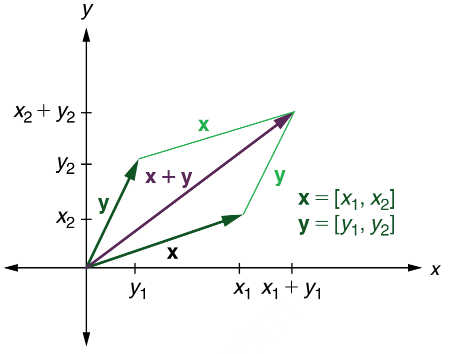


FIGURE 1.7 Addition of vectors in $\mathbb{R}^2$

Let $-\mathbf{y}$ represent the scalar multiple $-1\mathbf{y}$. We can now define **subtraction** of vectors in a natural way: if $\mathbf{x}$ and $\mathbf{y}$ are both vectors in $\mathbb{R}^n$, let $\mathbf{x} - \mathbf{y}$ be the vector $\mathbf{x} + (-\mathbf{y})$. A geometric interpretation of this is in Fig. 1.8 (movement $\mathbf{x}$ followed by movement $-\mathbf{y}$). An alternative interpretation is described in Exercise 11.

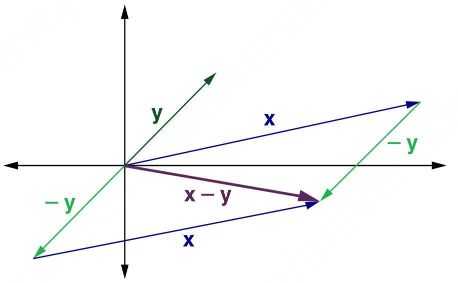


FIGURE 1.8 Subtraction of vectors in $\mathbb{R}^2$: $\mathbf{x} - \mathbf{y} = \mathbf{x} + (-\mathbf{y})$

## Fundamental Properties of Addition and Scalar Multiplication

Theorem 1.3 contains the basic properties of addition and scalar multiplication of vectors. The **commutative**, **associative**, and **distributive** laws are so named because they resemble the corresponding laws for real numbers.

**Theorem 1.3** *Let* $\mathbf{x} = [x_1, x_2, \ldots, x_n]$, $\mathbf{y} = [y_1, y_2, \ldots, y_n]$, *and* $\mathbf{z} = [z_1, z_2, \ldots, z_n]$ *be any vectors in* $\mathbb{R}^n$, *and let* $c$ *and* $d$ *be any real numbers (scalars). Let* $\mathbf{0}$ *represent the zero vector in* $\mathbb{R}^n$. *Then*

| | | |
|---|---|---|
| (1) | $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ | *Commutative Law of Addition* |
| (2) | $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$ | *Associative Law of Addition* |
| (3) | $\mathbf{0} + \mathbf{x} = \mathbf{x} + \mathbf{0} = \mathbf{x}$ | *Existence of Identity Element for Addition* |
| (4) | $\mathbf{x} + (-\mathbf{x}) = (-\mathbf{x}) + \mathbf{x} = \mathbf{0}$ | *Existence of Inverse Elements for Addition* |
| (5) | $c(\mathbf{x} + \mathbf{y}) = c\mathbf{x} + c\mathbf{y}$ | *Distributive Laws of Scalar Multiplication* |
| (6) | $(c + d)\mathbf{x} = c\mathbf{x} + d\mathbf{x}$ | *Over Addition* |
| (7) | $(cd)\mathbf{x} = c(d\mathbf{x})$ | *Associativity of Scalar Multiplication* |
| (8) | $1\mathbf{x} = \mathbf{x}$ | *Identity Property for Scalar Multiplication* |

In part (3), the vector $\mathbf{0}$ is called an **identity element** for addition because $\mathbf{0}$ does not change the identity of any vector to which it is added. A similar statement is true in part (8) for the scalar 1 with scalar multiplication. In part (4), the vector

$-\mathbf{x}$ is called the **additive inverse element of x** because it "cancels out $\mathbf{x}$" to produce the additive identity element (= the zero vector).

Each part of the theorem is proved by calculating the entries in each coordinate of the vectors and applying a corresponding law for real-number arithmetic. We illustrate this *coordinate-wise* technique by proving part (6). You are asked to prove other parts of the theorem in Exercise 22.

*Proof.* Proof of Part (6):

$$\begin{aligned}(c+d)\mathbf{x} &= (c+d)\,[x_1, x_2, \ldots, x_n] \\ &= [(c+d)x_1, (c+d)x_2, \ldots, (c+d)x_n] && \text{definition of scalar multiplication} \\ &= [cx_1 + dx_1, cx_2 + dx_2, \ldots, cx_n + dx_n] && \text{coordinate-wise use of distributive law in } \mathbb{R} \\ &= [cx_1, cx_2, \ldots, cx_n] + [dx_1, dx_2, \ldots, dx_n] && \text{definition of vector addition} \\ &= c\,[x_1, x_2, \ldots, x_n] + d\,[x_1, x_2, \ldots, x_n] && \text{definition of scalar multiplication} \\ &= c\mathbf{x} + d\mathbf{x}.\end{aligned}$$

□

The following theorem is very useful (the proof is left as Exercise 23):

**Theorem 1.4** *Let* $\mathbf{x}$ *be a vector in* $\mathbb{R}^n$*, and let* $c$ *be a scalar. If* $c\mathbf{x} = \mathbf{0}$*, then* $c = 0$ *or* $\mathbf{x} = \mathbf{0}$*.*

## Linear Combinations of Vectors

**Definition** Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be vectors in $\mathbb{R}^n$. Then the vector $\mathbf{v}$ is a **linear combination** of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ if and only if there are scalars $c_1, c_2, \ldots, c_k$ such that $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$.

Thus, a linear combination of vectors is a sum of scalar multiples of those vectors. For example, the vector $[-2, 8, 5, 0]$ is a linear combination of $[3, 1, -2, 2]$, $[1, 0, 3, -1]$, and $[4, -2, 1, 0]$ because $2[3, 1, -2, 2] + 4[1, 0, 3, -1] - 3[4, -2, 1, 0] = [-2, 8, 5, 0]$.

Note that any vector in $\mathbb{R}^3$ can be expressed in a unique way as a linear combination of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. For example, $[3, -2, 5] = 3[1, 0, 0] - 2[0, 1, 0] + 5[0, 0, 1] = 3\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$. In general, $[a, b, c] = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. Also, every vector in $\mathbb{R}^n$ can be expressed as a linear combination of the standard unit vectors $\mathbf{e}_1 = [1, 0, 0, \ldots, 0]$, $\mathbf{e}_2 = [0, 1, 0, \ldots, 0]$, $\ldots$, $\mathbf{e}_n = [0, 0, \ldots, 0, 1]$ (why?).

One helpful way to picture linear combinations of a set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in $\mathbb{R}^n$ is to imagine an $n$-dimensional machine that can move a given point in $\mathbb{R}^n$ in several directions simultaneously. We assume that the machine accomplishes this task by having $k$ dials that can be turned by hand, with each dial pre-programmed for a different vector from $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$. This is analogous to the familiar Etch A Sketch®[3] toy, which moves a point on a two-dimensional screen. We can think of this imaginary machine as a "Generalized" Etch A Sketch® (GEaS).

Suppose that turning the GEaS dial for $\mathbf{v}_1$ once clockwise results in a displacement of the given point along the vector $1\mathbf{v}$, while turning the GEaS dial for $\mathbf{v}_1$ once counterclockwise (that is, $-1$ times clockwise) results in a displacement of the given point along the vector $-1\mathbf{v}_1$. Similarly, for example, the $\mathbf{v}_1$ dial can be turned 4 times clockwise for a displacement of $4\mathbf{v}_1$, or $\frac{1}{3}$ of the way around counterclockwise for a displacement of $-\frac{1}{3}\mathbf{v}_1$. Assume that the GEaS dials for $\mathbf{v}_2, \ldots, \mathbf{v}_k$ behave in a similar fashion, producing displacements that are appropriate scalar multiples of $\mathbf{v}_2, \ldots, \mathbf{v}_k$, respectively. Then this GEaS will displace the given point along the linear combination $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$ when we simultaneously turn the first dial $c_1$ times clockwise, the second dial $c_2$ times clockwise, etc.

For example, suppose we program three dials of a GEaS for $\mathbb{R}^2$ with the vectors $[1, 3]$, $[4, -5]$, and $[2, -1]$. If we turn the first dial 2 times clockwise, the second dial $\frac{1}{2}$ of a turn counterclockwise, and the third dial 3 times clockwise, the overall displacement obtained is the linear combination $\mathbf{w} = 2[1, 3] - \frac{1}{2}[4, -5] + 3[2, -1] = [6, \frac{11}{2}]$, as shown in Fig. 1.9(a).

Next, consider the set of all displacements that can result from all possible linear combinations of a certain set of vectors. For example, the set of all linear combinations in $\mathbb{R}^3$ of $\mathbf{v}_1 = [2, 0, 1]$ and $\mathbf{v}_2 = [0, 1, -2]$ is the set of all vectors of the

[3] Etch A Sketch is a registered trademark of the Ohio Art Company.

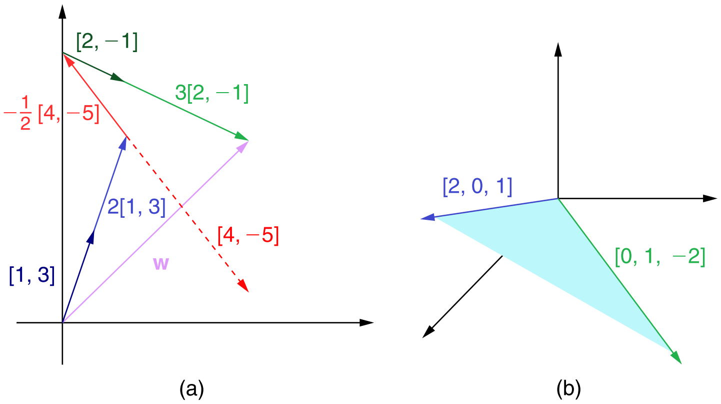


FIGURE 1.9 (a) The linear combination $\mathbf{w} = 2[1,3] - \frac{1}{2}[4,-5] + 3[2,-1] = [6, \frac{11}{2}]$; (b) The plane in $\mathbb{R}^3$ containing all linear combinations of [2, 0, 1] and [0, 1, −2]

form $c_1[2,0,1] + c_2[0,1,-2]$. If we use the origin as a common initial point, this is the set of all vectors with endpoints lying in the plane through the origin containing [2, 0, 1] and [0, 1, −2] (see Fig. 1.9(b)). In other words, from the origin, it is not possible to reach endpoints lying outside this plane by using a GEaS for $\mathbb{R}^3$ with dials corresponding to [2, 0, 1] and [0, 1, −2]. An interesting problem that we will explore in depth later is to determine exactly which endpoints can and cannot be reached from the origin for a given GEaS.

## Physical Applications of Addition and Scalar Multiplication

Addition and scalar multiplication of vectors are often used to solve problems in elementary physics. Recall the trigonometric fact that if $\mathbf{v}$ is a vector in $\mathbb{R}^2$ forming an angle of $\theta$ with the positive $x$-axis then $\mathbf{v} = [\|\mathbf{v}\|\cos\theta, \|\mathbf{v}\|\sin\theta]$, as in Fig. 1.10.

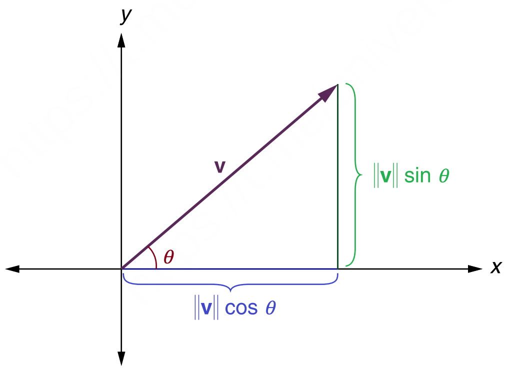


FIGURE 1.10 The vector $\mathbf{v} = [\|\mathbf{v}\|\cos\theta, \|\mathbf{v}\|\sin\theta]$ forming an angle of $\theta$ with the positive $x$-axis

### Example 3

**Resultant Velocity:** Suppose a man swims 5 km/hr in calm water. If he is swimming toward the east in a wide stream with a northwest current of 3 km/hr, what is his **resultant velocity** (net speed and direction)?

The velocities of the swimmer and current are shown as vectors in Fig. 1.11, where we have, for convenience, placed the swimmer at the origin. Now, $\mathbf{v}_1 = [5,0]$ and $\mathbf{v}_2 = [3\cos 135°, 3\sin 135°] = \left[-3\sqrt{2}/2, 3\sqrt{2}/2\right]$. Thus, the total (resultant) velocity of the swimmer is the sum of these velocities, $\mathbf{v}_1 + \mathbf{v}_2$, which is $\left[5 - 3\sqrt{2}/2, 3\sqrt{2}/2\right] \approx [2.88, 2.12]$. Hence, each hour the swimmer is traveling about 2.9 km east and 2.1 km north. The **resultant speed** of the swimmer is $\left\|\left[5 - 3\sqrt{2}/2, 3\sqrt{2}/2\right]\right\| \approx 3.58$ km/hr.

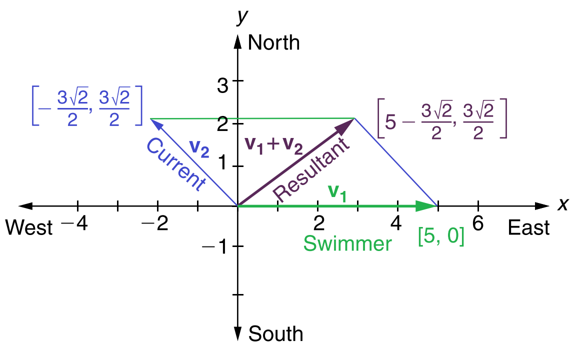


**FIGURE 1.11** Velocity $\mathbf{v}_1$ of swimmer, velocity $\mathbf{v}_2$ of current, and resultant velocity $\mathbf{v}_1 + \mathbf{v}_2$

**Example 4**

**Newton's Second Law:** Newton's famous **Second Law of Motion** asserts that the sum, $\mathbf{f}$, of the vector forces on an object is equal to the scalar multiple of the mass $m$ of the object times the vector acceleration $\mathbf{a}$ of the object; that is, $\mathbf{f} = m\mathbf{a}$. For example, suppose a mass of 5 kg (kilograms) in a three-dimensional coordinate system has two forces acting on it: a force $\mathbf{f}_1$ of 10 newtons[4] in the direction of the vector $[-2, 1, 2]$ and a force $\mathbf{f}_2$ of 20 newtons in the direction of the vector $[6, 3, -2]$. What is the acceleration of the object?

To find the vectors representing $\mathbf{f}_1$ and $\mathbf{f}_2$, we multiply the magnitude of each vector by a unit vector in that vector's direction. The magnitudes of $\mathbf{f}_1$ and $\mathbf{f}_2$ are 10 and 20, respectively. Next, we normalize the direction vectors $[-2, 1, 2]$ and $[6, 3, -2]$ to create unit vectors in those directions, obtaining $[-2, 1, 2]/\|[-2, 1, 2]\|$ and $[6, 3, -2]/\|[6, 3, -2]\|$, respectively. Therefore, $\mathbf{f}_1 = 10\left([-2, 1, 2]/\|[-2, 1, 2]\|\right)$, and $\mathbf{f}_2 = 20\left([6, 3, -2]/\|[6, 3, -2]\|\right)$. Now, the net force on the object is $\mathbf{f} = \mathbf{f}_1 + \mathbf{f}_2$. Thus, the net acceleration on the object is

$$\mathbf{a} = \frac{1}{m}\mathbf{f} = \frac{1}{m}(\mathbf{f}_1 + \mathbf{f}_2) = \frac{1}{5}\left(10\left(\frac{[-2, 1, 2]}{\|[-2, 1, 2]\|}\right) + 20\left(\frac{[6, 3, -2]}{\|[6, 3, -2]\|}\right)\right),$$

which equals $\frac{2}{3}[-2, 1, 2] + \frac{4}{7}[6, 3, -2] = \left[\frac{44}{21}, \frac{50}{21}, \frac{4}{21}\right]$. The length of $\mathbf{a}$ is approximately 3.18, so pulling out a factor of 3.18 from each coordinate, we can approximate $\mathbf{a}$ as $3.18[0.66, 0.75, 0.06]$, where $[0.66, 0.75, 0.06]$ is a *unit* vector. Hence, the acceleration is about 3.18 m/sec$^2$ in the direction $[0.66, 0.75, 0.06]$.

If the sum of the forces on an object is $\mathbf{0}$, then the object is in **equilibrium**; there is no acceleration in any direction (see Exercise 20).

## New Vocabulary

addition of vectors
additive inverse vector
associative law for vector addition
associative law for scalar multiplication
commutative law for vector addition
contraction of a vector
corollary
dilation of a vector
distance formula
distributive laws for vectors
equilibrium
initial point of a vector
length (norm, magnitude) of a vector
linear combination of vectors
normalization of a vector
opposite direction vectors
parallel vectors
real $n$-vector
resultant speed
resultant velocity
same direction vectors
scalar
scalar multiplication of a vector
standard unit vectors
subtraction of vectors
terminal point of a vector
unit vector
zero $n$-vector

[4] 1 newton = 1 kg-m/sec$^2$ (kilogram-meter/second$^2$), or the force needed to push 1 kg at a speed 1 m/sec (meter per second) faster every second.

## Highlights

- $n$-vectors are used to represent movement from one point to another in an $n$-dimensional coordinate system.
- The norm (length) of a vector $\mathbf{a} = [a_1, a_2, \ldots, a_n]$ in $\mathbb{R}^n$ is $\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$, the nonnegative distance from its initial point to its terminal point.
- If $c$ is a scalar, and $\mathbf{x}$ is a vector, then $\|c\mathbf{x}\| = |c|\,\|\mathbf{x}\|$.
- Multiplication of a nonzero vector by a nonzero scalar results in a vector that is parallel to the original.
- For any given nonzero vector $\mathbf{v}$, there is a *unique* unit vector $\mathbf{u}$ in the same direction, found by normalizing the given vector: $\mathbf{u} = (\frac{1}{||\mathbf{v}||})\mathbf{v}$.
- The sum and difference of two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^2$ can be found using the diagonals of parallelograms with adjacent sides $\mathbf{x}$ and $\mathbf{y}$.
- The commutative, associative, and distributive laws hold for addition of vectors in $\mathbb{R}^n$.
- If $c$ is a scalar, $\mathbf{x}$ is a vector, and $c\mathbf{x} = \mathbf{0}$, then $c = 0$ or $\mathbf{x} = \mathbf{0}$.
- A linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ is any vector of the form $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$, where $c_1, c_2, \ldots, c_k$ are scalars.
- Every vector in $\mathbb{R}^n$ is a linear combination of the standard unit vectors in $\mathbb{R}^n$.
- The linear combinations of a given set of vectors represent all possible displacements that can be created using an imaginary GEaS whose dials respectively correspond to the distinct vectors in the linear combination.
- Any nonzero vector $\mathbf{v}$ in $\mathbb{R}^2$ can be expressed as $[||\mathbf{v}|| \cos\theta, ||\mathbf{v}|| \sin\theta]$, where $\theta$ is the angle $\mathbf{v}$ forms with the positive $x$-axis.
- The resultant velocity of an object is the sum of its individual vector velocities.
- The sum, $\mathbf{f}$, of the vector forces on an object is equal to the mass $m$ of the object times the vector acceleration $\mathbf{a}$ of the object; that is, $\mathbf{f} = m\mathbf{a}$.

## Exercises for Section 1.1

**Note:** A star (★) next to an exercise indicates that the answer for that exercise appears in the back of the book, and the full solution appears in the Student Solutions Manual. A wedge (▶) next to an exercise indicates that the answer for this exercise appears in the Student Solutions Manual, but not in the back of this book. The wedge is typically reserved for problems that ask you to prove a theorem that appears in the text.

1. In each of the following cases, find a vector that represents a movement from the first (initial) point to the second (terminal) point. Then use this vector to find the distance between the given points.
   - ★ **(a)** $(-4, 3), (5, -1)$
   - **(b)** $(8, -2, 5), (-3, 2, 0)$
   - ★ **(c)** $(1, -2, 0, 2, 3), (0, -3, 2, -1, -1)$
2. In each of the following cases, draw a directed line segment in space that represents the movement associated with each of the vectors if the initial point is $(1, 1, 1)$. What is the terminal point in each case?
   - ★ **(a)** $[2, 3, 1]$
   - **(b)** $[-1, 4, 2]$
   - ★ **(c)** $[0, -3, -1]$
   - **(d)** $[2, -1, -1]$
3. In each of the following cases, find the initial point, given the vector and the terminal point.
   - ★ **(a)** $[-1, 4], (6, -9)$
   - **(b)** $[-4, 1, 6], (6, -4, 5)$
   - ★ **(c)** $[3, -4, 0, 1, -2], (2, -1, -1, 5, 4)$
4. In each of the following cases, find a point that is two-thirds of the distance from the first (initial) point to the second (terminal) point.
   - ★ **(a)** $(-4, 7, 2), (10, -10, 11)$
   - **(b)** $(5, -7, 2, -6), (-1, -10, 8, 4)$
5. In each of the following cases, find a unit vector in the same direction as the given vector. Is the resulting (normalized) vector longer or shorter than the original? Why?
   - ★ **(a)** $[3, -5, 6]$
   - **(b)** $[-4, 0, 7, 4]$
   - ★ **(c)** $[0.6, -0.8]$
   - **(d)** $\left[\frac{2}{21}, -\frac{5}{21}, \frac{8}{21}, -\frac{2}{21}, -\frac{1}{21}\right]$
6. Which of the following pairs of vectors are parallel?
   - ★ **(a)** $[12, -16], [9, -12]$
   - **(b)** $[16, -30], [24, -42]$
   - ★ **(c)** $[-2, 3, 1], [6, -4, -3]$
   - **(d)** $\left[-9, 24, -8, 15, \frac{18}{5}\right], \left[\frac{15}{4}, -10, \frac{10}{3}, -\frac{25}{4}, -\frac{3}{2}\right]$

7. If $\mathbf{x} = [-2, 4, 5]$, $\mathbf{y} = [-5, -3, 6]$, and $\mathbf{z} = [4, -1, 2]$, find the following:
   ★ (a) $3\mathbf{x}$
   (b) $-2\mathbf{z}$
   ★ (c) $\mathbf{x} + \mathbf{y}$
   (d) $\mathbf{x} - \mathbf{z}$
   ★ (e) $4\mathbf{y} - 5\mathbf{x}$
   (f) $3\mathbf{x} - 5\mathbf{y} + 4\mathbf{z}$
8. Given $\mathbf{x}$ and $\mathbf{y}$ as follows, calculate $\mathbf{x} + \mathbf{y}$, $\mathbf{x} - \mathbf{y}$, and $\mathbf{y} - \mathbf{x}$, and sketch $\mathbf{x}$, $\mathbf{y}$, $\mathbf{x} + \mathbf{y}$, $\mathbf{x} - \mathbf{y}$, and $\mathbf{y} - \mathbf{x}$ in the same coordinate system.
   ★ (a) $\mathbf{x} = [-1, 5]$, $\mathbf{y} = [2, -4]$
   (b) $\mathbf{x} = [10, -2]$, $\mathbf{y} = [-3, -7]$
   ★ (c) $\mathbf{x} = [2, 5, -3]$, $\mathbf{y} = [-1, 3, -2]$
   (d) $\mathbf{x} = [1, -2, 3]$, $\mathbf{y} = [-3, 2, -1]$
9. Show that the points $(2, 5, 5)$, $(6, -3, 2)$, and $(15, -5, 0)$ are the vertices of an isosceles triangle. Is this an equilateral triangle?
10. A certain clock has a minute hand that is 10 cm long. Find the vector representing the displacement of the tip of the minute hand of the clock:
   ★ (a) From 12 PM to 12:15 PM
   ★ (b) From 12 PM to 12:40 PM (Hint: use trigonometry)
   (c) From 12 PM to 1:30 PM
11. Show that if $\mathbf{x}$ and $\mathbf{y}$ are vectors in $\mathbb{R}^2$, then $\mathbf{x} + \mathbf{y}$ and $\mathbf{x} - \mathbf{y}$ are the two diagonals of the parallelogram whose sides are $\mathbf{x}$ and $\mathbf{y}$.
12. Consider the vectors in $\mathbb{R}^3$ in Fig. 1.12. Verify that $\mathbf{x} + (\mathbf{y} + \mathbf{z})$ is a diagonal of the parallelepiped with sides $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$. Do either of the vectors $(\mathbf{x} + \mathbf{y}) + \mathbf{z}$ and $(\mathbf{x} + \mathbf{z}) + \mathbf{y}$ represent the same diagonal vector? Why or why not?

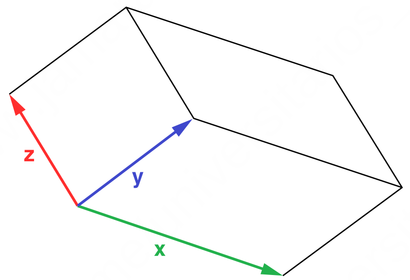


**FIGURE 1.12** Parallelepiped with sides $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$

★ 13. At a certain green on a golf course, a golfer takes three putts to sink the ball. If the first putt moved the ball 1 m (meter) southwest, the second putt moved the ball 0.5 m east, and the third putt moved the ball 0.2 m northwest, what single putt (expressed as a vector) would have had the same final result?
★ 14. A rower can propel a boat 4 km/hr on a calm river. If the rower rows northwestward against a current of 3 km/hr southward, what is the net velocity of the boat? What is its resultant speed?
15. A singer is walking 2 km/hr southeastward on a moving parade float that is being pulled northeastward at 5 km/hr. What is the net velocity of the singer? What is the singer's resultant speed?
★ 16. A woman rowing on a wide river wants the resultant (net) velocity of her boat to be 8 km/hr westward. If the current is moving 2 km/hr northeastward, what velocity vector should she maintain?
★ 17. Using Newton's Second Law of Motion, find the acceleration vector on a 20-kg object in a three-dimensional coordinate system when the following three forces are simultaneously applied:
   $\mathbf{f}_1$: A force of 4 newtons in the direction of the vector $[3, -12, 4]$
   $\mathbf{f}_2$: A force of 2 newtons in the direction of the vector $[0, -4, -3]$
   $\mathbf{f}_3$: A force of 6 newtons in the direction of the unit vector $\mathbf{k}$
18. Using Newton's Second Law of Motion, find the acceleration vector on a 4-kg object in a three-dimensional coordinate system when the following two forces are simultaneously applied:
   $\mathbf{f}_1$: A force of 21 newtons in the direction of the vector $[14, -5, 2]$
   $\mathbf{f}_2$: A force of 34 newtons in the direction of the vector $[12, 8, -9]$
19. Using Newton's Second Law of Motion, find the resultant sum of the forces on a 23-kg object in a three-dimensional coordinate system undergoing an acceleration of 7 m/sec$^2$ in the direction of the vector $[5, -3, 8]$.

★ **20.** Two forces, **a** and **b**, are simultaneously applied along cables attached to a weight, as in Fig. 1.13, to keep the weight in equilibrium by balancing the force of gravity (which is $m\mathbf{g}$, where $m$ is the mass of the weight and $\mathbf{g} = [0, -g]$ (with units in m/sec$^2$) is the downward acceleration due to gravity). Solve for the coordinates of forces **a** and **b** in terms of $m$ and $g$.

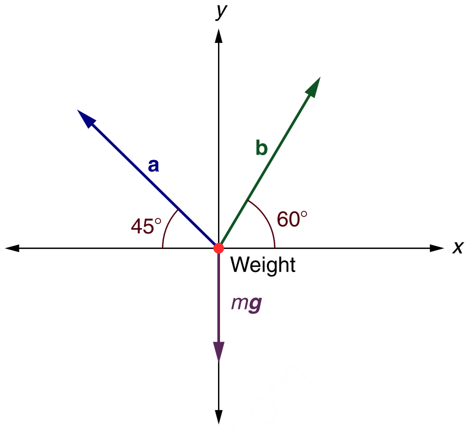


FIGURE 1.13 Forces in equilibrium

**21.** This exercise develops some properties of the length of a vector.
  **(a)** Prove that the length of each vector in $\mathbb{R}^n$ is nonnegative (that is, $\geq 0$).
  **(b)** Prove that the only vector in $\mathbb{R}^n$ of length 0 is the zero vector.

**22.** This exercise asks for proofs of various parts of Theorem 1.3.
  **(a)** Prove part (2) of Theorem 1.3.
  ▶ **(b)** Prove part (4) of Theorem 1.3.
  ▶ **(c)** Prove part (5) of Theorem 1.3.
  **(d)** Prove part (7) of Theorem 1.3.

▶ **23.** Prove Theorem 1.4.

**24.** If $\mathbf{x}$ is a vector in $\mathbb{R}^n$ and $c_1 \neq c_2$, show that $c_1\mathbf{x} = c_2\mathbf{x}$ implies that $\mathbf{x} = \mathbf{0}$ (zero vector).

★ **25.** True or False:
  **(a)** The length of $\mathbf{a} = [a_1, a_2, a_3]$ is $a_1^2 + a_2^2 + a_3^2$.
  **(b)** For any vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$ in $\mathbb{R}^n$, $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{z} + (\mathbf{y} + \mathbf{x})$.
  **(c)** $[2, 0, -3]$ is a linear combination of $[1, 0, 0]$ and $[0, 0, 1]$.
  **(d)** The vectors $[3, -5, 2]$ and $[6, -10, 5]$ are parallel.
  **(e)** Let $\mathbf{x} \in \mathbb{R}^n$, and let $d$ be a scalar. If $d\mathbf{x} = \mathbf{0}$, and $d \neq 0$, then $\mathbf{x} = \mathbf{0}$.
  **(f)** If two nonzero vectors in $\mathbb{R}^n$ are parallel, then they are in the same direction.
  **(g)** The properties in Theorem 1.3 are only true if all the vectors have their initial points at the origin.
  **(h)** $[9, -8, -4]$ is a unit vector.

## 1.2 The Dot Product

We now discuss another important vector operation: the dot product. After explaining several properties of the dot product, we show how to calculate the angle between vectors and how to "project" one vector onto another.

### Definition and Properties of the Dot Product

**Definition** Let $\mathbf{x} = [x_1, x_2, \ldots, x_n]$ and $\mathbf{y} = [y_1, y_2, \ldots, y_n]$ be two vectors in $\mathbb{R}^n$. The **dot (inner) product** of $\mathbf{x}$ and $\mathbf{y}$ is given by

$$\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n = \sum_{k=1}^{n} x_ky_k.$$

For example, if $\mathbf{x} = [2, -4, 3]$ and $\mathbf{y} = [1, 5, -2]$, then $\mathbf{x} \cdot \mathbf{y} = (2)(1) + (-4)(5) + (3)(-2) = -24$. Notice that the dot product involves two vectors and the result is a *scalar*, whereas scalar multiplication involves a scalar and a vector and the result is a *vector*. Also, the dot product is not defined for vectors having different numbers of coordinates. The next theorem states some elementary results involving the dot product.

**Theorem 1.5** *If* $\mathbf{x}$, $\mathbf{y}$, *and* $\mathbf{z}$ *are any vectors in* $\mathbb{R}^n$, *and if* $c$ *is any scalar, then*

| | | |
|---|---|---|
| **(1)** | $\mathbf{x}\cdot\mathbf{y}=\mathbf{y}\cdot\mathbf{x}$ | *Commutativity of Dot Product* |
| **(2)** | $\mathbf{x}\cdot\mathbf{x}=\|\mathbf{x}\|^2\geq 0$ | *Relationship Between Dot Product and Length* |
| **(3)** | $\mathbf{x}\cdot\mathbf{x}=0$ *if and only if* $\mathbf{x}=\mathbf{0}$ | |
| **(4)** | $c(\mathbf{x}\cdot\mathbf{y})=(c\mathbf{x})\cdot\mathbf{y}=\mathbf{x}\cdot(c\mathbf{y})$ | *Relationship Between Scalar Multiplication and Dot Product* |
| **(5)** | $\mathbf{x}\cdot(\mathbf{y}+\mathbf{z})=(\mathbf{x}\cdot\mathbf{y})+(\mathbf{x}\cdot\mathbf{z})$ | *Distributive Laws of Dot Product* |
| **(6)** | $(\mathbf{x}+\mathbf{y})\cdot\mathbf{z}=(\mathbf{x}\cdot\mathbf{z})+(\mathbf{y}\cdot\mathbf{z})$ | *Over Addition* |

The proofs of parts (1), (2), (4), (5), and (6) are done by expanding the expressions on each side of the equation and then showing they are equal. We illustrate this with the proof of part (5). The remaining proofs are left as Exercise 6.

*Proof.* Proof of Part (5): Let $\mathbf{x}=[x_1,x_2,\ldots,x_n]$, $\mathbf{y}=[y_1,y_2,\ldots,y_n]$, and $\mathbf{z}=[z_1,z_2,\ldots,z_n]$. Then,

$$\begin{aligned}\mathbf{x}\cdot(\mathbf{y}+\mathbf{z})&=[x_1,x_2,\ldots,x_n]\cdot([y_1,y_2,\ldots,y_n]+[z_1,z_2,\ldots,z_n])\\&=[x_1,x_2,\ldots,x_n]\cdot[y_1+z_1,y_2+z_2,\ldots,y_n+z_n]\\&=x_1(y_1+z_1)+x_2(y_2+z_2)+\cdots+x_n(y_n+z_n)\\&=(x_1y_1+x_2y_2+\cdots+x_ny_n)+(x_1z_1+x_2z_2+\cdots+x_nz_n).\end{aligned}$$

Also,

$$\begin{aligned}(\mathbf{x}\cdot\mathbf{y})+(\mathbf{x}\cdot\mathbf{z})&=([x_1,x_2,\ldots,x_n]\cdot[y_1,y_2,\ldots,y_n])+([x_1,x_2,\ldots,x_n]\cdot[z_1,z_2,\ldots,z_n])\\&=(x_1y_1+x_2y_2+\cdots+x_ny_n)+(x_1z_1+x_2z_2+\cdots+x_nz_n).\end{aligned}$$

Hence, $\mathbf{x}\cdot(\mathbf{y}+\mathbf{z})=(\mathbf{x}\cdot\mathbf{y})+(\mathbf{x}\cdot\mathbf{z})$. □

The properties in Theorem 1.5 allow us to simplify dot product expressions just as in elementary algebra. For example,

$$\begin{aligned}(5\mathbf{x}-4\mathbf{y})\cdot(-2\mathbf{x}+3\mathbf{y})&=[(5\mathbf{x}-4\mathbf{y})\cdot(-2\mathbf{x})]+[(5\mathbf{x}-4\mathbf{y})\cdot(3\mathbf{y})]\\&=[(5\mathbf{x})\cdot(-2\mathbf{x})]+[(-4\mathbf{y})\cdot(-2\mathbf{x})]+[(5\mathbf{x})\cdot(3\mathbf{y})]+[(-4\mathbf{y})\cdot(3\mathbf{y})]\\&=-10(\mathbf{x}\cdot\mathbf{x})+8(\mathbf{y}\cdot\mathbf{x})+15(\mathbf{x}\cdot\mathbf{y})-12(\mathbf{y}\cdot\mathbf{y})\\&=-10\,\|\mathbf{x}\|^2+23\,(\mathbf{x}\cdot\mathbf{y})-12\,\|\mathbf{y}\|^2.\end{aligned}$$

## Inequalities Involving the Dot Product

The following lemma is used in the proof of Theorem 1.7 below. (A lemma is a theorem whose main purpose is to assist in the proof of a more powerful result.)

**Lemma 1.6** *If* $\mathbf{a}$ *and* $\mathbf{b}$ *are unit vectors in* $\mathbb{R}^n$, *then* $-1\leq\mathbf{a}\cdot\mathbf{b}\leq 1$.

*Proof.* Notice that the term $\mathbf{a}\cdot\mathbf{b}$ appears in the expansion of $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})$, as well as in the expansion of $(\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})$. The first expansion gives

$$\begin{aligned}&(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})=\|\mathbf{a}+\mathbf{b}\|^2\geq 0 &&\text{using part (2) of Theorem 1.5}\\ \Longrightarrow\;&(\mathbf{a}\cdot\mathbf{a})+(\mathbf{b}\cdot\mathbf{a})+(\mathbf{a}\cdot\mathbf{b})+(\mathbf{b}\cdot\mathbf{b})\geq 0 &&\text{using parts (5) and (6) of Theorem 1.5}\\ \Longrightarrow\;&\|\mathbf{a}\|^2+2\,(\mathbf{a}\cdot\mathbf{b})+\|\mathbf{b}\|^2\geq 0 &&\text{by parts (1) and (2) of Theorem 1.5}\\ \Longrightarrow\;&1+2\,(\mathbf{a}\cdot\mathbf{b})+1\geq 0 &&\text{because } \mathbf{a} \text{ and } \mathbf{b} \text{ are unit vectors}\\ \Longrightarrow\;&\mathbf{a}\cdot\mathbf{b}\geq -1.\end{aligned}$$

A similar argument beginning with $(\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})=\|\mathbf{a}-\mathbf{b}\|^2\geq 0$ shows $\mathbf{a}\cdot\mathbf{b}\leq 1$ (see Exercise 8). Hence, $-1\leq\mathbf{a}\cdot\mathbf{b}\leq 1$. □

The next theorem gives an upper and lower bound on the dot product.

**Theorem 1.7 (Cauchy-Schwarz Inequality)** *If* $\mathbf{x}$ *and* $\mathbf{y}$ *are vectors in* $\mathbb{R}^n$, *then*

$$|\mathbf{x}\cdot\mathbf{y}| \leq (\|\mathbf{x}\|)(\|\mathbf{y}\|).$$

*Proof.* If $\mathbf{x}=\mathbf{0}$ or $\mathbf{y}=\mathbf{0}$, the theorem is certainly true. Hence, we need only examine the case when both $\|\mathbf{x}\|$ and $\|\mathbf{y}\|$ are nonzero. We need to prove $-(\|\mathbf{x}\|)(\|\mathbf{y}\|) \leq \mathbf{x}\cdot\mathbf{y} \leq (\|\mathbf{x}\|)(\|\mathbf{y}\|)$. This statement is true if and only if

$$-1 \leq \frac{\mathbf{x}\cdot\mathbf{y}}{(\|\mathbf{x}\|)(\|\mathbf{y}\|)} \leq 1.$$

But $(\mathbf{x}\cdot\mathbf{y})/((\|\mathbf{x}\|)(\|\mathbf{y}\|))$ is equal to $(\mathbf{x}/\|\mathbf{x}\|)\cdot(\mathbf{y}/\|\mathbf{y}\|)$. However, both $\mathbf{x}/\|\mathbf{x}\|$ and $\mathbf{y}/\|\mathbf{y}\|$ are *unit* vectors, so by Lemma 1.6, their dot product satisfies the required double inequality above. □

**Example 1**

Let $\mathbf{x}=[-1,4,2,0,-3]$ and let $\mathbf{y}=[2,1,-4,-1,0]$. We verify the Cauchy-Schwarz Inequality in this specific case. Now, $\mathbf{x}\cdot\mathbf{y}=-2+4-8+0+0=-6$. Also, $\|\mathbf{x}\|=\sqrt{1+16+4+0+9}=\sqrt{30}$, and $\|\mathbf{y}\|=\sqrt{4+1+16+1+0}=\sqrt{22}$. Then, $|\mathbf{x}\cdot\mathbf{y}| \leq ((\|\mathbf{x}\|)(\|\mathbf{y}\|))$, because $|-6|=6\leq\sqrt{30}\sqrt{22}=2\sqrt{165}\approx 25.7$.

Another useful result, sometimes known as **Minkowski's Inequality**, is

**Theorem 1.8 (Triangle Inequality)** *If* $\mathbf{x}$ *and* $\mathbf{y}$ *are vectors in* $\mathbb{R}^n$, *then*

$$\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\|+\|\mathbf{y}\|.$$

We can prove this theorem geometrically in $\mathbb{R}^2$ and $\mathbb{R}^3$ by noting that the length of $\mathbf{x}+\mathbf{y}$, one side of the triangle in Fig. 1.14, is never larger than the sum of the lengths of the other two sides, $\mathbf{x}$ and $\mathbf{y}$. The following algebraic proof extends this result to $\mathbb{R}^n$ for $n>3$.

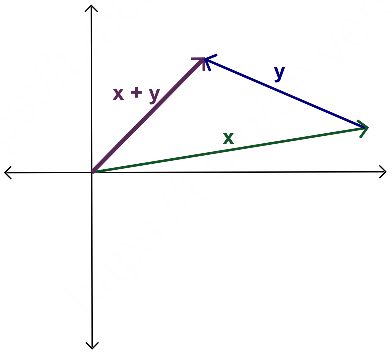


FIGURE 1.14 Triangle Inequality in $\mathbb{R}^2$: $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\|+\|\mathbf{y}\|$

*Proof.* It is enough to show that $\|\mathbf{x}+\mathbf{y}\|^2 \leq (\|\mathbf{x}\|+\|\mathbf{y}\|)^2$ (why?). But

$$\begin{aligned}
\|\mathbf{x}+\mathbf{y}\|^2 &= (\mathbf{x}+\mathbf{y})\cdot(\mathbf{x}+\mathbf{y})\\
&= (\mathbf{x}\cdot\mathbf{x})+2(\mathbf{x}\cdot\mathbf{y})+(\mathbf{y}\cdot\mathbf{y})\\
&= \|\mathbf{x}\|^2+2(\mathbf{x}\cdot\mathbf{y})+\|\mathbf{y}\|^2\\
&\leq \|\mathbf{x}\|^2+2|\mathbf{x}\cdot\mathbf{y}|+\|\mathbf{y}\|^2\\
&\leq \|\mathbf{x}\|^2+2(\|\mathbf{x}\|)(\|\mathbf{y}\|)+\|\mathbf{y}\|^2 \qquad \text{by the Cauchy-Schwarz Inequality}\\
&= (\|\mathbf{x}\|+\|\mathbf{y}\|)^2.
\end{aligned}$$

□

## The Angle Between Two Vectors

The dot product enables us to find the angle $\theta$ between two nonzero vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ that begin at the same initial point. There are actually *two* angles formed by the vectors $\mathbf{x}$ and $\mathbf{y}$, but we always choose the angle $\theta$ between two vectors to be the one measuring between 0 and $\pi$ radians, inclusive.

Consider the vector $\mathbf{x}-\mathbf{y}$ in Fig. 1.15, which begins at the terminal point of $\mathbf{y}$ and ends at the terminal point of $\mathbf{x}$. Because $0 \le \theta \le \pi$, it follows from the Law of Cosines that $\|\mathbf{x}-\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\,(\|\mathbf{x}\|)\,(\|\mathbf{y}\|)\cos\theta$. But,

$$\begin{aligned}\|\mathbf{x}-\mathbf{y}\|^2 &= (\mathbf{x}-\mathbf{y})\cdot(\mathbf{x}-\mathbf{y})\\ &= (\mathbf{x}\cdot\mathbf{x}) - 2\,(\mathbf{x}\cdot\mathbf{y}) + (\mathbf{y}\cdot\mathbf{y})\\ &= \|\mathbf{x}\|^2 - 2\,(\mathbf{x}\cdot\mathbf{y}) + \|\mathbf{y}\|^2.\end{aligned}$$

Equating these two expressions for $||\mathbf{x}-\mathbf{y}||^2$, and then canceling like terms yields $-2\,\|\mathbf{x}\|\,\|\mathbf{y}\|\cos\theta = -2(\mathbf{x}\cdot\mathbf{y})$. This implies $\|\mathbf{x}\|\,\|\mathbf{y}\|\cos\theta = \mathbf{x}\cdot\mathbf{y}$, and so

$$\cos\theta = \frac{\mathbf{x}\cdot\mathbf{y}}{(\|\mathbf{x}\|)\,(\|\mathbf{y}\|)}.$$

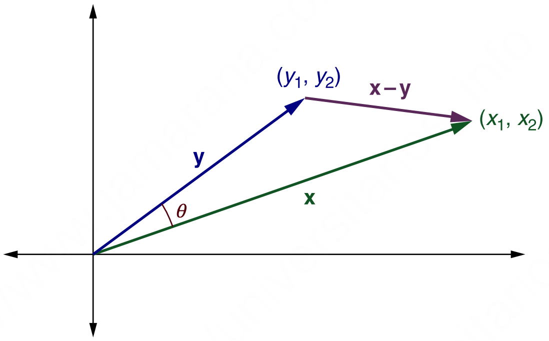


**FIGURE 1.15** The angle $\theta$ between two nonzero vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^2$

---

### Example 2

Suppose $\mathbf{x} = [6, -4]$ and $\mathbf{y} = [-2, 3]$ and $\theta$ is the angle between $\mathbf{x}$ and $\mathbf{y}$. Then,

$$\cos\theta = \frac{\mathbf{x}\cdot\mathbf{y}}{(\|\mathbf{x}\|)\,(\|\mathbf{y}\|)} = \frac{(6)(-2)+(-4)(3)}{\sqrt{52}\sqrt{13}} = -\frac{12}{13} \approx -0.9231.$$

Using the inverse cosine function on a calculator, we find that $\theta = \cos^{-1}(-\frac{12}{13}) \approx 2.74$ radians, or 157.4°. (Remember that $0 \le \theta \le \pi$.)

---

In higher-dimensional spaces, we are outside the geometry of everyday experience, and in such cases, we have not yet defined the angle between two vectors.[5] However, by the Cauchy-Schwarz Inequality, $(\mathbf{x}\cdot\mathbf{y})\,/\,(\|\mathbf{x}\|\,\|\mathbf{y}\|)$ always has a value between $-1$ and 1 (inclusive) for any nonzero vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$. Thus, this value equals $\cos\theta$ for a unique $\theta$ from 0 to $\pi$ radians. Hence, we can define the angle between two vectors in $\mathbb{R}^n$ so it is consistent with the situation in $\mathbb{R}^2$ and $\mathbb{R}^3$ as follows:

**Definition** Let $\mathbf{x}$ and $\mathbf{y}$ be two nonzero vectors in $\mathbb{R}^n$, for $n \ge 2$. Then the **angle between x and y** is the unique angle from 0 to $\pi$ radians whose cosine is $(\mathbf{x}\cdot\mathbf{y})\,/\,((\|\mathbf{x}\|)\,(\|\mathbf{y}\|))$.

[5] Although we cannot easily depict an angle between two vectors (with the same initial point) in four- or higher-dimensional space, the existence of such an angle is intuitively obvious. This is because we are accustomed to the notion in two- and three-dimensional space that three points (in this case, the common initial point of the two vectors, as well as the terminal points of these vectors) determine a plane in which the two vectors lie. In higher-dimensional spaces, a two-dimensional plane exists that contains both vectors, and in which an angle between these vectors makes sense.

**Example 3**

For $\mathbf{x} = [-1, 4, 2, 0, -3]$ and $\mathbf{y} = [2, 1, -4, -1, 0]$, we have $(\mathbf{x} \cdot \mathbf{y}) / ((\|\mathbf{x}\|)(\|\mathbf{y}\|)) = -6/\left(2\sqrt{165}\right) \approx -0.234$. Therefore, the angle $\theta$ between $\mathbf{x}$ and $\mathbf{y}$ is approximately $\cos^{-1}(-0.234) \approx 1.8$ radians, or $103.5°$.

The following theorem is an immediate consequence of the last definition and properties of the cosine function:

**Theorem 1.9** *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be nonzero vectors in* $\mathbb{R}^n$, *and let* $\theta$ *be the angle between* $\mathbf{x}$ *and* $\mathbf{y}$. *Then,*

**(1)** $\mathbf{x} \cdot \mathbf{y} > 0$ *if and only if* $0 \le \theta < \frac{\pi}{2}$ *radians* ($0°$ *or acute*).
**(2)** $\mathbf{x} \cdot \mathbf{y} = 0$ *if and only if* $\theta = \frac{\pi}{2}$ *radians* ($90°$).
**(3)** $\mathbf{x} \cdot \mathbf{y} < 0$ *if and only if* $\frac{\pi}{2} < \theta \le \pi$ *radians* (*obtuse or* $180°$).

## Special Cases: Orthogonal and Parallel Vectors

**Definition** Two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ are **orthogonal** if and only if $\mathbf{x} \cdot \mathbf{y} = 0$.

Note that by Theorem 1.9, two *nonzero* vectors are orthogonal if and only if they are perpendicular to each other.

**Example 4**

The vectors $\mathbf{x} = [2, -5]$ and $\mathbf{y} = [-10, -4]$ are orthogonal in $\mathbb{R}^2$ because $\mathbf{x} \cdot \mathbf{y} = 0$. By Theorem 1.9, $\mathbf{x}$ and $\mathbf{y}$ form a right angle, as shown in Fig. 1.16.

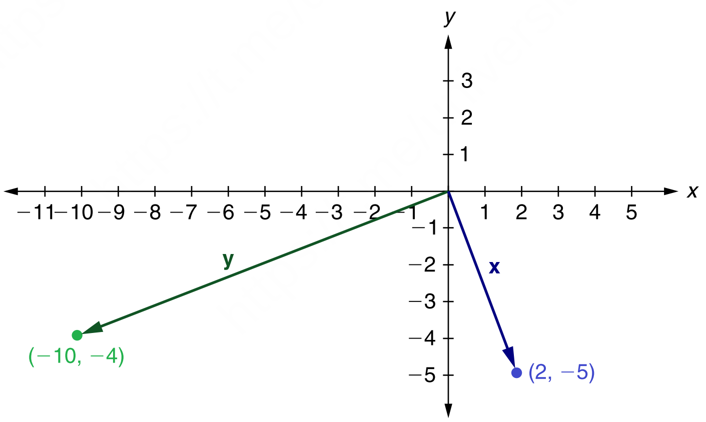


**FIGURE 1.16** The orthogonal vectors $\mathbf{x} = [2, -5]$ and $\mathbf{y} = [-10, -4]$

In $\mathbb{R}^3$, the set of vectors $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is **mutually orthogonal**; that is, the dot product of any two different vectors from this set equals zero. In general, in $\mathbb{R}^n$ the standard unit vectors $\mathbf{e}_1 = [1, 0, 0, \ldots, 0], \mathbf{e}_2 = [0, 1, 0, \ldots, 0], \ldots, \mathbf{e}_n = [0, 0, 0, \ldots, 1]$ form a mutually orthogonal set of vectors.

We now turn to parallel vectors. In this section, we defined the angle between two nonzero vectors $\mathbf{x}$ and $\mathbf{y}$ to be $0°$ or $180°$ when $(\mathbf{x} \cdot \mathbf{y})/(\|\mathbf{x}\|\|\mathbf{y}\|) = +1$ or $-1$, respectively. However, we still need to establish in higher dimensions that this geometric condition is, in fact, equivalent to our earlier algebraic definition of parallel vectors from Section 1.1—that is, two nonzero vectors $\mathbf{x}$ and $\mathbf{y}$ are parallel if one is a nonzero scalar multiple of the other.

The next theorem shows that the geometric condition for parallel vectors implies the algebraic condition. (The proof that the algebraic condition implies the geometric condition is left as Exercise 15.)

**Theorem 1.10** *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be nonzero vectors in* $\mathbb{R}^n$. *If* $\mathbf{x}\cdot\mathbf{y} = \pm\|\mathbf{x}\|\,\|\mathbf{y}\|$ *(that is,* $\cos\theta = \pm 1$, *where* $\theta$ *is the angle between* $\mathbf{x}$ *and* $\mathbf{y}$*), then* $\mathbf{x}$ *and* $\mathbf{y}$ *are parallel (that is,* $\mathbf{y} = c\mathbf{x}$ *for some* $c \neq 0$*).*

Before beginning the proof of Theorem 1.10, notice that if $\mathbf{y} = c\mathbf{x}$ for given nonzero vectors $\mathbf{x}$ and $\mathbf{y}$, then $\mathbf{x}\cdot\mathbf{y} = \mathbf{x}\cdot(c\mathbf{x}) = c\,(\mathbf{x}\cdot\mathbf{x}) = c\,\|\mathbf{x}\|^2$. Therefore, the value of $c$ must equal $\frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|^2}$. We use this insight in the following proof.

*Proof.* Suppose that $\mathbf{x}\cdot\mathbf{y} = \pm\|\mathbf{x}\|\,\|\mathbf{y}\|$. (Note that $\mathbf{x}\cdot\mathbf{y} \neq 0$ since both $||\mathbf{x}||$ and $||\mathbf{y}||$ are nonzero.) We want to prove that $\mathbf{y} = c\mathbf{x}$ for some $c \neq 0$. It is enough to show that $\mathbf{y} - c\mathbf{x} = \mathbf{0}$ for the particular value of $c$ suggested in the remark above. That is, we show that $\mathbf{y} - \left(\frac{\mathbf{x}\cdot\mathbf{y}}{||\mathbf{x}||^2}\right)\mathbf{x} = \mathbf{0}$. By part (3) of Theorem 1.5, it is enough to show that the dot product of $\mathbf{y} - \left(\frac{\mathbf{x}\cdot\mathbf{y}}{||\mathbf{x}||^2}\right)\mathbf{x}$ with itself equals 0:

$$
\begin{aligned}
&(\mathbf{y} - \frac{\mathbf{x}\cdot\mathbf{y}}{||\mathbf{x}||^2}\mathbf{x})\cdot(\mathbf{y} - \frac{\mathbf{x}\cdot\mathbf{y}}{||\mathbf{x}||^2}\mathbf{x}) \\
&= (\mathbf{y}\cdot\mathbf{y}) - 2\left(\frac{\mathbf{x}\cdot\mathbf{y}}{||\mathbf{x}||^2}\right)(\mathbf{x}\cdot\mathbf{y}) + \left(\frac{\mathbf{x}\cdot\mathbf{y}}{||\mathbf{x}||^2}\right)^2(\mathbf{x}\cdot\mathbf{x}) && \text{expanding the dot product} \\
&= ||\mathbf{y}||^2 - 2\left(\frac{(\mathbf{x}\cdot\mathbf{y})^2}{||\mathbf{x}||^2}\right) + \left(\frac{(\mathbf{x}\cdot\mathbf{y})^2}{||\mathbf{x}||^4}\right)||\mathbf{x}||^2 && \text{by Theorem 1.5, part (2)} \\
&= ||\mathbf{y}||^2 - 2\left(\frac{(\mathbf{x}\cdot\mathbf{y})^2}{||\mathbf{x}||^2}\right) + \left(\frac{(\mathbf{x}\cdot\mathbf{y})^2}{||\mathbf{x}||^2}\right) \\
&= ||\mathbf{y}||^2 - \frac{(\mathbf{x}\cdot\mathbf{y})^2}{||\mathbf{x}||^2} \\
&= \frac{(||\mathbf{x}||\,||\mathbf{y}||)^2}{||\mathbf{x}||^2} - \frac{(\mathbf{x}\cdot\mathbf{y})^2}{||\mathbf{x}||^2} && \text{finding a common denominator} \\
&= 0. && \text{since } \mathbf{x}\cdot\mathbf{y} = \pm\|\mathbf{x}\|\,\|\mathbf{y}\|
\end{aligned}
$$

Thus, $\mathbf{y} = c\mathbf{x}$, where $c = \frac{\mathbf{x}\cdot\mathbf{y}}{||\mathbf{x}||^2} \neq 0$. □

**Example 5**

Let $\mathbf{x} = [8, -20, 4]$ and $\mathbf{y} = [6, -15, 3]$. Then, if $\theta$ is the angle between $\mathbf{x}$ and $\mathbf{y}$,

$$\cos\theta = \frac{\mathbf{x}\cdot\mathbf{y}}{(\|\mathbf{x}\|)(\|\mathbf{y}\|)} = \frac{48 + 300 + 12}{\sqrt{480}\sqrt{270}} = \frac{360}{\sqrt{129600}} = 1.$$

Thus, by Theorem 1.10, $\mathbf{x}$ and $\mathbf{y}$ are parallel. (Notice also that $\mathbf{x}$ and $\mathbf{y}$ are parallel by the definition of parallel vectors in Section 1.1 because $[8, -20, 4] = \frac{4}{3}[6, -15, 3]$.)

## Projection Vectors

The projection of one vector onto another is useful in physics, engineering, computer graphics, and statistics. Suppose $\mathbf{a}$ and $\mathbf{b}$ are nonzero vectors, both in $\mathbb{R}^2$ or both in $\mathbb{R}^3$, drawn at the same initial point. Let $\theta$ represent the angle between $\mathbf{a}$ and $\mathbf{b}$. Drop a perpendicular line segment from the terminal point of $\mathbf{b}$ to the straight line $\ell$ containing the vector $\mathbf{a}$, as in Fig. 1.17.

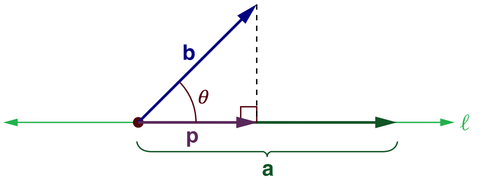


FIGURE 1.17 The projection $\mathbf{p}$ of the vector $\mathbf{b}$ onto $\mathbf{a}$ (when $\theta$ is acute)

By the projection **p** of **b** onto **a**, we mean the vector from the initial point of **a** to the point where the dropped perpendicular meets the line $\ell$. Note that **p** is in the same direction as **a** when $0 \le \theta < \frac{\pi}{2}$ radians (see Fig. 1.17) and in the opposite direction to **a** when $\frac{\pi}{2} < \theta \le \pi$ radians, as in Fig. 1.18.

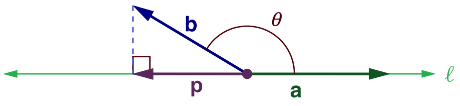


FIGURE 1.18 The projection **p** of **b** onto **a** (when $\theta$ is obtuse)

Using trigonometry, we see that when $0 \le \theta \le \frac{\pi}{2}$, the vector **p** has length $\|\mathbf{b}\|\cos\theta$ and is in the direction of the unit vector $\mathbf{a}/\,\|\mathbf{a}\|$. Also, when $\frac{\pi}{2} < \theta \le \pi$, **p** has length $-\|\mathbf{b}\|\cos\theta$ and is in the direction of the unit vector $-\mathbf{a}/\,\|\mathbf{a}\|$. Therefore, we can express **p** in all cases as

$$\mathbf{p} = (\|\mathbf{b}\|\cos\theta)\left(\frac{\mathbf{a}}{\|\mathbf{a}\|}\right).$$

But we know that $\cos\theta = (\mathbf{a}\cdot\mathbf{b})\,/\,(\|\mathbf{a}\|\,\|\mathbf{b}\|)$, and hence

$$\mathbf{p} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2}\right)\mathbf{a}.$$

The projection **p** of vector **b** onto **a** is often represented by $\mathbf{proj_a b}$.

---

**Example 6**

Let $\mathbf{a} = [4, 0, -3]$ and $\mathbf{b} = [3, 1, -7]$. Then

$$\mathbf{proj_a b} = \mathbf{p} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2}\right)\mathbf{a} = \frac{(4)(3)+(0)(1)+(-3)(-7)}{\left(\sqrt{16+0+9}\right)^2}\mathbf{a} = \frac{33}{25}\mathbf{a}$$
$$= \frac{33}{25}[4, 0, -3] = \left[\frac{132}{25}, 0, -\frac{99}{25}\right].$$

---

Next, we algebraically define projection vectors in $\mathbb{R}^n$, for all values of $n$, to be consistent with the geometric definition in $\mathbb{R}^2$ and $\mathbb{R}^3$.

**Definition** If **a** and **b** are vectors in $\mathbb{R}^n$, with $\mathbf{a} \ne \mathbf{0}$, then the **projection vector of b onto a** is

$$\mathbf{proj_a b} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2}\right)\mathbf{a}.$$

The projection vector can be used to decompose a given vector **b** into the sum of two **component vectors**. Suppose $\mathbf{a} \ne \mathbf{0}$. Notice that if $\mathbf{proj_a b} \ne \mathbf{0}$, then it is parallel to **a** by definition because it is a scalar multiple of **a** (see Fig. 1.19). Also, $\mathbf{b} - \mathbf{proj_a b}$ is orthogonal to **a** because

$$\begin{aligned}(\mathbf{b} - \mathbf{proj_a b})\cdot\mathbf{a} &= \mathbf{b}\cdot\mathbf{a} - (\mathbf{proj_a b})\cdot\mathbf{a}\\ &= \mathbf{b}\cdot\mathbf{a} - \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2}\right)(\mathbf{a}\cdot\mathbf{a})\\ &= \mathbf{b}\cdot\mathbf{a} - \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2}\right)\|\mathbf{a}\|^2\\ &= 0.\end{aligned}$$

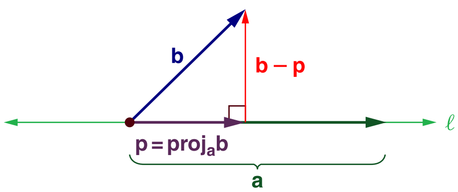


**FIGURE 1.19** Decomposition of a vector **b** into two components: one parallel to **a** and the other orthogonal to **a**

Because $\mathbf{proj_a b} + (\mathbf{b} - \mathbf{proj_a b}) = \mathbf{b}$, we have proved

**Theorem 1.11** *Let* **a** *be a nonzero vector in* $\mathbb{R}^n$*, and let* **b** *be any vector in* $\mathbb{R}^n$*. Then* **b** *can be decomposed as the sum of two component vectors,* $\mathbf{proj_a b}$ *and* $\mathbf{b} - \mathbf{proj_a b}$*, where the first (if nonzero) is parallel to* **a** *and the second is orthogonal to* **a**.

### Example 7

Consider $\mathbf{a} = [4, 0, -3]$ and $\mathbf{b} = [3, 1, -7]$ from Example 6, where we found the component of **b** in the direction of the vector **a** is $\mathbf{proj_a b} = [132/25, 0, -99/25]$. Then the component of **b** orthogonal to **a** (and **p** as well) is $\mathbf{b} - \mathbf{proj_a b} = [-57/25, 1, -76/25]$. We can easily check that $\mathbf{b} - \mathbf{p}$ is orthogonal to **a** as follows:

$$(\mathbf{b} - \mathbf{p}) \cdot \mathbf{a} = \left(-\frac{57}{25}\right)(4) + (1)(0) + \left(-\frac{76}{25}\right)(-3) = -\frac{228}{25} + \frac{228}{25} = 0.$$

## Application: Work

Suppose that a vector force **f** is exerted on an object and causes the object to undergo a vector displacement **d**. Let $\theta$ be the angle between these vectors. In physics, when measuring the work done on the object, only the component of the force that acts in the direction of movement is important. But the component of **f** in the direction of **d** is $\|\mathbf{f}\| \cos\theta$, as shown in Fig. 1.20. Thus, the **work** accomplished by the force is defined to be the product of this force component, $\|\mathbf{f}\| \cos\theta$, times the length $\|\mathbf{d}\|$ of the displacement, which equals $(\|\mathbf{f}\| \cos\theta) \|\mathbf{d}\| = \mathbf{f} \cdot \mathbf{d}$. That is, we can calculate the work simply by finding the dot product of **f** and **d**.

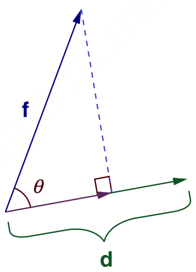


**FIGURE 1.20** Projection $\|\mathbf{f}\| \cos\theta$ of a vector force **f** onto a vector displacement **d**, with angle $\theta$ between **f** and **d**

Work is measured in *joules*, where 1 joule is the work done when a force of 1 newton (nt) moves an object 1 m.

### Example 8

Suppose that a force of 8 nt is exerted on an object in the direction of the vector $[1, -2, 1]$ and that the object travels 5 m in the direction of the vector $[2, -1, 0]$. Then, **f** is 8 times a unit vector in the direction of $[1, -2, 1]$ and **d** is 5 times a unit vector in the direction of $[2, -1, 0]$. Therefore, the total work performed is

$$\mathbf{f} \cdot \mathbf{d} = 8\left(\frac{[1, -2, 1]}{\|[1, -2, 1]\|}\right) \cdot 5\left(\frac{[2, -1, 0]}{\|[2, -1, 0]\|}\right) = \frac{40\,(2 + 2 + 0)}{\sqrt{6}\sqrt{5}} \approx 29.2 \text{ joules.}$$

## New Vocabulary

angle between two vectors
Cauchy-Schwarz Inequality
commutative law for dot product
distributive laws for dot product
dot (inner) product of vectors
lemma
mutually orthogonal vectors
orthogonal (perpendicular) vectors
projection of one vector onto another
Triangle Inequality
work (accomplished by a vector force)

## Highlights

- If $\mathbf{x} = [x_1, x_2, \ldots, x_n]$ and $\mathbf{y} = [y_1, y_2, \ldots, y_n]$ are two vectors in $\mathbb{R}^n$, the dot product of $\mathbf{x}$ and $\mathbf{y}$ is $\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$. The dot product of vectors is always a *scalar*.
- For any vector $\mathbf{x}$, we have $\mathbf{x} \cdot \mathbf{x} = \|\mathbf{x}\|^2 \geq 0$.
- The commutative and distributive laws hold for the dot product of vectors in $\mathbb{R}^n$.
- For any vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$, we have $|\mathbf{x} \cdot \mathbf{y}| \leq (\|\mathbf{x}\|)(\|\mathbf{y}\|)$ and $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.
- If $\theta$ is the angle between nonzero vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$, then $\cos\theta = \frac{\mathbf{x}\cdot\mathbf{y}}{(\|\mathbf{x}\|)(\|\mathbf{y}\|)}$.
- Two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ are orthogonal if and only if $\mathbf{x} \cdot \mathbf{y} = 0$.
- Two nonzero vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ are parallel if and only if $\mathbf{x} \cdot \mathbf{y} = \pm(\|\mathbf{x}\|)(\|\mathbf{y}\|)$.
- If $\mathbf{a}$ and $\mathbf{b}$ are vectors in $\mathbb{R}^n$, with $\mathbf{a} \neq \mathbf{0}$, then the projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\mathbf{proj_a b} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2}\right)\mathbf{a}$.
- If $\mathbf{a}$ and $\mathbf{b}$ are vectors in $\mathbb{R}^n$, with $\mathbf{a} \neq \mathbf{0}$, then $\mathbf{b}$ can be decomposed as the sum of two component vectors, $\mathbf{proj_a b}$ and $\mathbf{b} - \mathbf{proj_a b}$, where the first (if nonzero) is parallel to $\mathbf{a}$ and the second is orthogonal to $\mathbf{a}$.
- The work accomplished by a force $\mathbf{f}$ is equal to $\mathbf{f} \cdot \mathbf{d}$, the dot product of the force and the displacement $\mathbf{d}$.

## Exercises for Section 1.2

**Note**: Some exercises ask for proofs. If you have difficulty with these, try them again after working through Section 1.3, in which proof techniques are discussed.

1. Use the inverse cosine function on a calculator to find the angle $\theta$ (to the nearest degree) between the following given vectors $\mathbf{x}$ and $\mathbf{y}$:
   ★ (a) $\mathbf{x} = [-4, 3]$, $\mathbf{y} = [6, -1]$
   (b) $\mathbf{x} = [5, -8, 2]$, $\mathbf{y} = [3, -7, 0]$
   ★ (c) $\mathbf{x} = [7, -4, 2]$, $\mathbf{y} = [-6, -10, 1]$
   (d) $\mathbf{x} = [-5, 15, -35, 20]$, $\mathbf{y} = [2, -6, 14, -8]$
2. Show that points $A_1(8, -16, 7)$, $A_2(3, -7, 5)$, and $A_3(11, -1, 12)$ are the vertices of a right triangle. (Hint: Construct vectors between the points and check for an orthogonal pair.)
3. (a) Show that $[a, b]$ and $[-b, a]$ are orthogonal. Show that $[a, -b]$ and $[b, a]$ are orthogonal.
   (b) Show that the lines given by the equations $ax + by + c = 0$ and $bx - ay + d = 0$ (where $a, b, c, d \in \mathbb{R}$) are perpendicular by finding a vector in the direction of each line and showing that these vectors are orthogonal. (Hint: Watch out for the cases in which $a$ or $b$ equals zero.)
4. (a) Calculate (in joules) the total work performed by a force $\mathbf{f} = 2\mathbf{i} + 3\mathbf{j} - 7\mathbf{k}$ (nt) on an object which causes a displacement $\mathbf{d} = 8\mathbf{i} - 4\mathbf{j} - 9\mathbf{k}$ (m).
   ★ (b) Calculate (in joules) the total work performed by a force of 26 nt acting in the direction of the vector $-2\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ on an object displaced a total of 10 m in the direction of the vector $-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$.
   (c) Calculate (in joules) the total work performed by a force of 11 nt acting in the direction of the vector $-3\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}$ on an object displaced a total of 12 m in the direction of the vector $11\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$.
5. Why isn't it true that if $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$, then $\mathbf{x} \cdot (\mathbf{y} \cdot \mathbf{z}) = (\mathbf{x} \cdot \mathbf{y}) \cdot \mathbf{z}$?
6. ▶ This exercise asks for proofs of various parts of Theorem 1.5.
   (a) Prove part (1) of Theorem 1.5.
   (b) Prove part (2) of Theorem 1.5.
   (c) Prove the following half of part (3) of Theorem 1.5: For any vector $\mathbf{x}$ in $\mathbb{R}^n$: if $\mathbf{x} \cdot \mathbf{x} = 0$ then $\mathbf{x} = \mathbf{0}$.
   (d) Prove part (4) of Theorem 1.5.
   (e) Prove part (6) of Theorem 1.5.
7. ★ Does the Cancellation Law of algebra hold for the dot product; that is, assuming that $\mathbf{z} \neq \mathbf{0}$, does $\mathbf{x} \cdot \mathbf{z} = \mathbf{y} \cdot \mathbf{z}$ always imply that $\mathbf{x} = \mathbf{y}$?
8. ▶ Finish the proof of Lemma 1.6 by showing that for unit vectors $\mathbf{a}$ and $\mathbf{b}$, $(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) \geq 0$ implies $\mathbf{a} \cdot \mathbf{b} \leq 1$.

**9.** Prove that if $(\mathbf{x}+\mathbf{y})\cdot(\mathbf{x}-\mathbf{y})=0$, then $\|\mathbf{x}\|=\|\mathbf{y}\|$. (Hence, if the diagonals of a parallelogram are perpendicular, then the parallelogram is a rhombus.)

**10.** Prove that $\frac{1}{2}\left(\|\mathbf{x}+\mathbf{y}\|^2+\|\mathbf{x}-\mathbf{y}\|^2\right)=\|\mathbf{x}\|^2+\|\mathbf{y}\|^2$ for any vectors $\mathbf{x}$, $\mathbf{y}$ in $\mathbb{R}^n$. (This equation is known as the **Parallelogram Identity** because it asserts that the sum of the squares of the lengths of all four sides of a parallelogram equals the sum of the squares of the diagonals.)

**11.** Let $\mathbf{x}$ and $\mathbf{y}$ be vectors in $\mathbb{R}^n$.
- **(a)** Prove that if $\|\mathbf{x}+\mathbf{y}\|^2=\|\mathbf{x}\|^2+\|\mathbf{y}\|^2$ then $\mathbf{x}\cdot\mathbf{y}=0$.
- **(b)** Prove that if $\mathbf{x}\cdot\mathbf{y}=0$ then $\|\mathbf{x}+\mathbf{y}\|^2=\|\mathbf{x}\|^2+\|\mathbf{y}\|^2$.

**12.** Prove that if $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are mutually orthogonal vectors in $\mathbb{R}^n$, then $\|\mathbf{x}+\mathbf{y}+\mathbf{z}\|^2=\|\mathbf{x}\|^2+\|\mathbf{y}\|^2+\|\mathbf{z}\|^2$.

**13.** Prove that $\mathbf{x}\cdot\mathbf{y}=\frac{1}{4}\left(\|\mathbf{x}+\mathbf{y}\|^2-\|\mathbf{x}-\mathbf{y}\|^2\right)$, if $\mathbf{x}$ and $\mathbf{y}$ are vectors in $\mathbb{R}^n$. (This result, a form of the **Polarization Identity**, gives a way of defining the dot product using the norms of vectors.)

**14.** Given $\mathbf{x}, \mathbf{y}, \mathbf{z}$ in $\mathbb{R}^n$, with $\mathbf{x}$ orthogonal to both $\mathbf{y}$ and $\mathbf{z}$, prove that $\mathbf{x}$ is orthogonal to $c_1\mathbf{y}+c_2\mathbf{z}$, where $c_1, c_2 \in \mathbb{R}$.

**15.** Prove that if $\mathbf{x}$ and $\mathbf{y}$ are nonzero parallel vectors (that is, $\mathbf{y}=c\mathbf{x}$ for some $c\neq 0$), then $\mathbf{x}\cdot\mathbf{y}=\pm\|\mathbf{x}\|\,\|\mathbf{y}\|$ (that is, $\cos\theta=\pm 1$, where $\theta$ is the angle between $\mathbf{x}$ and $\mathbf{y}$). (Together with Theorem 1.10, this establishes that the algebraic and geometric conditions for parallel vectors are equivalent.)

★ **16.** This exercise determines some useful information about a cube.
- **(a)** If the side of a cube has length $s$, use vectors to calculate the length of the cube's diagonal.
- **(b)** Using vectors, find the angle that the diagonal makes with one of the sides of a cube.

**17.** Calculate $\mathbf{proj_a b}$ in each case, and verify $\mathbf{b}-\mathbf{proj_a b}$ is orthogonal to $\mathbf{a}$.
- ★ **(a)** $\mathbf{a}=[2,1,5]$, $\mathbf{b}=[1,4,-3]$
- **(b)** $\mathbf{a}=[4,-7,5]$, $\mathbf{b}=[6,8,10]$
- ★ **(c)** $\mathbf{a}=[1,0,-1,2]$, $\mathbf{b}=[3,-1,0,-1]$
- **(d)** $\mathbf{a}=[5,-7,3,1]$, $\mathbf{b}=[-1,1,4,-12]$

**18.** Let $\mathbf{a}$ and $\mathbf{b}$ be nonzero vectors in $\mathbb{R}^n$.
- **(a)** Suppose $\mathbf{a}$ and $\mathbf{b}$ are orthogonal vectors. What is $\mathbf{proj_a b}$? Why? Give a geometric interpretation in $\mathbb{R}^2$ and $\mathbb{R}^3$.
- **(b)** Suppose $\mathbf{a}$ and $\mathbf{b}$ are parallel vectors. What is $\mathbf{proj_a b}$? Why? Give a geometric interpretation in $\mathbb{R}^2$ and $\mathbb{R}^3$.

★ **19.** What are the projections of the general vector $[a,b,c]$ onto each of the vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ in turn?

**20.** Let $\mathbf{x}=[-6,2,7]$ (with units in newtons) represent the force on an object in a three-dimensional coordinate system. Decompose $\mathbf{x}$ into two component forces in directions parallel and orthogonal to each vector given.
- ★ **(a)** $[2,-3,4]$
- **(b)** $[5,11,-4]$
- ★ **(c)** $[3,-2,6]$

**21.** Show that if $\ell$ is any line through the origin in $\mathbb{R}^3$ and $\mathbf{x}$ is any vector with its initial point at the origin, then the **reflection** of $\mathbf{x}$ through the line $\ell$ (acting as a mirror) is equal to $2(\mathbf{proj_r x})-\mathbf{x}$, where $\mathbf{r}$ is any nonzero vector parallel to the line $\ell$ (see Fig. 1.21).

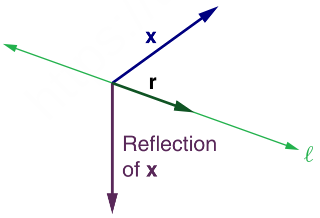


**FIGURE 1.21** Reflection of $\mathbf{x}$ through the line $\ell$

**22.** Prove the **Reverse Triangle Inequality**; that is, for any vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$, $\big|\|\mathbf{x}\|-\|\mathbf{y}\|\big|\leq\|\mathbf{x}+\mathbf{y}\|$. (Hint: Consider the cases $\|\mathbf{x}\|\leq\|\mathbf{y}\|$ and $\|\mathbf{x}\|\geq\|\mathbf{y}\|$ separately.)

**23.** This exercise generalizes the type of vector decomposition given in Theorem 1.11.
- ★ **(a)** Consider $\mathbf{x}=[4,-3,5]$ and $\mathbf{y}=[3,-6,-2]$. Prove that $\mathbf{y}=c\mathbf{x}+\mathbf{w}$ for some scalar $c$ and some vector $\mathbf{w}$ such that $\mathbf{w}$ is orthogonal to $\mathbf{x}$. (Hint: $\mathbf{y}=\mathbf{proj_x y}+(\mathbf{y}-\mathbf{proj_x y})$.)
- **(b)** Let $\mathbf{x}$ and $\mathbf{y}$ be nonzero vectors in $\mathbb{R}^n$. Prove that $\mathbf{y}=c\mathbf{x}+\mathbf{w}$ for some scalar $c$ and some vector $\mathbf{w}$ such that $\mathbf{w}$ is orthogonal to $\mathbf{x}$.
- **(c)** Show that the vector $\mathbf{w}$ and the scalar $c$ in part (b) are unique; that is, show that if $\mathbf{y}=c\mathbf{x}+\mathbf{w}$ and $\mathbf{y}=d\mathbf{x}+\mathbf{v}$, where $\mathbf{w}$ and $\mathbf{v}$ are both orthogonal to $\mathbf{x}$, then $c=d$ and $\mathbf{w}=\mathbf{v}$. (Hint: Compute $\mathbf{x}\cdot\mathbf{y}$.)

**24.** If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ such that $\mathbf{x}\cdot\mathbf{y}\neq 0$, prove that the angle between $\mathbf{x}$ and $\mathbf{y}$ equals the angle between $\mathbf{proj_x y}$ and $\mathbf{proj_y x}$.

**25.** In each part, you are given the length of one side of a parallelogram and the lengths of the two diagonals. Use the formulas in Exercises 10 and 13, as well as the formula for the measure of an angle between two vectors, to compute the measures of the four angles in the parallelogram. (Opposing angles are congruent.)

★ **(a)** One side has length 9; the lengths of the diagonals are 8 and 14.
★ **(b)** One side has length 6; the lengths of the diagonals are 11 and 17.
**(c)** One side has length 9; the lengths of the diagonals are 10 and 20.
**(d)** One side has length 15; the lengths of the diagonals are 17 and 19.

★ **26.** True or False:

**(a)** For any vectors $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^n$, and any scalar $d$, $\mathbf{x} \cdot (d\mathbf{y}) = (d\mathbf{x}) \cdot \mathbf{y}$.
**(b)** For all $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^n$ with $\mathbf{x} \neq \mathbf{0}$, $(\mathbf{x} \cdot \mathbf{y})/\|\mathbf{x}\| \leq \|\mathbf{y}\|$.
**(c)** For all $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^n$, $\|\mathbf{x} - \mathbf{y}\| \leq \|\mathbf{x}\| - \|\mathbf{y}\|$.
**(d)** If $\theta$ is the angle between vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$, and $\theta > \frac{\pi}{2}$, then $\mathbf{x} \cdot \mathbf{y} > 0$.
**(e)** The standard unit vectors in $\mathbb{R}^n$ are mutually orthogonal.
**(f)** If $\mathbf{proj_a b} = \mathbf{b}$, then $\mathbf{a}$ is perpendicular to $\mathbf{b}$.

## 1.3 An Introduction to Proof Techniques

In reading this book, you will spend much time studying the proofs of theorems, and for the exercises, you will often write proofs. Hence, in this section we discuss several methods of proving theorems in order to sharpen your skills in reading and writing proofs.

The "results" (not all new) proved in this section are intended only to illustrate various proof techniques. Therefore, they are not labeled as "theorems."

### Proof Technique: Direct Proof

The most straightforward proof method is **direct proof**, a logical step-by-step argument concluding with the statement to be proved. The following is a direct proof for a familiar result from Theorem 1.5:

**Result 1** *Let* $\mathbf{x}$ *be a vector in* $\mathbb{R}^n$. *Then* $\mathbf{x} \cdot \mathbf{x} = \|\mathbf{x}\|^2$.

*Proof.*

| | |
|---|---|
| Step 1: Let $\mathbf{x} = [x_1, \ldots, x_n]$ | because $\mathbf{x} \in \mathbb{R}^n$ |
| Step 2: $\mathbf{x} \cdot \mathbf{x} = x_1^2 + \cdots + x_n^2$ | definition of dot product |
| Step 3: $\|\mathbf{x}\| = \sqrt{x_1^2 + \cdots + x_n^2}$ | definition of $\|\mathbf{x}\|$ |
| Step 4: $\|\mathbf{x}\|^2 = x_1^2 + \cdots + x_n^2$ | squaring both sides of Step 3 |
| Step 5: $\mathbf{x} \cdot \mathbf{x} = \|\mathbf{x}\|^2$ | from Steps 2 and 4. |

□

Each step in a direct proof should follow immediately from a definition, a previous step, or a known fact. The reasons for each step should be clearly stated when necessary for the intended reader. However, the preceding type of presentation is infrequently used. A more typical paragraph version of the same argument is:

*Proof.* If $\mathbf{x}$ is a vector in $\mathbb{R}^n$, then we can express $\mathbf{x}$ as $[x_1, x_2, \ldots, x_n]$ for some real numbers $x_1, \ldots, x_n$. Now, $\mathbf{x} \cdot \mathbf{x} = x_1^2 + \cdots + x_n^2$, by definition of the dot product. However, $\|\mathbf{x}\| = \sqrt{x_1^2 + \cdots + x_n^2}$, by definition of the length of a vector. Therefore, $\|\mathbf{x}\|^2 = \mathbf{x} \cdot \mathbf{x}$, because both sides are equal to $x_1^2 + \cdots + x_n^2$. □

The paragraph form should contain the same information as the step-by-step form and be presented in such a way that a corresponding step-by-step proof occurs naturally to the reader. We present most proofs in this book in paragraph style. But you may want to begin writing proofs in the step-by-step format and then change to paragraph style once you have more experience with proofs.

Stating the definitions of the relevant terms is usually a good beginning when tackling a proof because it helps to clarify what you must prove. For example, the first four of the five steps in the step-by-step proof of Result 1 merely involve writing what each side of $\mathbf{x} \cdot \mathbf{x} = \|\mathbf{x}\|^2$ means. The final result then follows naturally.

## "If $A$ Then $B$" Proofs

Frequently, a theorem is given in the form "If $A$ then $B$," where $A$ and $B$ represent statements. An example is "If $\|\mathbf{x}\| = 0$, then $\mathbf{x} = \mathbf{0}$" for vectors $\mathbf{x}$ in $\mathbb{R}^n$, where $A$ is "$\|\mathbf{x}\| = 0$" and $B$ is "$\mathbf{x} = \mathbf{0}$." The entire "If $A$ then $B$" statement is called an **implication**; $A$ alone is the **premise**, and $B$ is the **conclusion**. The meaning of "If $A$ then $B$" is that, whenever $A$ is true, $B$ is true as well. Thus, the implication "If $\|\mathbf{x}\| = 0$, then $\mathbf{x} = \mathbf{0}$" means that, if we know $\|\mathbf{x}\| = 0$ for some particular vector $\mathbf{x}$ in $\mathbb{R}^n$, then we can conclude that $\mathbf{x}$ is the zero vector.

Note that the implication "If $A$ then $B$" asserts nothing about the truth or falsity of $B$ unless $A$ is true.[6] Therefore, to prove "If $A$ then $B$," we assume $A$ is true and try to prove $B$ is also true. This is illustrated in the proof of the next result, a part of Theorem 1.9.

**Result 2** *If $\mathbf{x}$ and $\mathbf{y}$ are nonzero vectors in $\mathbb{R}^n$ such that $\mathbf{x} \cdot \mathbf{y} > 0$, then the angle between $\mathbf{x}$ and $\mathbf{y}$ is $0°$ or acute.*

*Proof.* The premise in this result is "$\mathbf{x}$ and $\mathbf{y}$ are nonzero vectors and $\mathbf{x} \cdot \mathbf{y} > 0$." The conclusion is "the angle between $\mathbf{x}$ and $\mathbf{y}$ is $0°$ or acute." We begin by assuming that both parts of the premise are true.

| | |
|---|---|
| Step 1: $\mathbf{x}$ and $\mathbf{y}$ are nonzero | first part of premise |
| Step 2: $\|\mathbf{x}\| > 0$ and $\|\mathbf{y}\| > 0$ | Theorem 1.5, parts (2) and (3) |
| Step 3: $\mathbf{x} \cdot \mathbf{y} > 0$ | second part of premise |
| Step 4: $\cos\theta = \frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|}$, where $\theta$ is the angle between $\mathbf{x}$ and $\mathbf{y}$, and $0 \le \theta \le \pi$ | definition of the angle between two vectors |
| Step 5: $\cos\theta > 0$ | quotient of positive reals is positive |
| Step 6: $\theta$ is $0°$ or acute | since $\cos\theta \le 0$ when $\frac{\pi}{2} \le \theta \le \pi$ |

□

Beware! An implication is not always written in the form "If $A$ then $B$."

**Some Equivalent Forms for "If $A$ Then $B$"**

| | |
|---|---|
| $A$ implies $B$ | $B$, if $A$ |
| $A \Longrightarrow B$ | $A$ is a sufficient condition for $B$ |
| $A$ only if $B$ | $B$ is a necessary condition for $A$ |

Another common practice is to place some of the conditions of the premise before the "If … then." For example, Result 2 might be rewritten as

*Let $\mathbf{x}$ and $\mathbf{y}$ be nonzero vectors in $\mathbb{R}^n$. If $\mathbf{x} \cdot \mathbf{y} > 0$, then the angle between $\mathbf{x}$ and $\mathbf{y}$ is $0°$ or acute.*

The condition "$\mathbf{x}$ and $\mathbf{y}$ are nonzero vectors in $\mathbb{R}^n$" sets the stage for the implication to come. Such conditions are treated as given information along with the premise in the actual proof.

## Working "Backward" to Discover a Proof

A method often used when there is no obvious direct proof is to work "backward"—that is, to start with the desired conclusion and work in reverse toward the given facts. Although these "reversed" steps do not constitute a proof, they may provide sufficient insight to make construction of a "forward" proof easier, as we now illustrate.

[6] In formal logic, when $A$ is false, the implication "If $A$ then $B$" is considered true but worthless because it tells us absolutely nothing about $B$. For example, the implication "If every vector in $\mathbb{R}^3$ is a unit vector, then the inflation rate will be 8% next year" is considered true because the premise "every vector in $\mathbb{R}^3$ is a unit vector" is clearly false. However, the implication is useless. It tells us nothing about next year's inflation rate, which is free to take any value, such as 6%.

**Result 3** *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be nonzero vectors in* $\mathbb{R}^n$. *If* $\mathbf{x}\cdot\mathbf{y} \geq 0$, *then* $\|\mathbf{x}+\mathbf{y}\| > \|\mathbf{y}\|$.

We begin with the desired conclusion $\|\mathbf{x}+\mathbf{y}\| > \|\mathbf{y}\|$ and try to work "backward" toward the given fact $\mathbf{x}\cdot\mathbf{y} \geq 0$, as follows:

$$\begin{aligned}
\|\mathbf{x}+\mathbf{y}\| &> \|\mathbf{y}\| \\
\|\mathbf{x}+\mathbf{y}\|^2 &> \|\mathbf{y}\|^2 \\
(\mathbf{x}+\mathbf{y})\cdot(\mathbf{x}+\mathbf{y}) &> \|\mathbf{y}\|^2 \\
\mathbf{x}\cdot\mathbf{x}+2\mathbf{x}\cdot\mathbf{y}+\mathbf{y}\cdot\mathbf{y} &> \|\mathbf{y}\|^2 \\
\|\mathbf{x}\|^2+2\mathbf{x}\cdot\mathbf{y}+\|\mathbf{y}\|^2 &> \|\mathbf{y}\|^2 \\
\|\mathbf{x}\|^2+2\mathbf{x}\cdot\mathbf{y} &> 0.
\end{aligned}$$

At this point, we cannot easily continue going "backward." However, the last inequality is true if $\mathbf{x}\cdot\mathbf{y} \geq 0$. Therefore, we *reverse* the above steps to create the following "forward" proof of Result 3:

*Proof.*

| | |
|---|---|
| Step 1: $\|\mathbf{x}\|^2 > 0$ | $\mathbf{x}$ is nonzero |
| Step 2: $2\,(\mathbf{x}\cdot\mathbf{y}) \geq 0$ | because $\mathbf{x}\cdot\mathbf{y} \geq 0$ |
| Step 3: $\|\mathbf{x}\|^2 + 2(\mathbf{x}\cdot\mathbf{y}) > 0$ | from Steps 1 and 2 |
| Step 4: $\|\mathbf{x}\|^2 + 2(\mathbf{x}\cdot\mathbf{y}) + \|\mathbf{y}\|^2 > \|\mathbf{y}\|^2$ | |
| Step 5: $(\mathbf{x}\cdot\mathbf{x}) + 2(\mathbf{x}\cdot\mathbf{y}) + (\mathbf{y}\cdot\mathbf{y}) > \|\mathbf{y}\|^2$ | Theorem 1.5, part (2) |
| Step 6: $(\mathbf{x}+\mathbf{y})\cdot(\mathbf{x}+\mathbf{y}) > \|\mathbf{y}\|^2$ | Theorem 1.5, parts (5) and (6) |
| Step 7: $\|\mathbf{x}+\mathbf{y}\|^2 > \|\mathbf{y}\|^2$ | Theorem 1.5, part (2) |
| Step 8: $\|\mathbf{x}+\mathbf{y}\| > \|\mathbf{y}\|$ | take square root of both sides; length is always nonnegative |

□

When "working backward," your steps must be reversed for the final proof. Therefore, each step must be carefully examined to determine if it is "reversible." For example, if $t$ is a real number, then $t > 5 \Longrightarrow t^2 > 25$ is a valid step, but reversing this yields $t^2 > 25 \Longrightarrow t > 5$, which is certainly an invalid step if $t < -5$. Notice that we were very careful in Step 8 of the proof when we took the square root of both sides to ensure the step was indeed valid.

## "*A* If and Only If *B*" Proofs

Some theorems have the form "$A$ if and only if $B$." This is really a combination of two statements: "If $A$ then $B$" and "If $B$ then $A$." Both of these statements must be shown true to fully complete the proof of the original statement. In essence, we must show $A$ and $B$ are logically equivalent: the "If $A$ then $B$" half means that whenever $A$ is true, $B$ must follow; the "If $B$ then $A$" half means that whenever $B$ is true, $A$ must follow. Therefore, $A$ is true exactly when $B$ is true.

For instance, the statement in Result 2 above is actually still valid if the "if...then" is replaced by an "if and only if," as follows:

Let $\mathbf{x}$ and $\mathbf{y}$ be nonzero vectors in $\mathbb{R}^n$. Then,
$\mathbf{x}\cdot\mathbf{y} > 0$ *if and only if* the angle between $\mathbf{x}$ and $\mathbf{y}$ is $0°$ or acute.

One half of this proof has already been given (as the proof of Result 2), and the other half can be done by essentially reversing the steps in that proof, as follows: $\theta$ is $0°$ or acute $\Longrightarrow \cos\theta > 0 \Longrightarrow \mathbf{x}\cdot\mathbf{y} > 0$ (since both $||\mathbf{x}||, \|\mathbf{y}\| > 0$).

Note that two proofs are required to prove an "if and only if" type of statement, one for each of the implications involved. However, in many cases it is not possible simply to reverse the steps of one half in order to prove the other half. Sometimes the two halves must be proved very differently, as we see in the following proof of Result 4 (which is related to both Theorem 1.10 and Exercise 15 in Section 1.2).

**Result 4** *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be nonzero vectors in* $\mathbb{R}^n$. *Then* $\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \, \|\mathbf{y}\|$ *if and only if* $\mathbf{y} = c\mathbf{x}$, *for some* $c > 0$.

In an "if and only if" proof, it is usually good to begin by stating the two halves of the "if and only if" statement. This gives a clearer picture of what is given and what must be proved in each half. In Result 4, the two halves are

1. Suppose that $\mathbf{y} = c\mathbf{x}$ for some $c > 0$. Prove that $\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \, \|\mathbf{y}\|$. (This is a special case of Exercise 15 in Section 1.2.)
2. Suppose that $\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \, \|\mathbf{y}\|$. Prove that there is some $c > 0$ such that $\mathbf{y} = c\mathbf{x}$. (This is a special case of Theorem 1.10 in Section 1.2.)

The assumption "Let $\mathbf{x}$ and $\mathbf{y}$ be nonzero vectors in $\mathbb{R}^n$" is considered given information for both halves.

*Proof.* **Part 1:** We suppose that $\mathbf{y} = c\mathbf{x}$ for some $c > 0$. Then,

$$\begin{aligned}
\mathbf{x} \cdot \mathbf{y} &= \mathbf{x} \cdot (c\mathbf{x}) && \text{because } \mathbf{y} = c\mathbf{x} \\
&= c\,(\mathbf{x} \cdot \mathbf{x}) && \text{Theorem 1.5, part (4)} \\
&= c\,\|\mathbf{x}\|^2 && \text{Theorem 1.5, part (2)} \\
&= \|\mathbf{x}\|\,(c\,\|\mathbf{x}\|) && \text{associative law of multiplication for real numbers} \\
&= \|\mathbf{x}\|\,(|c|\,\|\mathbf{x}\|) && \text{because } c > 0 \\
&= \|\mathbf{x}\|\,\|c\mathbf{x}\| && \text{Theorem 1.1} \\
&= \|\mathbf{x}\|\,\|\mathbf{y}\| && \text{because } \mathbf{y} = c\mathbf{x}.
\end{aligned}$$

**Part 2:** Suppose that $\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \, \|\mathbf{y}\|$. (Notice that $\mathbf{x} \cdot \mathbf{y} > 0$ since both $||\mathbf{x}||$ and $||\mathbf{y}||$ are nonzero.) We want to prove that $\mathbf{y} = c\mathbf{x}$ for some $c > 0$. An argument virtually identical to the proof of Theorem 1.10 then finishes the proof by demonstrating that $\mathbf{y} = c\mathbf{x}$, where $c = \frac{\mathbf{x} \cdot \mathbf{y}}{||\mathbf{x}||^2} > 0$. □

Note that two proofs are required to prove an "if and only if" type of statement, one for each of the implications involved. Also, each half is not necessarily just a reversal of the steps in the other half. Sometimes the two halves must be proved very differently, as we did for Result 4.

Other common alternate forms for "if and only if" are:

**Some Equivalent Forms for "$A$ If and Only If $B$"**

$A$ iff $B$

$A \Longleftrightarrow B$

$A$ is a necessary and sufficient condition for $B$

## "If $A$ Then ($B$ or $C$)" Proofs

Sometimes we must prove a statement of the form "If $A$ then ($B$ or $C$)."[7] This is an implication whose conclusion has two parts. Note that $B$ is either true or false. Now, if $B$ is true, there is no need for a proof, because we only need to establish that *at least one of* $B$ or $C$ holds. For this reason, "If $A$ then ($B$ or $C$)" is equivalent to "If $A$ is true and $B$ is *false*, then $C$ is true." That is, we are allowed to assume that $B$ is false, and then use this extra information to prove $C$ is true. This strategy often makes the proof easier. As an example, consider the following result:

**Result 5** *If* $\mathbf{x}$ *is a nonzero vector in* $\mathbb{R}^2$, *then* $\mathbf{x} \cdot [1, 0] \neq 0$ *or* $\mathbf{x} \cdot [0, 1] \neq 0$.

[7] In this text, *or* is used in the **inclusive** sense. That is, "A or B" always means "A or B or both." For example, "$n$ is even or prime" means that $n$ could be even or $n$ could be prime or $n$ could be both. Therefore, "$n$ is even or prime" is true for $n = 2$, which is both even and prime, as well as for $n = 6$ (even but not prime) and $n = 7$ (prime but not even). However, in English, the word *or* is frequently used in the **exclusive** sense, as in "You may have the prize behind the curtain *or* the cash in my hand," where you are not meant to have both prizes. The "exclusive or" is rarely used in mathematics.

In this case, $A =$ "$\mathbf{x}$ is a nonzero vector in $\mathbb{R}^2$," $B =$ "$\mathbf{x}\cdot[1,0]\neq 0$," and $C =$ "$\mathbf{x}\cdot[0,1]\neq 0$." Assuming $B$ is false, we obtain the following statement equivalent to Result 5:

$$\text{If } \mathbf{x} \text{ is a nonzero vector in } \mathbb{R}^n \text{ and } \mathbf{x}\cdot[1,0]=0, \text{ then } \mathbf{x}\cdot[0,1]\neq 0.$$

Proving this (which can be done with a direct proof—try it!) has the effect of proving the original statement in Result 5.

Of course, an alternate way of proving "If $A$ then ($B$ or $C$)" is to assume instead that $C$ is false and use this assumption to prove $B$ is true.

## Proof Technique: Proof by Contrapositive

Related to the implication "If $A$ then $B$" is its **contrapositive**: "If not $B$, then not $A$." For example, for an integer $n$, the statement "If $n^2$ is even, then $n$ is even" has the contrapositive "If $n$ is odd (that is, not even), then $n^2$ is odd." A statement and its contrapositive are always logically equivalent; that is, they are either both true or both false together. Therefore, proving the contrapositive of any statement (known as **proof by contrapositive**) has the effect of proving the original statement as well. In many cases, the contrapositive is easier to prove. The following result illustrates this method:

**Result 6** *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be vectors in* $\mathbb{R}^n$. *If* $\mathbf{x}\cdot\mathbf{y}\neq ||\mathbf{x}||^2$, *then* $\mathbf{x}\neq\mathbf{y}$.

*Proof.* To prove this result, it is enough to give a direct proof of its contrapositive: if $\mathbf{x}=\mathbf{y}$, then $\mathbf{x}\cdot\mathbf{y}=||\mathbf{x}||^2$.

| | |
|---|---|
| Step 1: Suppose $\mathbf{x}=\mathbf{y}$. | premise of contrapositive |
| Step 2: Then $\mathbf{x}\cdot\mathbf{y}=\mathbf{x}\cdot\mathbf{x}$. | substitution |
| Step 3: But $\mathbf{x}\cdot\mathbf{x}=\|\mathbf{x}\|^2$. | Theorem 1.5, part (2) |
| Step 4: Therefore, $\mathbf{x}\cdot\mathbf{y}=\|\mathbf{x}\|^2$. | Steps 3 and 4 |

Since we have proven the contrapositive, the original statement ("If $\mathbf{x}\cdot\mathbf{y}\neq||\mathbf{x}||^2$, then $\mathbf{x}\neq\mathbf{y}$") is proven as well. □

Because many theorems are implications, and because the contrapositive of an implication is logically equivalent to the original statement, many theorems can be restated in an alternate, valid manner by using the contrapositive. That is, from such a theorem we automatically get a second theorem, its contrapositive, for free! Frequently, the contrapositive has a form that is equally useful in applications. (Of course, theorems such as Theorem 1.3 and Theorem 1.5 which list properties that hold true in all cases do not have useful contrapositives.) Exercise 26 asks you to construct the contrapositives of several Theorems and Results, thereby providing additional insight into the meaning of these statements.

## Converse and Inverse

Along with the contrapositive, there are two other related statements of interest—the **converse** and **inverse**:

| | |
|---|---|
| **Original Statement** | If $A$ then $B$ |
| **Contrapositive** | If not $B$ then not $A$ |
| **Converse** | If $B$ then $A$ |
| **Inverse** | If not $A$ then not $B$ |

Notice that, when "If $A$ then $B$" and its converse "If $B$ then $A$" are combined together, they form the familiar "$A$ if and only if $B$" statement.

Although the converse and inverse may resemble the contrapositive, take care: neither the converse nor the inverse is logically equivalent to the original statement. However, the converse and inverse of a statement are equivalent to each other, and are both true or both false together. For example, for a vector $\mathbf{x}$ in $\mathbb{R}^n$, consider the statement "If $\mathbf{x}=\mathbf{0}$, then $||\mathbf{x}||\neq 1$."

| | | |
|---|---|---|
| Original Statement | If $\mathbf{x}=\mathbf{0}$, then $\|\|\mathbf{x}\|\|\neq 1$. | equivalent to each other |
| Contrapositive | If $\|\|\mathbf{x}\|\|=1$, then $\mathbf{x}\neq\mathbf{0}$. | |
| Converse | If $\|\|\mathbf{x}\|\|\neq 1$, then $\mathbf{x}=\mathbf{0}$. | equivalent to each other |
| Inverse | If $\mathbf{x}\neq\mathbf{0}$, then $\|\|\mathbf{x}\|\|=1$. | |

Notice that in this case the original statement and its contrapositive are both true, while the converse and the inverse are both false. (Why?)

Beware! It is possible in some cases for a statement and its converse to have the same truth value. For example, the statement "If $\mathbf{x} = \mathbf{0}$, then $\|\mathbf{x}\| = 0$" is a true statement, and its converse "If $\|\mathbf{x}\| = 0$, then $\mathbf{x} = \mathbf{0}$" is also a true statement. The moral here is that a statement and its converse are logically independent, and thus, proving the converse (or inverse) is never acceptable as a valid proof of the original statement.

Finally, when constructing the contrapositive, converse, or inverse of an "If $A$ then $B$" statement, you should not change the accompanying conditions. For instance, consider the condition "Let $\mathbf{x}$ and $\mathbf{y}$ be nonzero vectors in $\mathbb{R}^n$" of Result 3. The contrapositive, converse, and inverse should all begin with this condition. For example, the contrapositive of Result 3 is "Let $\mathbf{x}$ and $\mathbf{y}$ be nonzero vectors in $\mathbb{R}^n$. If $\|\mathbf{x}+\mathbf{y}\| \le \|\mathbf{y}\|$, then $\mathbf{x}\cdot\mathbf{y} < 0$."

## Proof Technique: Proof by Contradiction

Another common proof method is **proof by contradiction**, in which we assume the statement to be proved is false and use this assumption to contradict a known fact. In effect, we prove a result by showing that if it were false it would be inconsistent with some other true statement, as in the proof of the following:

**Result 7** *Let $S = \{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ be a set of mutually orthogonal nonzero vectors in $\mathbb{R}^n$. Then no vector in $S$ can be expressed as a linear combination of the other vectors in $S$.*

Recall that a set[8] $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ of nonzero vectors is mutually orthogonal if and only if $\mathbf{x}_i \cdot \mathbf{x}_j = 0$ whenever $i \neq j$.

*Proof.* To prove this by contradiction, we assume it is false; that is, *some* vector in $S$ *can* be expressed as a linear combination of the other vectors in $S$. That is, some $\mathbf{x}_i = a_1\mathbf{x}_1 + \cdots + a_{i-1}\mathbf{x}_{i-1} + a_{i+1}\mathbf{x}_{i+1} + \cdots + a_k\mathbf{x}_k$, for some $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_k \in \mathbb{R}$. We then show this assumption leads to a contradiction:

$$\begin{aligned}
\mathbf{x}_i \cdot \mathbf{x}_i &= \mathbf{x}_i \cdot (a_1\mathbf{x}_1 + \cdots + a_{i-1}\mathbf{x}_{i-1} + a_{i+1}\mathbf{x}_{i+1} + \cdots + a_k\mathbf{x}_k) \\
&= a_1(\mathbf{x}_i \cdot \mathbf{x}_1) + \cdots + a_{i-1}(\mathbf{x}_i \cdot \mathbf{x}_{i-1}) + a_{i+1}(\mathbf{x}_i \cdot \mathbf{x}_{i+1}) + \cdots + a_k(\mathbf{x}_i \cdot \mathbf{x}_k) \\
&= a_1(0) + \cdots + a_{i-1}(0) + a_{i+1}(0) + \cdots + a_k(0) = 0.
\end{aligned}$$

Hence, $\mathbf{x}_i = \mathbf{0}$, by part (3) of Theorem 1.5. This equation contradicts the given fact that $\mathbf{x}_1, \ldots, \mathbf{x}_k$ are all nonzero vectors. Thus, the original statement is not false, but true, which completes the proof. □

A mathematician generally constructs a proof by contradiction by assuming that the given statement is false and then investigates where this assumption leads until some absurdity appears. Of course, any "blind alleys" encountered in the investigation should not appear in the final proof.

In the preceding proof we assumed that some chosen vector $\mathbf{x}_i$ could be expressed as a linear combination of the other vectors. However, we could easily have renumbered the vectors so that $\mathbf{x}_i$ becomes $\mathbf{x}_1$, and the other vectors are $\mathbf{x}_2$ through $\mathbf{x}_k$. A mathematician would express this by writing: "We assume some vector in $S$ can be expressed as a linear combination of the others. Without loss of generality, choose $\mathbf{x}_1$ to be this vector." This phrase "**without loss of generality**" implies here that the vectors have been suitably rearranged if necessary, so that $\mathbf{x}_1$ now has the desired property. Then our assumption in the proof of Result 7 could be stated more simply as: $\mathbf{x}_1 = a_2\mathbf{x}_2 + \cdots + a_k\mathbf{x}_k$. The proof is now more straightforward to express, since we do not have to skip over subscript "$i$":

$$\begin{aligned}
\mathbf{x}_1 \cdot \mathbf{x}_1 &= \mathbf{x}_1 \cdot (a_2\mathbf{x}_2 + \cdots + a_k\mathbf{x}_k) \\
&= a_2(\mathbf{x}_1 \cdot \mathbf{x}_2) + \cdots + a_k(\mathbf{x}_1 \cdot \mathbf{x}_k) \\
&= a_2(0) + \cdots + a_k(0) = 0, \text{ etc.}
\end{aligned}$$

[8] Throughout this text, we use the standard convention that whenever braces are used to list the elements of a set, the elements of the set are assumed to be distinct.

## Proof Technique: Proof by Induction

The method of **proof by induction** is used to show that a statement is true for all values of an integer variable greater than or equal to some initial value $i$. For example, $A =$ "For every integer $n \geq 1$, $1^2 + 2^2 + \cdots + n^2 = n(n+1)(2n+1)/6$" can be proved by induction for all integers $n$ greater than or equal to the initial value $i = 1$. You may have seen such a proof in your calculus course.

There are two steps in any induction proof, the **Base Step** and the **Inductive Step**.

**(1)** **Base Step:** Prove that the desired statement is true for the initial value $i$ of the (integer) variable.
**(2)** **Inductive Step:** Prove that if the statement is true for an integer value $k$ of the variable (with $k \geq i$), then the statement is true for the next integer value $k + 1$ as well.

These two steps together show that the statement is true for every integer greater than or equal to the initial value $i$ because the Inductive Step sets up a "chain of implications," as in Fig. 1.22. First, the Base Step implies that the initial statement, $A_i$, is true. But $A_i$ is the premise for the first implication in the chain. Hence, the Inductive Step tells us that the conclusion of this implication, $A_{i+1}$, must also be true. However, $A_{i+1}$ is the premise of the second implication; hence, the Inductive Step tells us that the conclusion $A_{i+2}$ must be true. In this way, the statement is true for each integer value $\geq i$.

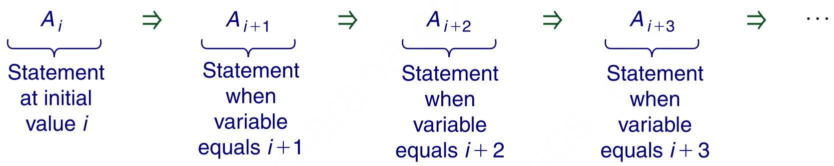


**FIGURE 1.22** Chain of implications set up by the Inductive Step

The process of induction can be likened to knocking down a line of dominoes—one domino for each integer greater than or equal to the initial value. Keep in mind that the Base Step is needed to knock over the first domino and thus start the entire process. Without the Base Step, we cannot be sure that the given statement is true for any integer value at all. The next proof illustrates the induction technique:

**Result 8** *Let* $\mathbf{z}$, *and* $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ *(for* $n \geq 1$*) be vectors in* $\mathbb{R}^m$, *and let* $c_1, c_2, \ldots, c_n \in \mathbb{R}$. *Then,*

$$(c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n) \cdot \mathbf{z} = c_1(\mathbf{x}_1 \cdot \mathbf{z}) + c_2(\mathbf{x}_2 \cdot \mathbf{z}) + \cdots + c_n(\mathbf{x}_n \cdot \mathbf{z}).$$

This is a generalization of part (6) of Theorem 1.5, where a linear combination replaces a single addition of vectors.

*Proof.* The integer induction variable is $n$, with initial value $i = 1$.

**Base Step:** The Base Step is typically proved by plugging in the initial value and verifying the result is true in that case. When $n = 1$, the left-hand side of the equation in Result 8 has only one term: $(c_1\mathbf{x}_1) \cdot \mathbf{z}$, while the right-hand side yields $c_1(\mathbf{x}_1 \cdot \mathbf{z})$. But $(c_1\mathbf{x}_1) \cdot \mathbf{z} = c_1(\mathbf{x}_1 \cdot \mathbf{z})$ by part (4) of Theorem 1.5, and so we have completed the Base Step.

**Inductive Step:** Assume in what follows that $c_1, c_2, \ldots, c_k, c_{k+1} \in \mathbb{R}$, $\mathbf{z}, \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k, \mathbf{x}_{k+1} \in \mathbb{R}^m$, and $k \geq 1$. The Inductive Step requires us to prove the following:

$$\text{If } (c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k) \cdot \mathbf{z} = c_1(\mathbf{x}_1 \cdot \mathbf{z}) + c_2(\mathbf{x}_2 \cdot \mathbf{z}) + \cdots + c_k(\mathbf{x}_k \cdot \mathbf{z}),$$
$$\text{then } (c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k + c_{k+1}\mathbf{x}_{k+1}) \cdot \mathbf{z} = c_1(\mathbf{x}_1 \cdot \mathbf{z}) + c_2(\mathbf{x}_2 \cdot \mathbf{z}) + \cdots + c_k(\mathbf{x}_k \cdot \mathbf{z}) + c_{k+1}(\mathbf{x}_{k+1} \cdot \mathbf{z}).$$

We assume that the premise is true, and use it to prove that one side of the desired conclusion is equal to the other side:

$$\begin{aligned}
&(c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k + c_{k+1}\mathbf{x}_{k+1}) \cdot \mathbf{z} \\
&= ((c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k) + (c_{k+1}\mathbf{x}_{k+1})) \cdot \mathbf{z} \\
&= (c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k) \cdot \mathbf{z} + (c_{k+1}\mathbf{x}_{k+1}) \cdot \mathbf{z} \\
&\qquad \text{by part (6) of Theorem 1.5, where } \mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k, \text{ and } \mathbf{y} = c_{k+1}\mathbf{x}_{k+1}
\end{aligned}$$

$$\begin{aligned} &= (c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k)\cdot\mathbf{z} + c_{k+1}(\mathbf{x}_{k+1}\cdot\mathbf{z}) && \text{by part (4) of Theorem 1.5} \\ &= c_1(\mathbf{x}_1\cdot\mathbf{z}) + c_2(\mathbf{x}_2\cdot\mathbf{z}) + \cdots + c_k(\mathbf{x}_k\cdot\mathbf{z}) + c_{k+1}(\mathbf{x}_{k+1}\cdot\mathbf{z}) && \text{by the induction premise.} \end{aligned}$$

Thus, we have proven the conclusion and completed the Inductive Step. Because we have completed both parts of the induction proof, the proof is finished. □

Note that in the Inductive Step we are proving an implication, and so we get the powerful advantage of assuming the premise of that implication. This premise is called the **inductive hypothesis**. In Result 8, the inductive hypothesis is

$$(c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k)\cdot\mathbf{z} = c_1(\mathbf{x}_1\cdot\mathbf{z}) + c_2(\mathbf{x}_2\cdot\mathbf{z}) + \cdots + c_k(\mathbf{x}_k\cdot\mathbf{z}).$$

It allows us to make the crucial substitution for $(c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k)\cdot\mathbf{z}$ in the very last line of the proof of the Inductive Step. A successful proof by induction ultimately depends on using the inductive hypothesis to reach the final conclusion.

## Proof Technique: Reducing to a Previous Result

Frequently, a result that needs to be proved can be restated in such a way that it follows immediately from an earlier theorem. This proof technique is commonly referred to as *reducing to a previous result*. In effect, the earlier result acts as a lemma so that the new result does not have to be proved from scratch. Many exercises in this text employ this strategy. Such problems involve multiple parts, where it is expected that certain later parts are to be proved using one or more earlier parts.

The following is a typical example of such an exercise (all vectors are assumed to be in $\mathbb{R}^n$). Part (a) is proven directly, and then part (b) is proven using the result from part (a).

**(a)** Prove that if $\mathbf{v}$ is orthogonal to each of $\mathbf{v}_1, \ldots, \mathbf{v}_k$, then $\mathbf{v}$ is orthogonal to every linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$.

**(b)** Suppose $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ is a mutually orthogonal set of vectors, $\mathbf{v}$ is any vector, and that

$$\mathbf{w} = \mathbf{v} - \sum_{i=1}^{k} \mathbf{proj}_{\mathbf{w}_i}\mathbf{v}.$$

Use part (a) to prove that $\mathbf{w}$ is orthogonal to $\mathbf{w}_1$.

**Proof of Part (a):** Let $c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$ be a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$. Then,

$$\begin{aligned} (c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k)\cdot\mathbf{v} &= c_1(\mathbf{v}_1\cdot\mathbf{v}) + \cdots + c_k(\mathbf{v}_k\cdot\mathbf{v}) && \text{by Result 8} \\ &= c_1(0) + \cdots + c_k(0) = 0. \end{aligned}$$

Therefore, $\mathbf{v}$ is orthogonal to $c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$. □

**Proof of Part (b):** The strategy here is to rewrite the sum that defines $\mathbf{w}$ as a linear combination of vectors, each of which is individually orthogonal to $\mathbf{w}_1$. Then we can deduce from part (a) that $\mathbf{w}_1$ is orthogonal to the linear combination of these vectors, which is $\mathbf{w}$.

To accomplish this, we regroup the sum by separating the first term from the others as follows:

$$\mathbf{w} = (\mathbf{v} - \mathbf{proj}_{\mathbf{w}_1}\mathbf{v}) - \sum_{i=2}^{k} \mathbf{proj}_{\mathbf{w}_i}\mathbf{v}.$$

Notice that $\mathbf{w}_1$ is orthogonal to the term $(\mathbf{v} - \mathbf{proj}_{\mathbf{w}_1}\mathbf{v})$ by Theorem 1.11. Also, each term $\mathbf{proj}_{\mathbf{w}_i}\mathbf{v}$, by definition, is a scalar multiple of $\mathbf{w}_i$, which is orthogonal to $\mathbf{w}_1$. Hence, we have expressed $\mathbf{w}$ as a linear combination of vectors orthogonal to $\mathbf{w}_1$, and so by part (a), $\mathbf{w}_1$ is orthogonal to $\mathbf{w}$. (Stated more explicitly, $(\mathbf{v} - \mathbf{proj}_{\mathbf{w}_1}\mathbf{v})$ in part (b) corresponds to the vector $c_1\mathbf{v}_1$ in part (a), while each $-\mathbf{proj}_{\mathbf{w}_i}\mathbf{v}$ in part (b) corresponds to $c_i\mathbf{v}_i$ in part (a).) □

Exercise 19 in this section and Exercises 18 and 22 in Section 1.5 utilize the proof technique of reducing to a previous result.

## Negating Statements With Quantifiers and Connectives

When considering some statement $A$, we are frequently interested in its **negation**, "not $A$." For example, negation is used in constructing a contrapositive, as well as in proof by contradiction. Of course, "not $A$" is true precisely when $A$ is false, and "not $A$" is false precisely when $A$ is true. That is, $A$ and "not $A$" always have opposite truth values. Negating a simple statement is usually easy. However, when a statement involves **quantifiers** (such as *all*, *some*, or *none*) or involves **connectives** (such as *and* or *or*), the negation process can be tricky.

We first discuss negating statements with quantifiers. As an example, suppose $S$ represents some set of vectors in $\mathbb{R}^3$ and

$$A = \text{“All vectors in } S \text{ are unit vectors.”}$$

The correct negation of $A$ is

$$\text{not } A = \text{“Some vector in } S \text{ is not a unit vector.”}$$

These statements have opposite truth values in all cases. Students frequently err in giving

$$B = \text{“No vector in } S \text{ is a unit vector”}$$

as the negation of $A$. This is incorrect, because if $S$ contained unit and non-unit vectors, then both $A$ and $B$ would be false. Hence, $A$ and $B$ do not have opposite truth values in all cases.

Next consider

$$C = \text{“There is a real number } c \text{ such that } \mathbf{y} = c\mathbf{x}, \text{”}$$

referring to specific vectors $\mathbf{x}$ and $\mathbf{y}$. Then

$$\text{not } C = \text{“No real number } c \text{ exists such that } \mathbf{y} = c\mathbf{x}.\text{”}$$

Alternately,

$$\text{not } C = \text{“For every real number } c, \mathbf{y} \neq c\mathbf{x}.\text{”}$$

There are two types of quantifiers. **Universal quantifiers** (such as *every*, *all*, *no*, and *none*) say that a statement is true or false in every instance, and **existential quantifiers** (such as *some* and *there exists*) claim that there is at least one instance in which the statement is satisfied. The statements $A$ and "not $C$" in the preceding examples involve universal quantifiers; "not $A$" and $C$ use existential quantifiers. These examples follow a general pattern.

**Rules for Negating Statements With Quantifiers**
The negation of a statement involving a universal quantifier uses an existential quantifier.
The negation of a statement involving an existential quantifier uses a universal quantifier.

Hence, negating a statement changes the type of quantifier used. For example, notice at the beginning of the proof of Result 7 when we assumed its conclusion ("*no* vector in $S$ is a linear combination...") to be false, it was equivalent to assuming that "*some* vector in $S$ can be expressed as a linear combination... ."

Next, consider negating with the connectives *and* or *or*. The formal rules for negating such statements are known as **DeMorgan's Laws**.

**Rules for Negating Statements With Connectives (DeMorgan's Laws)**
The negation of "$A$ or $B$" is "(not $A$) and (not $B$)."
The negation of "$A$ and $B$" is "(not $A$) or (not $B$)."

Note that when negating, *or* is converted to *and* and vice versa.

Table 1.1 illustrates the rules for negating quantifiers and connectives. In the table, $S$ refers to a set of vectors in $\mathbb{R}^3$, and $n$ represents a positive integer. Only some of the statements in Table 1.1 are true. Regardless, each statement has the opposite truth value of its negation.

**TABLE 1.1** Several statements and their negations

| Original statement | Negation of the statement |
|---|---|
| $n$ is an even number or a prime. | $n$ is odd and not prime. |
| $\mathbf{x}$ is a unit vector and $\mathbf{x} \in S$. | $\|\mathbf{x}\| \neq 1$ or $\mathbf{x} \notin S$. |
| Some prime numbers are odd. | Every prime number is even. |
| There is a unit vector in $S$. | No elements of $S$ are unit vectors. |
| There is a vector $\mathbf{x}$ in $S$ with $\mathbf{x} \cdot [1, 1, -1] = 0$. | For every vector $\mathbf{x}$ in $S$, $\mathbf{x} \cdot [1, 1, -1] \neq 0$. |
| All numbers divisible by 4 are even. | Some number divisible by 4 is odd. |
| Every vector in $S$ is a unit vector or is parallel to $[1, -2, 1]$. | There is a non-unit vector in $S$ that is not parallel to $[1, -2, 1]$. |
| For every nonzero vector $\mathbf{x}$ in $\mathbb{R}^3$, there is a vector in $S$ that is parallel to $\mathbf{x}$. | There is a nonzero vector $\mathbf{x}$ in $\mathbb{R}^3$ that is not parallel to any vector in $S$. |
| There is a real number $K$ such that for every $\mathbf{x} \in S$, $\|\mathbf{x}\| \leq K$. | For every real number $K$ there is a vector $\mathbf{x} \in S$ such that $\|\mathbf{x}\| > K$. |

## Disproving Statements

Frequently we must prove that a given statement is false rather than true. To disprove a statement $A$, we must instead prove "not $A$." There are two cases.

**Case 1: Statements involving universal quantifiers**: A statement $A$ with a universal quantifier is disproved by finding a single **counterexample** that makes $A$ false. For example, consider

$$B = \text{"For all } \mathbf{x} \text{ and } \mathbf{y} \text{ in } \mathbb{R}^3, \ \ \|\mathbf{x}+\mathbf{y}\| = \|\mathbf{x}\| + \|\mathbf{y}\|.\text{"}$$

We disprove $B$ by finding a counterexample—that is, a specific case where $B$ is false. Letting $\mathbf{x} = [3, 0, 0]$ and $\mathbf{y} = [0, 0, 4]$, we get $\|\mathbf{x}+\mathbf{y}\| = \|[3, 0, 4]\| = 5$. However, $\|\mathbf{x}\| = 3$ and $\|\mathbf{y}\| = 4$, so $\|\mathbf{x}+\mathbf{y}\| \neq \|\mathbf{x}\| + \|\mathbf{y}\|$, and $B$ is disproved.

Sometimes we want to disprove an implication "If $A$ then $B$." This implication involves a universal quantifier because it asserts "In all cases in which $A$ is true, $B$ is also true." Therefore,

> Disproving "If $A$ then $B$" entails finding a specific counterexample for which $A$ is true but $B$ is false.

To illustrate, consider

$$C = \text{"If } \mathbf{x} \text{ and } \mathbf{y} \text{ are unit vectors in } \mathbb{R}^4, \text{ then } \mathbf{x}\cdot\mathbf{y} = 1.\text{"}$$

To disprove $C$, we must find a counterexample in which the premise "$\mathbf{x}$ and $\mathbf{y}$ are unit vectors in $\mathbb{R}^4$" is true and the conclusion "$\mathbf{x}\cdot\mathbf{y} = 1$" is false. Consider $\mathbf{x} = [1, 0, 0, 0]$ and $\mathbf{y} = [0, 1, 0, 0]$, which are unit vectors in $\mathbb{R}^4$. Then $\mathbf{x}\cdot\mathbf{y} = 0 \neq 1$. This counterexample disproves $C$.

**Case 2: Statements involving existential quantifiers**: Recall that an existential quantifier changes to a universal quantifier under negation. For example, consider

$$D = \text{"There is a nonzero vector } \mathbf{x} \text{ in } \mathbb{R}^2 \text{ such that } \mathbf{x}\cdot[1, 0] = 0 \text{ and } \mathbf{x}\cdot[0, 1] = 0.\text{"}$$

To disprove $D$, we must prove[9]

$$\text{not } D = \text{"For every nonzero vector } \mathbf{x} \text{ in } \mathbb{R}^2, \ \mathbf{x}\cdot[1, 0] \neq 0 \text{ or } \mathbf{x}\cdot[0, 1] \neq 0.\text{"}$$

We cannot prove this statement by giving a single example. Instead, we must show "not $D$" is true for *every* nonzero vector in $\mathbb{R}^2$. This can be done with a direct proof. (You were asked to supply its proof earlier, since "not $D$" is actually Result 5.)

The moral here is that we cannot disprove a statement having an existential quantifier with a counterexample. Instead, a proof of the negation must be given.

[9] Notice that along with the change in the quantifier, the *and* connective changes to *or*.

## New Vocabulary

Base Step of an induction proof
conclusion of an "If..then" proof
connectives
contrapositive of a statement
converse of a statement
counterexample
DeMorgan's Laws
direct proof
existential quantifier
"If..then" proof
"If and only if" proof
"If A then (B or C)" proof
implication
induction
inductive hypothesis for the Inductive Step
Inductive Step of an induction proof
inverse of a statement
negation of a statement
premise of an "If..then" proof
proof by contradiction
proof by contrapositive
proof by induction
proof by reducing to a previous result
quantifiers
universal quantifier
without loss of generality

## Highlights

- There are various types of proofs, including: direct proof, "If $A$ then $B$" proof, "$A$ if and only if $B$" proof, "If $A$ then ($B$ or $C$)" proof, proof by contrapositive, proof by contradiction, proof by reducing to a previous result, and proof by induction.
- When proving that an equation is true, a useful strategy is to begin with one half of the equation and work toward the other half.
- A useful strategy when proving a statement is to begin with the definitions of the terms used in the statement.
- A useful strategy for trying to prove an "If $A$ then $B$" statement is to assume the premise $A$ and derive the conclusion $B$.
- A useful strategy for trying to prove a given statement is to work "backward" to discover a proof, then write the proof in a "forward" (correct) manner.
- In an "$A$ if and only if $B$" proof, there are two proofs to be done: we must assume $A$ and prove $B$, and we also must assume $B$ and prove $A$.
- In an "If $A$ then ($B$ or $C$)" proof, a typical strategy is to assume $A$ and "not $B$" and prove $C$. Alternately, we can assume $A$ and "not $C$" and prove $B$.
- A statement is logically equivalent to its contrapositive, but *not* to its converse or inverse. For any theorem that represents an implication, its contrapositive is also a theorem that represents a valid alternate statement of the original theorem.
- An "If $A$ then $B$" statement can be proven by contrapositive by assuming "not $B$" and proving "not $A$."
- In an induction proof, both the Base Step and the Inductive Step must be proven. In carrying out the Inductive Step, assume the statement is true for some integer value (say, $k$) of the given variable (this is the induction hypothesis), and then prove the statement is true for the next integer value ($k+1$).
- When negating a statement, universal quantifiers change to existential quantifiers, and vice versa.
- When negating a statement, "and" is replaced by "or," and vice versa.
- To disprove an "If $A$ then $B$" statement, it is enough to find a counterexample for which $A$ is true and $B$ is false.

## Exercises for Section 1.3

1. This exercise involves the technique of direct proof.
   (a) Give a direct proof that, if $\mathbf{x}$ and $\mathbf{y}$ are vectors in $\mathbb{R}^n$, then $\|9\mathbf{x}+5\mathbf{y}\| \le 9\,(\|\mathbf{x}\|+\|\mathbf{y}\|)$.
   ★ (b) Can you generalize your proof in part (a) to draw any conclusions about $\|c\mathbf{x}+d\mathbf{y}\|$, where $c, d \in \mathbb{R}$? What about $\|c\mathbf{x}-d\mathbf{y}\|$?
2. This exercise involves the technique of direct proof.
   (a) Give a direct proof that if an integer has the form $6j+5$, then it also has the form $3k-1$, where $j$ and $k$ are integers.
   ★ (b) Find a counterexample to show that the converse of part (a) is not true.
3. Let $\mathbf{x}$ and $\mathbf{y}$ be nonzero vectors in $\mathbb{R}^n$. Prove $\mathbf{proj_x y}=\mathbf{0}$ if and only if $\mathbf{proj_y x}=\mathbf{0}$.
4. Let $\mathbf{x}$ and $\mathbf{y}$ be nonzero vectors in $\mathbb{R}^n$. Prove $\|\mathbf{x}+\mathbf{y}\| = \|\mathbf{x}\|+\|\mathbf{y}\|$ if and only if $\mathbf{y}=c\mathbf{x}$ for some $c>0$. (Hint: Be sure to prove both halves of this statement. Result 4 may make one half of the proof easier.)

**5.** Prove the following statements of the form "If $A$, then $B$ or $C$."
  **(a)** If $\|\mathbf{x}+\mathbf{y}\| = \|\mathbf{x}\|$, then $\mathbf{y} = \mathbf{0}$ or $\mathbf{x}$ is not orthogonal to $\mathbf{y}$.
  **(b)** If $\mathbf{proj_x y} = \mathbf{x}$, then $\mathbf{x}$ is a unit vector or $\mathbf{x}\cdot\mathbf{y} \neq 1$.

★ **6.** Consider the statement $A$ = "If $\mathbf{x}\cdot\mathbf{y} = \|\mathbf{x}\|^2$, then $\mathbf{x} = \mathbf{y}$."
  **(a)** Show that $A$ is false by exhibiting a counterexample.
  **(b)** State the contrapositive of $A$.
  **(c)** Does your counterexample from part (a) also show that the contrapositive from part (b) is false?

★ **7.** Prove the following by contrapositive: Assume that $\mathbf{x}$ and $\mathbf{y}$ are vectors in $\mathbb{R}^n$. If $\mathbf{x}\cdot\mathbf{y} \neq 0$, then $\|\mathbf{x}+\mathbf{y}\|^2 \neq \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$.

**8.** Prove the following by contrapositive: Assume $\mathbf{x}$ and $\mathbf{y}$ are vectors in $\mathbb{R}^n$. If the angle between $\mathbf{x}+\mathbf{y}$ and $\mathbf{x}-\mathbf{y}$ is not acute and not $0°$, then $||\mathbf{x}|| \leq ||\mathbf{y}||$.

**9.** State the contrapositive, converse, and inverse of each of the following statements for vectors in $\mathbb{R}^n$:
  ★ **(a)** If $\mathbf{x}$ is a unit vector, then $\mathbf{x}$ is nonzero.
  **(b)** Let $\mathbf{x}$ and $\mathbf{y}$ be nonzero vectors. If $\mathbf{x}$ is parallel to $\mathbf{y}$, then $\mathbf{y} = \mathbf{proj_x y}$.
  ★ **(c)** Let $\mathbf{x}$ and $\mathbf{y}$ be nonzero vectors. If $\mathbf{proj_x y} = \mathbf{0}$, then $\mathbf{proj_y x} = \mathbf{0}$.

**10.** This exercise explores the converse of Result 3.
  **(a)** State the converse of Result 3.
  **(b)** Show that this converse is false by finding a counterexample.

**11.** Each of the following statements has the opposite truth value as its converse; that is, one of them is true, and the other is false. In each case,
  **(i)** State the converse of the given statement.
  **(ii)** Which is true—the statement or its converse?
  **(iii)** Prove the one from part (ii) that is true.
  **(iv)** Disprove the other one by finding a counterexample.

  **(a)** Let $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ be vectors in $\mathbb{R}^n$. If $\mathbf{x}\cdot\mathbf{y} = \mathbf{x}\cdot\mathbf{z}$, then $\mathbf{y} = \mathbf{z}$.
  ★ **(b)** Let $\mathbf{x}$ and $\mathbf{y}$ be vectors in $\mathbb{R}^n$. If $\mathbf{x}\cdot\mathbf{y} = 0$, then $\|\mathbf{x}+\mathbf{y}\| \geq \|\mathbf{y}\|$.
  **(c)** Assume that $\mathbf{x}$ and $\mathbf{y}$ are vectors in $\mathbb{R}^n$ with $n > 1$. If $\mathbf{x}\cdot\mathbf{y} = 0$, then $\mathbf{x} = \mathbf{0}$ or $\mathbf{y} = \mathbf{0}$.

★ **12.** Let $\mathbf{x}$ and $\mathbf{y}$ be vectors in $\mathbb{R}^n$ such that each coordinate of both $\mathbf{x}$ and $\mathbf{y}$ is equal to either 1 or $-1$. Prove by contradiction that if $\mathbf{x}$ is orthogonal to $\mathbf{y}$, then $n$ is even.

**13.** Prove by contradiction: There does not exist a nonzero vector orthogonal to both $[6, 5]$ and $[-2, 3]$.

**14.** Suppose $\mathbf{x}$ and $\mathbf{y}$ are vectors in $\mathbb{R}^n$, the angle between $\mathbf{x}$ and $\mathbf{y}$ is acute, and $||\mathbf{proj_x y}|| \neq \mathbf{x}\cdot\mathbf{y}$. Prove by contradiction that $\mathbf{x}$ is not a unit vector.

**15.** Prove by induction: If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{n-1}, \mathbf{x}_n$ (for $n \geq 1$) are vectors in $\mathbb{R}^m$, then $\mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_{n-1} + \mathbf{x}_n = \mathbf{x}_n + \mathbf{x}_{n-1} + \cdots + \mathbf{x}_2 + \mathbf{x}_1$.

**16.** Prove by induction: For each integer $m \geq 1$, let $\mathbf{x}_1, \ldots, \mathbf{x}_m$ be vectors in $\mathbb{R}^n$. Then, $\|\mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_m\| \leq \|\mathbf{x}_1\| + \|\mathbf{x}_2\| + \cdots + \|\mathbf{x}_m\|$.

★ **17.** Let $\mathbf{x}_1, \ldots, \mathbf{x}_k$ be a mutually orthogonal set of nonzero vectors in $\mathbb{R}^n$. Use induction to show that

$$\left\| \sum_{i=1}^{k} \mathbf{x}_i \right\|^2 = \sum_{i=1}^{k} \|\mathbf{x}_i\|^2 .$$

**18.** Prove by induction: Let $\mathbf{x}_1, \ldots, \mathbf{x}_k$ be unit vectors in $\mathbb{R}^n$, and let $a_1, \ldots, a_k$ be real numbers. Then, for every $\mathbf{y}$ in $\mathbb{R}^n$,

$$\left( \sum_{i=1}^{k} a_i \mathbf{x}_i \right) \cdot \mathbf{y} \leq \left( \sum_{i=1}^{k} |a_i| \right) \|\mathbf{y}\| .$$

**19.** Suppose $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is a mutually orthogonal set of unit vectors in $\mathbb{R}^n$, and that $\mathbf{v} = a_1\mathbf{u}_1 + \cdots + a_k\mathbf{u}_k$.
  **(a)** Prove that $a_i = \mathbf{v}\cdot\mathbf{u}_i$, for each $i$ with $1 \leq i \leq k$.
  **(b)** Use part (a) to prove that $\|\mathbf{v}\|^2 = (\mathbf{v}\cdot\mathbf{u}_1)^2 + \cdots + (\mathbf{v}\cdot\mathbf{u}_k)^2$. (The proof for part (b) involves the proof technique of reducing to a previous result.)

★ **20.** Which steps in the following argument cannot be "reversed"? Why? Assume that $y = f(x)$ is a nonzero function and that $d^2y/dx^2$ exists for all $x$.

$$\text{Step 1: } y = x^2 + 2 \implies y^2 = x^4 + 4x^2 + 4$$

$$\text{Step 2: } y^2 = x^4 + 4x^2 + 4 \implies 2y\frac{dy}{dx} = 4x^3 + 8x$$

$$\text{Step 3: } 2y\frac{dy}{dx} = 4x^3 + 8x \implies \frac{dy}{dx} = \frac{4x^3 + 8x}{2y}$$

$$\text{Step 4: } \frac{dy}{dx} = \frac{4x^3 + 8x}{2y} \implies \frac{dy}{dx} = \frac{4x^3 + 8x}{2\left(x^2 + 2\right)}$$

$$\text{Step 5: } \frac{dy}{dx} = \frac{4x^3 + 8x}{2\left(x^2 + 2\right)} \implies \frac{dy}{dx} = 2x$$

$$\text{Step 6: } \frac{dy}{dx} = 2x \implies \frac{d^2y}{dx^2} = 2$$

**21.** State the negation of each of the following statements involving quantifiers and connectives. (The statements are not necessarily true.)
- ★ **(a)** There is a unit vector in $\mathbb{R}^3$ perpendicular to $[1, -2, 3]$.
- **(b)** $\mathbf{x} = \mathbf{0}$ or $\mathbf{x} \cdot \mathbf{y} > 0$, for all vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$.
- ★ **(c)** $\mathbf{x} \neq \mathbf{0}$ and $\|\mathbf{x} + \mathbf{y}\| = \|\mathbf{y}\|$, for some vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$.
- **(d)** For every vector $\mathbf{x}$ in $\mathbb{R}^n$, $\mathbf{x} \cdot \mathbf{x} > 0$.
- ★ **(e)** For every $\mathbf{x} \in \mathbb{R}^3$, there is a nonzero $\mathbf{y} \in \mathbb{R}^3$ such that $\mathbf{x} \cdot \mathbf{y} = 0$.
- **(f)** There is an $\mathbf{x} \in \mathbb{R}^4$ such that for every $\mathbf{y} \in \mathbb{R}^4$, $\mathbf{x} \cdot \mathbf{y} = 0$.

**22.** State the contrapositive, converse, and inverse of the following statements involving connectives. (The statements are not necessarily true.)
- ★ **(a)** If $\mathbf{x} \cdot \mathbf{y} = 0$, then $\mathbf{x} = \mathbf{0}$ or $\|\mathbf{x} - \mathbf{y}\| > \|\mathbf{y}\|$.
- **(b)** If $\|\mathbf{x} - \mathbf{y}\| \leq \|\mathbf{y}\|$ and $\mathbf{x} \cdot \mathbf{y} = 0$, then $\mathbf{x} = \mathbf{0}$.

**23.** Prove the following by contrapositive: Let $\mathbf{x}$ be a vector in $\mathbb{R}^n$. If $\mathbf{x} \cdot \mathbf{y} = 0$ for every vector $\mathbf{y}$ in $\mathbb{R}^n$, then $\mathbf{x} = \mathbf{0}$.

**24.** Disprove the following: If $\mathbf{x}$ and $\mathbf{y}$ are vectors in $\mathbb{R}^n$, then $\|\mathbf{x} - \mathbf{y}\| \leq \|\mathbf{x}\| - \|\mathbf{y}\|$.

**25.** Use Result 3 to disprove the following: there is a vector $\mathbf{x}$ in $\mathbb{R}^3$ such that $\mathbf{x} \cdot [1, -2, 2] = 0$ and $\|\mathbf{x} + [1, -2, 2]\| < 3$.

**26.** In each part, write the contrapositive of the given theorem or result. Each answer represents a new theorem that is equivalent to the original.
- ★ **(a)** Theorem 1.4.
- **(b)** Result 3.
- ★ **(c)** Result 7. (Hint: First rephrase Result 7 as an implication as follows: "Let $S = \{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ be a set of vectors in $\mathbb{R}^n$. If $S$ is a mutually orthogonal set and all vectors in $S$ are nonzero, then no vector in $S$ can be expressed as a linear combination of the other vectors in $S$.")
- **(d)** Theorem 1.9, part (2). (Hint: The contrapositive of an "if and only if" is another "if and only if," whose two parts are the respective contrapositives of the two parts of the original statement.)

★ **27.** True or False:
- **(a)** After "working backward" to complete a proof, it is enough to reverse your steps to give a valid "forward" proof.
- **(b)** "If $A$ then $B$" has the same truth value as "If not $B$ then not $A$."
- **(c)** The converse of "$A$ only if $B$" is "If $B$ then $A$."
- **(d)** "$A$ if and only if $B$" is logically equivalent to "$A$ is a necessary condition for $B$."
- **(e)** "$A$ if and only if $B$" is logically equivalent to "$A$ is a necessary condition for $B$" together with "$B$ is a sufficient condition for $A$."
- **(f)** The converse and inverse of a statement must have opposite truth values.
- **(g)** A proof of a given statement by induction is valid if, whenever the statement is true for any integer $k$, it is also true for the next integer $k + 1$.
- **(h)** When negating a statement, universal quantifiers change to existential quantifiers, and vice versa.
- **(i)** The negation of "$A$ and $B$" is "not $A$ and not $B$."

## 1.4 Fundamental Operations With Matrices

We now introduce a new algebraic structure: the matrix. Matrices are two-dimensional arrays created by arranging vectors into rows and columns. We examine several fundamental types of matrices, as well as three basic operations on matrices and their properties.

### Definition of a Matrix

**Definition** An $m \times n$ **matrix** is a rectangular array of real numbers, arranged in $m$ rows and $n$ columns. The elements of a matrix are called the **entries**. The expression $m \times n$ represents the **size** of the matrix.

For example, each of the following is a matrix, listed with its correct size:

$$\mathbf{A} = \underbrace{\begin{bmatrix} 2 & 3 & -1 \\ 4 & 0 & -5 \end{bmatrix}}_{2\times 3 \text{ matrix}} \quad \mathbf{B} = \underbrace{\begin{bmatrix} 4 & -2 \\ 1 & 7 \\ -5 & 3 \end{bmatrix}}_{3\times 2 \text{ matrix}} \quad \mathbf{C} = \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}}_{3\times 3 \text{ matrix}}$$

$$\mathbf{D} = \underbrace{\begin{bmatrix} 7 \\ 1 \\ -2 \end{bmatrix}}_{3\times 1 \text{ matrix}} \quad \mathbf{E} = \underbrace{\begin{bmatrix} 4 & -3 & 0 \end{bmatrix}}_{1\times 3 \text{ matrix}} \quad \mathbf{F} = \underbrace{\begin{bmatrix} 4 \end{bmatrix}}_{1\times 1 \text{ matrix}}$$

Here are some conventions to remember regarding matrices.

- We use a single (or subscripted) bold capital letter to represent a matrix (such as $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}_1$, $\mathbf{C}_2$) in contrast to the lowercase bold letters used to represent vectors. The capital letters $\mathbf{I}$ and $\mathbf{O}$ are usually reserved for special types of matrices discussed later.
- The size of a matrix is always specified by stating the number of rows first. For example, a $3 \times 4$ matrix always has three rows and four columns, never four rows and three columns.
- An $m \times n$ matrix can be thought of either as a collection of $m$ row vectors, each having $n$ coordinates, or as a collection of $n$ column vectors, each having $m$ coordinates. A matrix with just one row (or column) is essentially equivalent to a vector with coordinates in row (or column) form.
- We often write $a_{ij}$ to represent the entry in the $i$th row and $j$th column of a matrix $\mathbf{A}$. For example, in the previous matrix $\mathbf{A}$, $a_{23}$ is the entry $-5$ in the second row and third column. A typical $3 \times 4$ matrix $\mathbf{C}$ has entries symbolized by

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \end{bmatrix}.$$

- $\mathcal{M}_{mn}$ represents the set of all matrices with real-number entries having $m$ rows and $n$ columns. For example, $\mathcal{M}_{34}$ is the set of all matrices having three rows and four columns. A typical matrix in $\mathcal{M}_{34}$ has the form of the preceding matrix $\mathbf{C}$.
- The **main diagonal** entries of a matrix $\mathbf{A}$ are $a_{11}, a_{22}, a_{33}, \ldots$, those that lie on a diagonal line drawn down to the right, beginning from the upper-left corner of the matrix.

Matrices occur naturally in many contexts. For example, two-dimensional tables (having rows and columns) of real numbers are matrices. The following table represents a $50 \times 3$ matrix for the U.S. states with integer entries:

| | **Population (2020)** | **Area (sq. mi.)** | **Year Admitted to U.S.** |
|---|---|---|---|
| **Alabama** | 4908620 | 52420 | 1819 |
| **Alaska** | 734002 | 665384 | 1959 |
| **Arizona** | 7378490 | 113990 | 1912 |
| ⋮ | ⋮ | ⋮ | ⋮ |
| **Wyoming** | 567025 | 97813 | 1890 |

Two matrices **A** and **B** are **equal** if and only if they have the same size and all of their corresponding entries are equal. That is, if **A** and **B** are both $m \times n$ matrices, then $\mathbf{A} = \mathbf{B}$ if $a_{ij} = b_{ij}$ for all $i$, $1 \le i \le m$, and for all $j$, $1 \le j \le n$.

Note that the following may be considered equal as vectors but not as matrices:

$$[3, -2, 4] \qquad \text{and} \qquad \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix},$$

since the former is a $1 \times 3$ matrix, but the latter is a $3 \times 1$ matrix.

## Special Types of Matrices

We now describe a few important types of matrices.

A **square matrix** is an $n \times n$ matrix; that is, a matrix having the same number of rows as columns. For example, the following matrices are square:

$$\mathbf{A} = \begin{bmatrix} 5 & 0 \\ 9 & -2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

A **diagonal matrix** is a square matrix in which all entries that are not on the main diagonal are zero. That is, **D** is diagonal if and only if it is square and $d_{ij} = 0$ for $i \ne j$. For example, the following are diagonal matrices:

$$\mathbf{E} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{G} = \begin{bmatrix} -4 & 0 \\ 0 & 5 \end{bmatrix}.$$

However, the following matrices

$$\mathbf{H} = \begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{J} = \begin{bmatrix} 0 & 4 & 3 \\ -7 & 0 & 6 \\ 5 & -2 & 0 \end{bmatrix}$$

are *not* diagonal. (The main diagonal entries have been printed in color in each case.) We use $\mathcal{D}_n$ to represent the **set of all** $n \times n$ **diagonal matrices**.

An **identity matrix** is a diagonal matrix with all main diagonal entries equal to 1. That is, an $n \times n$ matrix **A** is an identity matrix if and only if $a_{ij} = 0$ for $i \ne j$ and $a_{ii} = 1$ for $1 \le i \le n$. The $n \times n$ identity matrix is represented by $\mathbf{I}_n$. For example, the following are identity matrices:

$$\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{I}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

If the size of the identity matrix is clear from the context, **I** alone may be used.

An **upper triangular matrix** is a square matrix with all entries *below* the main diagonal equal to zero. That is, an $n \times n$ matrix **A** is upper triangular if and only if $a_{ij} = 0$ for $i > j$. For example, the following are upper triangular:

$$\mathbf{P} = \begin{bmatrix} 6 & 9 & 11 \\ 0 & -2 & 3 \\ 0 & 0 & 5 \end{bmatrix} \quad \text{and} \quad \mathbf{Q} = \begin{bmatrix} 7 & -2 & 2 & 0 \\ 0 & -4 & 9 & 5 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

Similarly, a **lower triangular matrix** is one in which all entries *above* the main diagonal equal zero; for example,

$$\mathbf{R} = \begin{bmatrix} 3 & 0 & 0 \\ 9 & -2 & 0 \\ 14 & -6 & 1 \end{bmatrix}$$

is lower triangular. We use $\mathcal{U}_n$ to represent the **set of all** $n \times n$ **upper triangular matrices** and $\mathcal{L}_n$ to represent the **set of all** $n \times n$ **lower triangular matrices**.

A **zero matrix** is any matrix all of whose entries are zero. $\mathbf{O}_{mn}$ represents the $m \times n$ zero matrix, and $\mathbf{O}_n$ represents the $n \times n$ zero matrix. For example,

$$\mathbf{O}_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{O}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

are zero matrices. If the size of the zero matrix is clear from the context, **O** alone may be used.

## Addition and Scalar Multiplication With Matrices

**Definition** Let **A** and **B** both be $m \times n$ matrices. The **sum** of **A** and **B** is the $m \times n$ matrix $(\mathbf{A} + \mathbf{B})$ whose $(i, j)$ entry is equal to $a_{ij} + b_{ij}$.

As with vectors, matrices are summed simply by adding their corresponding entries together. For example,

$$\begin{bmatrix} 6 & -3 & 2 \\ -7 & 0 & 4 \end{bmatrix} + \begin{bmatrix} 5 & -6 & -3 \\ -4 & -2 & -4 \end{bmatrix} = \begin{bmatrix} 11 & -9 & -1 \\ -11 & -2 & 0 \end{bmatrix}.$$

Notice that the definition does not allow addition of matrices with different sizes. For example, the following matrices cannot be added:

$$\mathbf{A} = \begin{bmatrix} -2 & 3 & 0 \\ 1 & 4 & -5 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 6 & 7 \\ -2 & 5 \\ 4 & -1 \end{bmatrix},$$

since **A** is a $2 \times 3$ matrix, and **B** is a $3 \times 2$ matrix.

**Definition** Let **A** be an $m \times n$ matrix, and let $c$ be a scalar. Then the matrix $c\mathbf{A}$, the **scalar multiplication** of $c$ and **A**, is the $m \times n$ matrix whose $(i, j)$ entry is equal to $ca_{ij}$.

As with vectors, scalar multiplication with matrices is done by multiplying every entry by the given scalar. For example, if $c = -2$ and

$$\mathbf{A} = \begin{bmatrix} 4 & -1 & 6 & 7 \\ 2 & 4 & 9 & -5 \end{bmatrix}, \text{ then } -2\mathbf{A} = \begin{bmatrix} -8 & 2 & -12 & -14 \\ -4 & -8 & -18 & 10 \end{bmatrix}.$$

Note that if **A** is any $m \times n$ matrix, then $0\mathbf{A} = \mathbf{O}_{mn}$.

Let $-\mathbf{A}$ represent the matrix $-1\mathbf{A}$, the scalar multiple of **A** by $(-1)$. For example, if

$$\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 10 & 6 \end{bmatrix}, \text{ then } -1\mathbf{A} = -\mathbf{A} = \begin{bmatrix} -3 & 2 \\ -10 & -6 \end{bmatrix}.$$

Also, we define **subtraction** of matrices as $\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$.

As with vectors, sums of scalar multiples of matrices are called **linear combinations.** For example, $-2\mathbf{A} + 6\mathbf{B} - 3\mathbf{C}$ is a linear combination of **A**, **B**, and **C**.

## Fundamental Properties of Addition and Scalar Multiplication

The properties in the next theorem are similar to the vector properties of Theorem 1.3.

**Theorem 1.12** *Let* **A**, **B**, *and* **C** *be* $m \times n$ *matrices (elements of* $\mathcal{M}_{mn}$*), and let* $c$ *and* $d$ *be scalars. Then*

(1) $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$ — *Commutative Law of Addition*
(2) $\mathbf{A}+(\mathbf{B}+\mathbf{C})=(\mathbf{A}+\mathbf{B})+\mathbf{C}$ — *Associative Law of Addition*
(3) $\mathbf{O}_{mn}+\mathbf{A}=\mathbf{A}+\mathbf{O}_{mn}=\mathbf{A}$ — *Existence of Identity Element for Addition*
(4) $\mathbf{A}+(-\mathbf{A})=(-\mathbf{A})+\mathbf{A}=\mathbf{O}_{mn}$ — *Existence of Inverse Elements for Addition*
(5) $c(\mathbf{A}+\mathbf{B})=c\mathbf{A}+c\mathbf{B}$ — *Distributive Laws of Scalar*
(6) $(c+d)\mathbf{A}=c\mathbf{A}+d\mathbf{A}$ — *Multiplication Over Addition*
(7) $(cd)\mathbf{A}=c(d\mathbf{A})$ — *Associativity of Scalar Multiplication*
(8) $1(\mathbf{A})=\mathbf{A}$ — *Identity Property for Scalar Multiplication*

To prove each property, calculate corresponding entries on both sides and show they agree by applying an appropriate law of real numbers. We prove part (1) as an example and leave some of the remaining proofs as Exercise 9.

*Proof.* Proof of Part (1): For any $i, j$, where $1 \le i \le m$ and $1 \le j \le n$, the $(i, j)$ entry of $(\mathbf{A}+\mathbf{B})$ is the sum of the entries $a_{ij}$ and $b_{ij}$ from **A** and **B**, respectively. Similarly, the $(i, j)$ entry of $\mathbf{B}+\mathbf{A}$ is the sum of $b_{ij}$ and $a_{ij}$. But $a_{ij}+b_{ij}=b_{ij}+a_{ij}$, by the commutative property of addition for real numbers. Hence, $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$, because their corresponding entries agree. □

## The Transpose of a Matrix and Its Properties

**Definition** If **A** is an $m \times n$ matrix, then its **transpose**, $\mathbf{A}^T$, is the $n \times m$ matrix whose $(i, j)$ entry is the same as the $(j, i)$ entry of **A**.

Thus, transposing **A** moves the $(i, j)$ entry of **A** to the $(j, i)$ entry of $\mathbf{A}^T$. Notice that the entries on the main diagonal do not move as we convert **A** to $\mathbf{A}^T$. However, all entries above the main diagonal are moved below it, and vice versa. For example,

$$\text{if } \mathbf{A}=\begin{bmatrix} 6 & 10 \\ -2 & 4 \\ 3 & 0 \\ 1 & 8 \end{bmatrix} \quad \text{and} \quad \mathbf{B}=\begin{bmatrix} 1 & 5 & -3 \\ 0 & -4 & 6 \\ 0 & 0 & -5 \end{bmatrix},$$

$$\text{then } \mathbf{A}^T=\begin{bmatrix} 6 & -2 & 3 & 1 \\ 10 & 4 & 0 & 8 \end{bmatrix} \quad \text{and} \quad \mathbf{B}^T=\begin{bmatrix} 1 & 0 & 0 \\ 5 & -4 & 0 \\ -3 & 6 & -5 \end{bmatrix}.$$

Notice that the transpose changes the rows of **A** into the columns of $\mathbf{A}^T$. Similarly, the columns of **A** become the rows of $\mathbf{A}^T$. Also note that the transpose of an upper triangular matrix (such as **B**) is lower triangular, and vice versa.

Three useful properties of the transpose are given in the next theorem. We prove the first half of part (2) and leave the others as Exercise 10.

**Theorem 1.13** *Let* **A** *and* **B** *both be* $m \times n$ *matrices, and let* $c$ *be a scalar. Then*

(1) $(\mathbf{A}^T)^T=\mathbf{A}$
(2) $(\mathbf{A}\pm\mathbf{B})^T=\mathbf{A}^T\pm\mathbf{B}^T$
(3) $(c\mathbf{A})^T=c(\mathbf{A}^T)$

*Proof.* Proof for Addition in Part (2): Notice that $(\mathbf{A}+\mathbf{B})^T$ and $(\mathbf{A}^T+\mathbf{B}^T)$ are both $n \times m$ matrices (why?). We need to show that the $(i, j)$ entries of both are equal, for $1 \le i \le n$ and $1 \le j \le m$. But,

$$(i, j) \text{ entry of } (\mathbf{A}+\mathbf{B})^T = (j, i) \text{ entry of } (\mathbf{A}+\mathbf{B}) = a_{ji} + b_{ji}, \quad \text{while}$$
$$(i, j) \text{ entry of } (\mathbf{A}^T+\mathbf{B}^T) = ((i, j) \text{ entry of } \mathbf{A}^T) + ((i, j) \text{ entry of } \mathbf{B}^T) = a_{ji} + b_{ji}.$$

□

## Symmetric and Skew-Symmetric Matrices

**Definition** A matrix $\mathbf{A}$ is **symmetric** if and only if $\mathbf{A} = \mathbf{A}^T$. A matrix $\mathbf{A}$ is **skew-symmetric** if and only if $\mathbf{A} = -\mathbf{A}^T$.

In Exercise 5, you are asked to show that any symmetric or skew-symmetric matrix is a square matrix.

### Example 1

Consider the following matrices:

$$\mathbf{A} = \begin{bmatrix} 2 & 6 & 4 \\ 6 & -1 & 0 \\ 4 & 0 & -3 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 0 & -1 & 3 & 6 \\ 1 & 0 & 2 & -5 \\ -3 & -2 & 0 & 4 \\ -6 & 5 & -4 & 0 \end{bmatrix}.$$

$\mathbf{A}$ is symmetric and $\mathbf{B}$ is skew-symmetric, because their respective transposes are

$$\mathbf{A}^T = \begin{bmatrix} 2 & 6 & 4 \\ 6 & -1 & 0 \\ 4 & 0 & -3 \end{bmatrix} \quad \text{and} \quad \mathbf{B}^T = \begin{bmatrix} 0 & 1 & -3 & -6 \\ -1 & 0 & -2 & 5 \\ 3 & 2 & 0 & -4 \\ 6 & -5 & 4 & 0 \end{bmatrix},$$

which equal $\mathbf{A}$ and $-\mathbf{B}$, respectively. However, neither of the following is symmetric or skew-symmetric (why?):

$$\mathbf{C} = \begin{bmatrix} 3 & -2 & 1 \\ 2 & 4 & 0 \\ -1 & 0 & -2 \end{bmatrix} \quad \text{and} \quad \mathbf{D} = \begin{bmatrix} 1 & -2 \\ 3 & 4 \\ 5 & -6 \end{bmatrix}.$$

Notice that an $n \times n$ matrix $\mathbf{A}$ is symmetric if and only if $a_{ij} = a_{ji}$, and $\mathbf{A}$ is skew-symmetric if and only if $a_{ij} = -a_{ji}$, for all $i, j$ such that $1 \le i, j \le n$. In other words, the entries above the main diagonal are reflected into equal (for symmetric) or opposite (for skew-symmetric) entries below the diagonal. Since the main diagonal elements are reflected into themselves, *all of the main diagonal elements of a skew-symmetric matrix must be zeroes* ($a_{ii} = -a_{ii}$ only if $a_{ii} = 0$).

Notice that any diagonal matrix is equal to its transpose, and so such matrices are automatically symmetric. Another useful result is the following:

**Theorem 1.14** *If* $\mathbf{A}$ *is a square matrix, then*
(1) $\mathbf{A} + \mathbf{A}^T$ *is symmetric, and*
(2) $\mathbf{A} - \mathbf{A}^T$ *is skew-symmetric.*

*Proof.* Let $\mathbf{A}$ be a square matrix.

**Part (1):** $(\mathbf{A}+\mathbf{A}^T)$ is symmetric since

$$\begin{aligned} (\mathbf{A}+\mathbf{A}^T)^T &= \mathbf{A}^T + (\mathbf{A}^T)^T && \text{by part (2) of Theorem 1.13} \\ &= \mathbf{A}^T + \mathbf{A} && \text{by part (1) of Theorem 1.13} \\ &= \mathbf{A} + \mathbf{A}^T && \text{by part (1) of Theorem 1.12} \end{aligned}$$

**Part (2):** $(\mathbf{A}-\mathbf{A}^T)$ is skew-symmetric since

$$\begin{aligned}(\mathbf{A}-\mathbf{A}^T)^T &= \mathbf{A}^T-(\mathbf{A}^T)^T && \text{by part (2) of Theorem 1.13}\\ &= \mathbf{A}^T-\mathbf{A} && \text{by part (1) of Theorem 1.13}\\ &= -\mathbf{A}+\mathbf{A}^T && \text{by part (1) of Theorem 1.12}\\ &= -(\mathbf{A}-\mathbf{A}^T) && \text{by part (5) of Theorem 1.12}\end{aligned}$$

□

**Example 2**

For the square matrix $\mathbf{C}=\begin{bmatrix}-4 & 3 & -2\\ 5 & -1 & 6\\ -3 & 8 & 1\end{bmatrix}$ (neither symmetric nor skew-symmetric),

$$\mathbf{C}+\mathbf{C}^T=\begin{bmatrix}-4 & 3 & -2\\ 5 & -1 & 6\\ -3 & 8 & 1\end{bmatrix}+\begin{bmatrix}-4 & 5 & -3\\ 3 & -1 & 8\\ -2 & 6 & 1\end{bmatrix}=\begin{bmatrix}-8 & 8 & -5\\ 8 & -2 & 14\\ -5 & 14 & 2\end{bmatrix}$$

is symmetric, while

$$\mathbf{C}-\mathbf{C}^T=\begin{bmatrix}-4 & 3 & -2\\ 5 & -1 & 6\\ -3 & 8 & 1\end{bmatrix}-\begin{bmatrix}-4 & 5 & -3\\ 3 & -1 & 8\\ -2 & 6 & 1\end{bmatrix}=\begin{bmatrix}0 & -2 & 1\\ 2 & 0 & -2\\ -1 & 2 & 0\end{bmatrix}$$

is skew-symmetric.

The next result follows easily from Theorem 1.14.

**Theorem 1.15** *Every square matrix* $\mathbf{A}$ *can be decomposed uniquely as the sum of two matrices* $\mathbf{S}$ *and* $\mathbf{V}$, *where* $\mathbf{S}$ *is symmetric and* $\mathbf{V}$ *is skew-symmetric.*

An outline of the proof of Theorem 1.15 is given in Exercise 12, which also states that $\mathbf{S}=\frac{1}{2}\left(\mathbf{A}+\mathbf{A}^T\right)$ and $\mathbf{V}=\frac{1}{2}\left(\mathbf{A}-\mathbf{A}^T\right)$.

**Example 3**

We can decompose the matrix

$$\mathbf{A}=\begin{bmatrix}-4 & 2 & 5\\ 6 & 3 & 7\\ -1 & 0 & 2\end{bmatrix}$$

as the sum of a symmetric matrix $\mathbf{S}$ and a skew-symmetric matrix $\mathbf{V}$, where

$$\mathbf{S}=\frac{1}{2}\left(\mathbf{A}+\mathbf{A}^T\right)=\frac{1}{2}\left(\begin{bmatrix}-4 & 2 & 5\\ 6 & 3 & 7\\ -1 & 0 & 2\end{bmatrix}+\begin{bmatrix}-4 & 6 & -1\\ 2 & 3 & 0\\ 5 & 7 & 2\end{bmatrix}\right)=\begin{bmatrix}-4 & 4 & 2\\ 4 & 3 & \frac{7}{2}\\ 2 & \frac{7}{2} & 2\end{bmatrix}$$

and

$$\mathbf{V}=\frac{1}{2}\left(\mathbf{A}-\mathbf{A}^T\right)=\frac{1}{2}\left(\begin{bmatrix}-4 & 2 & 5\\ 6 & 3 & 7\\ -1 & 0 & 2\end{bmatrix}-\begin{bmatrix}-4 & 6 & -1\\ 2 & 3 & 0\\ 5 & 7 & 2\end{bmatrix}\right)=\begin{bmatrix}0 & -2 & 3\\ 2 & 0 & \frac{7}{2}\\ -3 & -\frac{7}{2} & 0\end{bmatrix}.$$

Notice that $\mathbf{S}$ and $\mathbf{V}$ really are, respectively, symmetric and skew-symmetric and that $\mathbf{S}+\mathbf{V}$ really does equal $\mathbf{A}$.

## New Vocabulary

additive inverse of a matrix
associative law for matrix addition
associative law for scalar multiplication
commutative law for matrix addition
diagonal matrix
distributive laws for matrices
identity matrix
identity property for scalar multiplication
lower triangular matrix
main diagonal entries
matrix
size of a matrix
skew-symmetric matrix
square matrix
symmetric matrix
trace of a square matrix
transpose of a matrix
upper triangular matrix
zero matrix

## Highlights

- An $m \times n$ matrix can be thought of as a collection of $m$ row vectors in $\mathbb{R}^n$, or a collection of $n$ column vectors in $\mathbb{R}^m$.
- Special types of matrices include square matrices, diagonal matrices, upper and lower triangular matrices, identity matrices, and zero matrices.
- Matrix addition and scalar multiplication satisfy commutative, associative, and distributive laws.
- For any matrix $\mathbf{A}$, $(\mathbf{A}^T)^T = \mathbf{A}$. That is, a double transpose of a matrix is equal to the original matrix.
- For matrices $\mathbf{A}$ and $\mathbf{B}$ of the same size, $(\mathbf{A} \pm \mathbf{B})^T = \mathbf{A}^T \pm \mathbf{B}^T$. That is, the transpose of a sum (or difference) of matrices is equal to the sum (or difference) of the transposes.
- For any matrix $\mathbf{A}$ and any scalar $c$, $(c\mathbf{A})^T = c(\mathbf{A}^T)$. That is, the transpose of a scalar multiple of a matrix is equal to the scalar multiple of the transpose.
- A matrix $\mathbf{A}$ is symmetric if and only if $\mathbf{A} = \mathbf{A}^T$. All entries above the main diagonal of a symmetric matrix are reflected into equal entries below the diagonal.
- A matrix $\mathbf{A}$ is skew-symmetric if and only if $\mathbf{A} = -\mathbf{A}^T$. All entries above the main diagonal of a skew-symmetric matrix are reflected into opposite entries below the diagonal. All main diagonal entries of a skew-symmetric matrix are zero.
- If $\mathbf{A}$ is any square matrix, then $\mathbf{A} + \mathbf{A}^T$ is symmetric, and $\mathbf{A} - \mathbf{A}^T$ is skew-symmetric.
- Every square matrix $\mathbf{A}$ is the sum in a unique way of a symmetric matrix $\mathbf{S} = \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^T\right)$ and a skew-symmetric matrix $\mathbf{V} = \frac{1}{2}\left(\mathbf{A} - \mathbf{A}^T\right)$.

## Exercises for Section 1.4

1. Compute the following, if possible, for the matrices

$$\mathbf{A} = \begin{bmatrix} -4 & 2 & 3 \\ 0 & 5 & -1 \\ 6 & 1 & -2 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 6 & -1 & 0 \\ 2 & 2 & -4 \\ 3 & -1 & 1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 5 & -1 \\ -3 & 4 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} -7 & 1 & -4 \\ 3 & -2 & 8 \end{bmatrix} \quad \mathbf{E} = \begin{bmatrix} 3 & -3 & 5 \\ 1 & 0 & -2 \\ 6 & 7 & -2 \end{bmatrix} \quad \mathbf{F} = \begin{bmatrix} 8 & -1 \\ 2 & 0 \\ 5 & -3 \end{bmatrix}.$$

★ (a) $\mathbf{A} + \mathbf{B}$
(b) $2\mathbf{A} - 3\mathbf{E}$
★ (c) $4\mathbf{A}$
(d) $\mathbf{E} + 4\mathbf{F}$
★ (e) $\mathbf{C} + 3\mathbf{F} - \mathbf{E}$
(f) $\mathbf{A} + \mathbf{B} - \mathbf{E}$
★ (g) $2\mathbf{A} - 3\mathbf{E} - \mathbf{B}$
(h) $4\mathbf{D} - 7\mathbf{C}$
★ (i) $\mathbf{A}^T + \mathbf{E}^T$
(j) $(\mathbf{A} + 2\mathbf{B})^T$
(k) $6\mathbf{F} - 5\mathbf{D}^T$
★ (l) $2\mathbf{C}^T - 3\mathbf{F}$
(m) $8\left(\mathbf{D}^T - \mathbf{B}^T\right)$
★ (n) $\left((\mathbf{B} - \mathbf{A})^T + \mathbf{E}^T\right)^T$

★ **2.** Indicate which of the following matrices are square, diagonal, upper or lower triangular, symmetric, or skew-symmetric. Calculate the transpose for each matrix.

$$\mathbf{A}=\begin{bmatrix}-1 & 4\\ 0 & 1\\ 6 & 0\end{bmatrix}\quad \mathbf{B}=\begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix}\quad \mathbf{C}=\begin{bmatrix}-1 & 1\\ -1 & 1\end{bmatrix}\quad \mathbf{D}=\begin{bmatrix}-1\\ 4\\ 2\end{bmatrix}$$

$$\mathbf{E}=\begin{bmatrix}0 & 0 & 6\\ 0 & -6 & 0\\ -6 & 0 & 0\end{bmatrix}\quad \mathbf{F}=\begin{bmatrix}1 & 0 & 0 & 1\\ 0 & 0 & 1 & 1\\ 0 & 1 & 0 & 0\\ 1 & 1 & 0 & 1\end{bmatrix}\quad \mathbf{G}=\begin{bmatrix}6 & 0 & 0\\ 0 & 6 & 0\\ 0 & 0 & 6\end{bmatrix}$$

$$\mathbf{H}=\begin{bmatrix}0 & -1 & 6 & 2\\ 1 & 0 & -7 & 1\\ -6 & 7 & 0 & -4\\ -2 & -1 & 4 & 0\end{bmatrix}\quad \mathbf{J}=\begin{bmatrix}0 & 1 & 0 & 0\\ 1 & 0 & 1 & 1\\ 0 & 1 & 1 & 1\\ 0 & 1 & 1 & 0\end{bmatrix}\quad \mathbf{K}=\begin{bmatrix}1 & 2 & 3 & 4\\ -2 & 1 & 5 & 6\\ -3 & -5 & 1 & 7\\ -4 & -6 & -7 & 1\end{bmatrix}$$

$$\mathbf{L}=\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}\quad \mathbf{M}=\begin{bmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 1 & 1 & 0\end{bmatrix}\quad \mathbf{N}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\mathbf{P}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\quad \mathbf{Q}=\begin{bmatrix}-2 & 0 & 0\\ 4 & 0 & 0\\ -1 & 2 & 3\end{bmatrix}\quad \mathbf{R}=\begin{bmatrix}6 & 2\\ 3 & -2\\ -1 & 0\end{bmatrix}$$

**3.** Decompose each of the following as the sum of a symmetric and a skew-symmetric matrix:

★ **(a)** $\begin{bmatrix}3 & -1 & 4\\ 0 & 2 & 5\\ 1 & -3 & 2\end{bmatrix}$

**(b)** $\begin{bmatrix}8 & 2 & 0\\ -2 & -3 & 4\\ 5 & 4 & -1\end{bmatrix}$

**(c)** $\begin{bmatrix}6 & 3 & 0 & -5\\ -3 & 4 & 2 & 1\\ 0 & -2 & -8 & 7\\ 5 & -1 & -7 & 9\end{bmatrix}$

**(d)** $\begin{bmatrix}7 & -3 & 6 & 2\\ -9 & 0 & 3 & 4\\ 12 & 11 & 10 & 9\\ -6 & -14 & 7 & 19\end{bmatrix}$

**4.** Prove that if $\mathbf{A}^T=\mathbf{B}^T$, then $\mathbf{A}=\mathbf{B}$.

**5.** This exercise involves properties of particular types of matrices.
- **(a)** Prove that any symmetric or skew-symmetric matrix is square.
- **(b)** Prove that every diagonal matrix is symmetric.
- **(c)** Show that $(\mathbf{I}_n)^T=\mathbf{I}_n$. (Hint: Use part (b).)
- ★ **(d)** Describe completely every matrix that is both diagonal and skew-symmetric.
- **(e)** Describe completely every matrix that is both upper triangular and lower triangular.

**6.** Assume that **A** and **B** are square matrices of the same size.
- **(a)** If **A** and **B** are diagonal, prove that $\mathbf{A}+\mathbf{B}$ is diagonal.
- **(b)** If **A** and **B** are symmetric, prove that $\mathbf{A}+\mathbf{B}$ is symmetric.

**7.** Use induction to prove that, if $\mathbf{A}_1,\ldots,\mathbf{A}_n$ are upper triangular matrices of the same size, then $\boldsymbol{\Sigma}_{i=1}^{n}\mathbf{A}_i$ is upper triangular.

**8.** This exercise explores properties of symmetric and skew-symmetric matrices.
 **(a)** If $\mathbf{A}$ is a symmetric matrix, show that $\mathbf{A}^T$ and $c\mathbf{A}$ are also symmetric.
 **(b)** If $\mathbf{A}$ is a skew-symmetric matrix, show that $\mathbf{A}^T$ and $c\mathbf{A}$ are also skew-symmetric.

**9.** This exercise asks for proofs of various parts of Theorem 1.12.
 **(a)** Prove part (4) of Theorem 1.12.
 **(b)** Prove part (5) of Theorem 1.12.
 **(c)** Prove part (7) of Theorem 1.12.

► **10.** This exercise asks for proofs of various parts of Theorem 1.13.
 **(a)** Prove part (1) of Theorem 1.13.
 **(b)** Prove part (2) of Theorem 1.13 for the subtraction case.
 **(c)** Prove part (3) of Theorem 1.13.

**11.** Let $\mathbf{A}$ be an $m \times n$ matrix. Prove that if $c\mathbf{A} = \mathbf{O}_{mn}$, the $m \times n$ zero matrix, then $c = 0$ or $\mathbf{A} = \mathbf{O}_{mn}$.

**12.** This exercise provides an outline for the proof of Theorem 1.15. Let $\mathbf{A}$ be an $n \times n$ matrix, $\mathbf{S} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T)$, and $\mathbf{V} = \frac{1}{2}(\mathbf{A} - \mathbf{A}^T)$.
 **(a)** Show that $\mathbf{A} = \mathbf{S} + \mathbf{V}$, and show that $\mathbf{S}$ is symmetric and $\mathbf{V}$ is skew-symmetric. (Hint: Use Theorem 1.14 together with Exercise 8 above.)
 **(b)** Suppose $\mathbf{S}_1$ and $\mathbf{S}_2$ are symmetric matrices and $\mathbf{V}_1$ and $\mathbf{V}_2$ are skew-symmetric matrices such that $\mathbf{S}_1 + \mathbf{V}_1 = \mathbf{S}_2 + \mathbf{V}_2$. Derive a second equation involving $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{V}_1$, and $\mathbf{V}_2$ by taking the transpose of both sides of the equation and simplifying.
 **(c)** Prove that $\mathbf{S}_1 = \mathbf{S}_2$ by adding the two equations from part (b) together, and then prove that $\mathbf{V}_1 = \mathbf{V}_2$.
 **(d)** Explain how parts (a) through (c) together prove Theorem 1.15.

**13.** The **trace** of a square matrix $\mathbf{A}$ is the sum of the elements along the main diagonal.
 ★ **(a)** Find the trace of each square matrix in Exercise 2.
 **(b)** If $\mathbf{A}$ and $\mathbf{B}$ are both $n \times n$ matrices, prove that:
  **(i)** $\text{trace}(\mathbf{A} + \mathbf{B}) = \text{trace}(\mathbf{A}) + \text{trace}(\mathbf{B})$
  **(ii)** $\text{trace}(c\mathbf{A}) = c(\text{trace}(\mathbf{A}))$
  **(iii)** $\text{trace}(\mathbf{A}) = \text{trace}(\mathbf{A}^T)$
 ★ **(c)** Suppose that $\text{trace}(\mathbf{A}) = \text{trace}(\mathbf{B})$ for two $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$. Does $\mathbf{A} = \mathbf{B}$? Prove your answer.

★ **14.** True or False:
 **(a)** A $5 \times 6$ matrix has exactly 6 entries on its main diagonal.
 **(b)** The transpose of a lower triangular matrix is upper triangular.
 **(c)** No skew-symmetric matrix is diagonal.
 **(d)** If $\mathbf{V}$ is a skew-symmetric matrix, then $-\mathbf{V}^T = \mathbf{V}$.
 **(e)** For all scalars $c$, and $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$, $(c(\mathbf{A}^T + \mathbf{B}))^T = c\mathbf{B}^T + c\mathbf{A}$.

## 1.5 Matrix Multiplication

In this section we introduce another useful operation, matrix multiplication, which is a generalization of the dot product of vectors.

### Definition of Matrix Multiplication

Two matrices $\mathbf{A}$ and $\mathbf{B}$ can be multiplied (in that order) if and only if the number of columns of $\mathbf{A}$ is equal to the number of rows of $\mathbf{B}$. In that case,

| Size of product $\mathbf{AB}$ = (number of rows of $\mathbf{A}$) $\times$ (number of columns of $\mathbf{B}$). |
|---|

That is, if $\mathbf{A}$ is an $m \times n$ matrix, then $\mathbf{AB}$ is defined only when the number of rows of $\mathbf{B}$ is $n$—that is, when $\mathbf{B}$ is an $n \times p$ matrix, for some integer $p$. In this case, $\mathbf{AB}$ is an $m \times p$ matrix, because $\mathbf{A}$ has $m$ rows and $\mathbf{B}$ has $p$ columns. The actual entries of $\mathbf{AB}$ are given by the following definition:

**Definition** If **A** is an $m \times n$ matrix and **B** is an $n \times p$ matrix, their matrix product $\mathbf{C} = \mathbf{AB}$ is the $m \times p$ matrix whose $(i, j)$ entry is the dot product of the $i$th row of **A** with the $j$th column of **B**. That is,

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}}_{m\times n \text{ matrix } \mathbf{A}} \underbrace{\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2p} \\ b_{31} & b_{32} & \cdots & b_{3j} & \cdots & b_{3p} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nj} & \cdots & b_{np} \end{bmatrix}}_{n\times p \text{ matrix } \mathbf{B}}$$

$$= \underbrace{\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1j} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2j} & \cdots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{i1} & c_{i2} & \cdots & c_{ij} & \cdots & c_{ip} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mj} & \cdots & c_{mp} \end{bmatrix}}_{m\times p \text{ matrix } \mathbf{C}}$$

where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$.

Since the number of columns in **A** equals the number of rows in **B** in this definition, each row of **A** contains the same number of entries as each column of **B**. Thus, it is possible to perform the dot products needed to calculate $\mathbf{C} = \mathbf{AB}$.

## Example 1

Consider

$$\mathbf{A} = \begin{bmatrix} 5 & -1 & 4 \\ -3 & 6 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 9 & 4 & -8 & 2 \\ 7 & 6 & -1 & 0 \\ -2 & 5 & 3 & -4 \end{bmatrix}.$$

Since **A** is a $2 \times 3$ matrix and **B** is a $3 \times 4$ matrix, the number of columns of **A** equals the number of rows of **B** (three in each case). Therefore, **A** and **B** can be multiplied, and the product matrix $\mathbf{C} = \mathbf{AB}$ is a $2 \times 4$ matrix, because **A** has two rows and **B** has four columns. To calculate each entry of **C**, we take the dot product of the appropriate row of **A** with the appropriate column of **B**. For example, to find $c_{11}$, we take the dot product of the 1st row of **A** with the 1st column of **B**:

$$c_{11} = [5, -1, 4] \cdot \begin{bmatrix} 9 \\ 7 \\ -2 \end{bmatrix} = (5)(9) + (-1)(7) + (4)(-2) = 45 - 7 - 8 = 30.$$

To find $c_{23}$, we take the dot product of the 2nd row of **A** with the 3rd column of **B**:

$$c_{23} = [-3, 6, 0] \cdot \begin{bmatrix} -8 \\ -1 \\ 3 \end{bmatrix} = (-3)(-8) + (6)(-1) + (0)(3) = 24 - 6 + 0 = 18.$$

The other entries are computed similarly, yielding

$$\mathbf{C} = \mathbf{AB} = \begin{bmatrix} 30 & 34 & -27 & -6 \\ 15 & 24 & 18 & -6 \end{bmatrix}.$$

**Example 2**

Consider the following five matrices:

$$\mathbf{D} = \underbrace{\begin{bmatrix} -2 & 1 \\ 0 & 5 \\ 4 & -3 \end{bmatrix}}_{3\times 2 \text{ matrix}}, \quad \mathbf{E} = \underbrace{\begin{bmatrix} 1 & -6 \\ 0 & 2 \end{bmatrix}}_{2\times 2 \text{ matrix}}, \quad \mathbf{F} = \underbrace{\begin{bmatrix} -4 & 2 & 1 \end{bmatrix}}_{1\times 3 \text{ matrix}},$$

$$\mathbf{G} = \underbrace{\begin{bmatrix} 7 \\ -1 \\ 5 \end{bmatrix}}_{3\times 1 \text{ matrix}}, \text{ and } \mathbf{H} = \underbrace{\begin{bmatrix} 5 & 0 \\ 1 & -3 \end{bmatrix}}_{2\times 2 \text{ matrix}}.$$

The only possible products of two of these matrices that are defined are

$$\mathbf{DE} = \begin{bmatrix} -2 & 14 \\ 0 & 10 \\ 4 & -30 \end{bmatrix}, \quad \mathbf{DH} = \begin{bmatrix} -9 & -3 \\ 5 & -15 \\ 17 & 9 \end{bmatrix}, \quad \mathbf{GF} = \begin{bmatrix} -28 & 14 & 7 \\ 4 & -2 & -1 \\ -20 & 10 & 5 \end{bmatrix},$$

$$\mathbf{EE} = \begin{bmatrix} 1 & -18 \\ 0 & 4 \end{bmatrix}, \quad \mathbf{EH} = \begin{bmatrix} -1 & 18 \\ 2 & -6 \end{bmatrix}, \quad \mathbf{HE} = \begin{bmatrix} 5 & -30 \\ 1 & -12 \end{bmatrix}, \quad \mathbf{HH} = \begin{bmatrix} 25 & 0 \\ 2 & 9 \end{bmatrix},$$

$$\mathbf{FG} = [-25] \ (1 \times 1 \text{ matrix}), \text{ and } \mathbf{FD} = [12 \ \ 3] \ (1 \times 2 \text{ matrix}). \quad \text{(Verify!)}$$

Example 2 points out that the order in which matrix multiplication is performed is extremely important. In fact, for two given matrices, we have seen the following:

- Neither product may be defined (for example, neither **DG** nor **GD**).
- One product may be defined but not the other. (**DE** is defined, but not **ED**.)
- Both products may be defined, but the resulting sizes may not agree. (**FG** is $1 \times 1$, but **GF** is $3 \times 3$.)
- Both products may be defined, and the resulting sizes may agree, but the entries may differ. (**EH** and **HE** are both $2 \times 2$, but have different entries.)

In unusual cases where $\mathbf{AB} = \mathbf{BA}$, we say that **A** and **B** **commute**, or that "**A commutes** with **B**." But as we have seen, there is no general commutative law for matrix multiplication, although there is a commutative law for addition.

If **A** is any $2 \times 2$ matrix, then $\mathbf{AI}_2 = \mathbf{I}_2\mathbf{A}(= \mathbf{A})$, where $\mathbf{I}_2$ is the identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. For example, if $\mathbf{A} = \begin{bmatrix} -4 & 2 \\ 5 & 6 \end{bmatrix}$, then $\begin{bmatrix} -4 & 2 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -4 & 2 \\ 5 & 6 \end{bmatrix}$. In Exercise 17, we generalize this to show that if **A** is any $m \times n$ matrix, then $\mathbf{AI}_n = \mathbf{I}_m\mathbf{A} = \mathbf{A}$. This is why **I** is called the (**multiplicative**) **identity** matrix—because it preserves the "identity" of any matrices multiplied by it. In particular, for an $n \times n$ matrix **A**, $\mathbf{AI}_n = \mathbf{I}_n\mathbf{A} = \mathbf{A}$, and so **A** commutes with $\mathbf{I}_n$.

## Application: Shipping Cost and Profit

Matrix products are vital in modeling certain geometric transformations (as we will see in Sections 5.2 and 8.8). They are also widely used in graph theory, coding theory, physics, and chemistry. Here is a simple application in business.

**Example 3**

Suppose four popular DVDs—say, W, X, Y, and Z—are being sold online by a video company that operates three different warehouses. After purchase, the shipping cost is added to the price of the DVDs when they are mailed to the customer. The number of each type of DVD shipped from each warehouse during the past week is shown in the following matrix **A**. The shipping cost and profit collected for each DVD sold is shown in matrix **B**.

**A** =

| | DVD W | DVD X | DVD Y | DVD Z |
|---|---|---|---|---|
| Warehouse 1 | 130 | 160 | 240 | 190 |
| Warehouse 2 | 210 | 180 | 320 | 240 |
| Warehouse 3 | 170 | 200 | 340 | 220 |

| | Shipping Cost | Profit |
|---|---|---|
| DVD W | \$3 | \$3 |
| DVD X | \$4 | \$2 |
| DVD Y | \$3 | \$4 |
| DVD Z | \$2 | \$2 |

$$\mathbf{B} = \begin{bmatrix} \$3 & \$3 \\ \$4 & \$2 \\ \$3 & \$4 \\ \$2 & \$2 \end{bmatrix}$$

The product **AB** represents the combined total shipping costs and profits last week.

| | Total Shipping Cost | Total Profit |
|---|---|---|
| Warehouse 1 | \$2130 | \$2050 |
| Warehouse 2 | \$2790 | \$2750 |
| Warehouse 3 | \$2770 | \$2710 |

$$\mathbf{AB} = \begin{bmatrix} \$2130 & \$2050 \\ \$2790 & \$2750 \\ \$2770 & \$2710 \end{bmatrix}$$

In particular, the entry in the 2nd row and 2nd column of **AB** is calculated by taking the dot product of the 2nd row of **A** with the 2nd column of **B**; that is,

$$(210)(\$3) + (180)(\$2) + (320)(\$4) + (240)(\$2) = \$2750.$$

In this case, we are multiplying the number of each type of DVD sold from Warehouse 2 times the profit per DVD, which equals the total profit for Warehouse 2.

Often we need to find only a particular row or column of a matrix product:

> If the product **AB** is defined, then the $k$th row of **AB** is the product ($k$th row of **A**)**B**.
> Also, the $l$th column of **AB** is the product **A**($l$th column of **B**).

Thus, in Example 3, if we only want the results for Warehouse 3, we only need to compute the 3rd row of **AB**. This is

$$\underbrace{\begin{bmatrix} 170 & 200 & 340 & 220 \end{bmatrix}}_{\text{3rd row of } \mathbf{A}} \underbrace{\begin{bmatrix} \$3 & \$3 \\ \$4 & \$2 \\ \$3 & \$4 \\ \$2 & \$2 \end{bmatrix}}_{\mathbf{B}} = \underbrace{\begin{bmatrix} \$2770 & \$2710 \end{bmatrix}}_{\text{3rd row of } \mathbf{AB}}.$$

## Linear Combinations Using Matrix Multiplication

Forming a linear combination of the rows or columns of a matrix can be done very easily using matrix multiplication, as illustrated in the following example.

### Example 4

Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 3 & -2 & 6 & 5 \\ -1 & 4 & -1 & -3 \\ 2 & -5 & 3 & -6 \end{bmatrix}.$$

In order to create a linear combination of the *rows* of **A** such as 7(1st row of **A**) − 8(2nd row of **A**) + 9(3rd row of **A**), we only need to multiply **A** on the *left* by the vector of coefficients $[7, -8, 9]$. That is,

$$\begin{aligned} &[7, -8, 9] \begin{bmatrix} 3 & -2 & 6 & 5 \\ -1 & 4 & -1 & -3 \\ 2 & -5 & 3 & -6 \end{bmatrix} \\ &= [7(3) + (-8)(-1) + 9(2), 7(-2) + (-8)(4) + 9(-5), 7(6) + (-8)(-1) + 9(3), 7(5) + (-8)(-3) + 9(-6)] \\ &= 7[3, -2, 6, 5] + (-8)[-1, 4, -1, -3] + 9[2, -5, 3, -6] = [47, -91, 77, 5] \\ &= 7(\text{1st row of } \mathbf{A}) - 8(\text{2nd row of } \mathbf{A}) + 9(\text{3rd row of } \mathbf{A}). \end{aligned}$$

Similarly, we can create a linear combination of the *columns* of **A** such as 10(1st column of **A**) − 11(2nd column of **A**) + 12(3rd column of **A**) − 13(4th column of **A**) by multiplying **A** on the *right* by the vector of coefficients [10, −11, 12, −13]. This gives

$$\begin{aligned}
&\begin{bmatrix} 3 & -2 & 6 & 5 \\ -1 & 4 & -1 & -3 \\ 2 & -5 & 3 & -6 \end{bmatrix}\begin{bmatrix} 10 \\ -11 \\ 12 \\ -13 \end{bmatrix} \\
&= \begin{bmatrix} 3(10)+(-2)(-11)+6(12)+5(-13) \\ (-1)(10)+4(-11)+(-1)(12)+(-3)(13) \\ 2(10)+(-5)(-11)+3(12)+(-6)(-13) \end{bmatrix} \\
&= 10\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} - 11\begin{bmatrix} -2 \\ 4 \\ -5 \end{bmatrix} + 12\begin{bmatrix} 6 \\ -1 \\ 3 \end{bmatrix} - 13\begin{bmatrix} 5 \\ -3 \\ -6 \end{bmatrix} = \begin{bmatrix} 59 \\ -27 \\ 189 \end{bmatrix} \\
&= 10(\text{1st column of } \mathbf{A}) - 11(\text{2nd column of } \mathbf{A}) + 12(\text{3rd column of } \mathbf{A}) - 13(\text{4th column of } \mathbf{A}).
\end{aligned}$$

## Fundamental Properties of Matrix Multiplication

The following theorem lists some other important properties of matrix multiplication:

**Theorem 1.16** *Suppose that* **A**, **B**, *and* **C** *are matrices for which the following sums and products are defined. Let c be a scalar. Then*

(1) $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$ *Associative Law of Multiplication*

(2) $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$ *Distributive Laws of Matrix Multiplication*

(3) $(\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC}+\mathbf{BC}$ *Over Addition*

(4) $c(\mathbf{AB}) = (c\mathbf{A})\mathbf{B} = \mathbf{A}(c\mathbf{B})$ *Associative Law of Scalar and Matrix Multiplication*

The proof of part (1) of Theorem 1.16 is more difficult than the others, and so it is included in Appendix A for the interested reader. You are asked to provide the proofs of parts (2), (3), and (4) in Exercise 15.

Other expected properties do not hold for matrix multiplication (such as the commutative law). For example, the **cancellation laws** of algebra do not hold in general. That is, if $\mathbf{AB} = \mathbf{AC}$, with $\mathbf{A} \neq \mathbf{O}$, it does not necessarily follow that $\mathbf{B} = \mathbf{C}$. For example, if

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -1 & 0 \\ 5 & 2 \end{bmatrix}, \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} 3 & 1 \\ -3 & 0 \end{bmatrix},$$

then

$$\mathbf{AB} = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 9 & 6 \end{bmatrix}$$

and

$$\mathbf{AC} = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 9 & 6 \end{bmatrix}.$$

Here, $\mathbf{AB} = \mathbf{AC}$, but $\mathbf{B} \neq \mathbf{C}$. Similarly, if $\mathbf{DF} = \mathbf{EF}$, for some matrices **D**, **E**, **F** with $\mathbf{F} \neq \mathbf{O}$, it does not necessarily follow that $\mathbf{D} = \mathbf{E}$.

If the zero matrix **O** is multiplied times any matrix **A**, or if **A** is multiplied times **O**, the result is **O** (see Exercise 16). But, beware! If $\mathbf{AB} = \mathbf{O}$, it is not necessarily true that $\mathbf{A} = \mathbf{O}$ or $\mathbf{B} = \mathbf{O}$. For example, if

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix},$$

$$\text{then } \mathbf{AB} = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

## Powers of Square Matrices

Any square matrix can be multiplied by itself because the number of rows is the same as the number of columns. In fact, square matrices are the only matrices that can be multiplied by themselves (why?). The various nonnegative powers of a square matrix are defined in a natural way.

**Definition** Let $\mathbf{A}$ be any $n \times n$ matrix. Then the (nonnegative) powers of $\mathbf{A}$ are given by $\mathbf{A}^0 = \mathbf{I}_n$, $\mathbf{A}^1 = \mathbf{A}$, and for $k \geq 2$, $\mathbf{A}^k = \left(\mathbf{A}^{k-1}\right)(\mathbf{A})$.

### Example 5

Suppose that $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ -4 & 3 \end{bmatrix}$. Then

$$\mathbf{A}^2 = (\mathbf{A})(\mathbf{A}) = \begin{bmatrix} 2 & 1 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ -20 & 5 \end{bmatrix}, \text{ and}$$

$$\mathbf{A}^3 = (\mathbf{A}^2)(\mathbf{A}) = \begin{bmatrix} 0 & 5 \\ -20 & 5 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} -20 & 15 \\ -60 & -5 \end{bmatrix}.$$

### Example 6

The identity matrix $\mathbf{I}_n$ is square, and so $\mathbf{I}_n^k$ exists, for all $k \geq 0$. However, since $\mathbf{I}_n\mathbf{A} = \mathbf{A}$, for any $n \times n$ matrix $\mathbf{A}$, we have $\mathbf{I}_n\mathbf{I}_n = \mathbf{I}_n$. Thus, $\mathbf{I}_n^k = \mathbf{I}_n$, for all $k \geq 0$.

The next theorem asserts that two familiar laws of exponents in algebra are still valid for powers of a square matrix. The proof is left as Exercise 20.

**Theorem 1.17** *If $\mathbf{A}$ is a square matrix, and if $s$ and $t$ are nonnegative integers, then*

(1) $\mathbf{A}^{s+t} = (\mathbf{A}^s)(\mathbf{A}^t)$

(2) $(\mathbf{A}^s)^t = \mathbf{A}^{st} = (\mathbf{A}^t)^s$.

As an example of part (1) of this theorem, we have $\mathbf{A}^{4+6} = (\mathbf{A}^4)(\mathbf{A}^6) = \mathbf{A}^{10}$. As an example of part (2), we have $(\mathbf{A}^3)^2 = \mathbf{A}^{(3)(2)} = (\mathbf{A}^2)^3 = \mathbf{A}^6$.

One law of exponents in elementary algebra that does not carry over to matrix algebra is $(xy)^q = x^q y^q$. In fact, if $\mathbf{A}$ and $\mathbf{B}$ are square matrices of the same size, usually $(\mathbf{AB})^q \neq \mathbf{A}^q\mathbf{B}^q$, if $q$ is an integer $\geq 2$. Even in the simplest case $q = 2$, usually $(\mathbf{AB})(\mathbf{AB}) \neq (\mathbf{AA})(\mathbf{BB})$ because the *order* of matrix multiplication is important.

### Example 7

Let

$$\mathbf{A} = \begin{bmatrix} 2 & -4 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 3 & 2 \\ -1 & 5 \end{bmatrix}.$$

Then

$$(\mathbf{AB})^2 = \begin{bmatrix} 10 & -16 \\ 0 & 17 \end{bmatrix}^2 = \begin{bmatrix} 100 & -432 \\ 0 & 289 \end{bmatrix}.$$

However,

$$\mathbf{A}^2\mathbf{B}^2 = \begin{bmatrix} 0 & -20 \\ 5 & 5 \end{bmatrix}\begin{bmatrix} 7 & 16 \\ -8 & 23 \end{bmatrix} = \begin{bmatrix} 160 & -460 \\ -5 & 195 \end{bmatrix}.$$

Hence, in this particular case, $(\mathbf{AB})^2 \neq \mathbf{A}^2\mathbf{B}^2$.

## The Transpose of a Matrix Product

**Theorem 1.18** *If* $\mathbf{A}$ *is an* $m \times n$ *matrix and* $\mathbf{B}$ *is an* $n \times p$ *matrix, then* $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.

This result may seem unusual at first because you might expect $(\mathbf{AB})^T$ to equal $\mathbf{A}^T\mathbf{B}^T$. But notice that $\mathbf{A}^T\mathbf{B}^T$ may not be defined, because $\mathbf{A}^T$ is an $n \times m$ matrix and $\mathbf{B}^T$ is a $p \times n$ matrix. Instead, the transpose of the product of two matrices is the product of their transposes *in reverse order*.

*Proof.* Because $\mathbf{AB}$ is an $m \times p$ matrix and $\mathbf{B}^T$ is a $p \times n$ matrix and $\mathbf{A}^T$ is an $n \times m$ matrix, it follows that $(\mathbf{AB})^T$ and $\mathbf{B}^T\mathbf{A}^T$ are both $p \times m$ matrices. Hence, we only need to show the $(i, j)$ entries of $(\mathbf{AB})^T$ and $\mathbf{B}^T\mathbf{A}^T$ are equal, for $1 \le i \le p$ and $1 \le j \le m$. Now,

$$\begin{aligned}(i, j) \text{ entry of } (\mathbf{AB})^T &= (j, i) \text{ entry of } \mathbf{AB} \\ &= [j\text{th row of } \mathbf{A}] \cdot [i\text{th column of } \mathbf{B}].\end{aligned}$$

However,

$$\begin{aligned}(i, j) \text{ entry of } \mathbf{B}^T\mathbf{A}^T &= [i\text{th row of } \mathbf{B}^T] \cdot [j\text{th column of } \mathbf{A}^T] \\ &= [i\text{th column of } \mathbf{B}] \cdot [j\text{th row of } \mathbf{A}] \\ &= [j\text{th row of } \mathbf{A}] \cdot [i\text{th column of } \mathbf{B}],\end{aligned}$$

by part (1) of Theorem 1.5. Thus, the $(i, j)$ entries of $(\mathbf{AB})^T$ and $\mathbf{B}^T\mathbf{A}^T$ agree. □

### Example 8

For the matrices $\mathbf{A}$ and $\mathbf{B}$ of Example 7, we have

$$\mathbf{AB} = \begin{bmatrix} 10 & -16 \\ 0 & 17 \end{bmatrix}, \quad \mathbf{B}^T = \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}, \quad \text{and} \quad \mathbf{A}^T = \begin{bmatrix} 2 & 1 \\ -4 & 3 \end{bmatrix}.$$

Hence

$$\mathbf{B}^T\mathbf{A}^T = \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ -16 & 17 \end{bmatrix} = (\mathbf{AB})^T.$$

Notice, however, that

$$\mathbf{A}^T\mathbf{B}^T = \begin{bmatrix} 2 & 1 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 8 & 3 \\ -6 & 19 \end{bmatrix} \neq (\mathbf{AB})^T.$$

- ♦ **Supplemental Material:** You have now covered the prerequisites for Section 7.1, "Complex $n$-Vectors and Matrices."
- ♦ **Application**: You have now covered the prerequisites for Section 8.1, "Graph Theory."

## New Vocabulary

commuting matrices
idempotent matrix
identity matrix for multiplication
multiplication of matrices
power of a square matrix

## Highlights

- Two matrices $\mathbf{A}$ and $\mathbf{B}$ can be multiplied if and only if the number of columns of $\mathbf{A}$ is equal to the number of rows of $\mathbf{B}$, and in that case, the product $\mathbf{AB}$ has the same number of rows as $\mathbf{A}$ and the same number of columns as $\mathbf{B}$, and the $(i, j)$ entry of $\mathbf{AB}$ equals ($i$th row of $\mathbf{A}$) · ($j$th column of $\mathbf{B}$).
- Assuming that both $\mathbf{AB}$ and $\mathbf{BA}$ are defined, $\mathbf{AB}$ is not necessarily equal to $\mathbf{BA}$. That is, in matrix multiplication, the *order* of the matrices is important.

- The $k$th row of **AB** is equal to ($k$th row of **A**)**B**, and the $l$th column of **AB** is equal to **A**($l$th column of **B**).
- If **A** is an $m \times n$ matrix, **B** is a $1 \times m$ matrix, and **C** is an $n \times 1$ matrix, then **BA** gives a linear combination of the rows of **A**, and **AC** gives a linear combination of the columns of **A**.
- The associative and distributive laws hold for matrix multiplication, but *not* the commutative law.
- The cancellation laws do not generally hold for matrix multiplication. That is, either $\mathbf{AB} = \mathbf{AC}$ or $\mathbf{BA} = \mathbf{CA}$ does not necessarily imply $\mathbf{B} = \mathbf{C}$.
- If the zero matrix **O** is multiplied times any matrix **A**, or if **A** is multiplied times **O**, the result is **O**. But, if $\mathbf{AB} = \mathbf{O}$, it does not necessarily follow that $\mathbf{A} = \mathbf{O}$ or $\mathbf{B} = \mathbf{O}$.
- The usual laws of exponents hold for powers of square matrices, except that a power of a matrix product is usually not equal to the product of the individual powers of the matrices; that is, in general, $(\mathbf{AB})^q \neq \mathbf{A}^q\mathbf{B}^q$. In particular, $\mathbf{ABAB} = (\mathbf{AB})^2 \neq \mathbf{A}^2\mathbf{B}^2 = \mathbf{AABB}$.
- If **A** is an $m \times n$ matrix and **B** is an $n \times p$ matrix, then $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$. That is, the transpose of a matrix product is found by multiplying the transposes of the matrices in *reverse* order.

## Exercises for Section 1.5

**Note**: Exercises 1 through 3 refer to the following matrices:

$$\mathbf{A} = \begin{bmatrix} -2 & 3 \\ 6 & 5 \\ 1 & -4 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} -5 & 3 & 6 \\ 3 & 8 & 0 \\ -2 & 0 & 4 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 11 & -2 \\ -4 & -2 \\ 3 & -1 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} -1 & 4 & 3 & 7 \\ 2 & 1 & 7 & 5 \\ 0 & 5 & 5 & -2 \end{bmatrix} \quad \mathbf{E} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \quad \mathbf{F} = \begin{bmatrix} 9 & -3 \\ 5 & -4 \\ 2 & 0 \\ 8 & -3 \end{bmatrix}$$

$$\mathbf{G} = \begin{bmatrix} 5 & 1 & 0 \\ 0 & -2 & -1 \\ 1 & 0 & 3 \end{bmatrix} \quad \mathbf{H} = \begin{bmatrix} 6 & 3 & 1 \\ 1 & -15 & -5 \\ -2 & -1 & 10 \end{bmatrix} \quad \mathbf{J} = \begin{bmatrix} 8 \\ -1 \\ 4 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} 2 & 1 & -5 \\ 0 & 2 & 7 \end{bmatrix} \quad \mathbf{L} = \begin{bmatrix} 10 & 9 \\ 8 & 7 \end{bmatrix} \quad \mathbf{M} = \begin{bmatrix} 7 & -1 \\ 11 & 3 \end{bmatrix}$$

$$\mathbf{N} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \mathbf{P} = \begin{bmatrix} 3 & -1 \\ 4 & 7 \end{bmatrix} \quad \mathbf{Q} = \begin{bmatrix} 1 & 4 & -1 & 6 \\ 8 & 7 & -3 & 3 \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} -3 & 6 & -2 \end{bmatrix} \quad \mathbf{S} = \begin{bmatrix} 6 & -4 & 3 & 2 \end{bmatrix} \quad \mathbf{T} = \begin{bmatrix} 4 & -1 & 7 \end{bmatrix}$$

1. Which of these products are possible? If possible, then calculate the product.
   - (a) **AB**
   - ★ (b) **BA**
   - ★ (c) **JM**
   - (d) **GD**
   - ★ (e) **RJ**
   - ★ (f) **JR**
   - ★ (g) **RT**
   - (h) $\mathbf{SD}^T$
   - (i) **KL**
   - ★ (j) $\mathbf{F}^2$
   - (k) $\mathbf{G}^2$
   - ★ (l) $\mathbf{E}^3$
   - (m) $(\mathbf{TJ})^4$
   - ★ (n) $\mathbf{D}(\mathbf{FK})$
   - (o) $(\mathbf{LQ})\mathbf{E}$
2. Determine whether these pairs of matrices commute.
   - ★ (a) **L** and **M**
   - (b) **G** and **H**
   - ★ (c) **A** and **K**
   - ★ (d) **N** and **P**
   - (e) **F** and **Q**

3. Find only the indicated row or column of each given matrix product.

★ (a) The 2nd row of **BG**

(b) The 4th column of **DE**

★ (c) The 1st column of **SE**

(d) The 1st row of **QF**

★ 4. Assuming that all of the following products exist, which of these equations are always valid? If valid, specify which theorems (and parts, if appropriate) apply.

(a) $(\mathbf{RG})\mathbf{H} = \mathbf{R}(\mathbf{GH})$

(b) $\mathbf{LP} = \mathbf{PL}$

(c) $\mathbf{E}(\mathbf{FK}) = (\mathbf{EF})\mathbf{K}$

(d) $\mathbf{K}(\mathbf{A} + \mathbf{C}) = \mathbf{KA} + \mathbf{KC}$

(e) $(\mathbf{QF})^T = \mathbf{F}^T\mathbf{Q}^T$

(f) $\mathbf{L}(\mathbf{ML}) = \mathbf{L}^2\mathbf{M}$

(g) $\mathbf{GC} + \mathbf{HC} = (\mathbf{G} + \mathbf{H})\mathbf{C}$

(h) $\mathbf{R}(\mathbf{J} + \mathbf{T}^T) = \mathbf{RJ} + \mathbf{RT}^T$

(i) $(\mathbf{AK})^T = \mathbf{A}^T\mathbf{K}^T$

(j) $(\mathbf{Q} + \mathbf{F}^T)\mathbf{E}^T = \mathbf{QE}^T + (\mathbf{EF})^T$

★ 5. The following matrices detail the number of employees at four different retail outlets and their wages and benefits (per year). Calculate the total salaries and fringe benefits paid by each outlet per year to its employees.

| | Executives | Salespersons | Others |
|---|---|---|---|
| Outlet 1 | 3 | 7 | 8 |
| Outlet 2 | 2 | 4 | 5 |
| Outlet 3 | 6 | 14 | 18 |
| Outlet 4 | 3 | 6 | 9 |

| | Salary | Fringe Benefits |
|---|---|---|
| Executives | \$30000 | \$7500 |
| Salespersons | \$22500 | \$4500 |
| Others | \$15000 | \$3000 |

6. The following matrices detail the typical amount spent on tickets, food, and souvenirs at a Summer Festival by a person from each of four age groups, and the total attendance by these different age groups during each month of the festival. Calculate the total amount spent on tickets, food, and souvenirs each month.

| | Tickets | Food | Souvenirs |
|---|---|---|---|
| Children | \$8 | \$7 | \$10 |
| Teens | \$10 | \$18 | \$8 |
| Adults | \$12 | \$16 | \$3 |
| Seniors | \$9 | \$12 | \$3 |

| | Children | Teens | Adults | Seniors |
|---|---|---|---|---|
| June Attendance | 32300 | 54600 | 121500 | 46400 |
| July Attendance | 37400 | 62700 | 136000 | 52900 |
| August Attendance | 29800 | 48500 | 98200 | 44100 |

★ 7. Matrix **A** gives the percentage of nitrogen, phosphates, and potash in three fertilizers. Matrix **B** gives the amount (in tons) of each type of fertilizer spread on three different fields. Use matrix operations to find the total amount of nitrogen, phosphates, and potash on each field.

**A** =

| | Nitrogen | Phosphates | Potash |
|---|---|---|---|
| Fertilizer 1 | 10% | 10% | 5% |
| Fertilizer 2 | 25% | 5% | 5% |
| Fertilizer 3 | 0% | 10% | 20% |

**B** =

| | Field 1 | Field 2 | Field 3 |
|---|---|---|---|
| Fertilizer 1 | 5 | 2 | 4 |
| Fertilizer 2 | 2 | 1 | 1 |
| Fertilizer 3 | 3 | 1 | 3 |

**8.** Matrix **A** gives the numbers of four different types of computer modules that are needed to assemble various rockets. Matrix **B** gives the amounts of four different types of computer chips that compose each module. Use matrix operations to find the total amount of each type of computer chip needed for each rocket.

$\mathbf{A} =$

| | Module A | Module B | Module C | Module D |
|---|---|---|---|---|
| Rocket 1 | 25 | 10 | 5 | 19 |
| Rocket 2 | 24 | 8 | 7 | 16 |
| Rocket 3 | 32 | 11 | 8 | 22 |
| Rocket 4 | 27 | 12 | 6 | 17 |

$\mathbf{B} =$

| | Module A | Module B | Module C | Module D |
|---|---|---|---|---|
| Chip 1 | 42 | 39 | 52 | 28 |
| Chip 2 | 25 | 23 | 48 | 31 |
| Chip 3 | 37 | 33 | 29 | 26 |
| Chip 4 | 52 | 44 | 35 | 54 |

★ **9.** This exercise asks for "roots" of identity matrices.
- **(a)** Find a nondiagonal matrix **A** such that $\mathbf{A}^2 = \mathbf{I}_2$.
- **(b)** Find a nondiagonal matrix **A** such that $\mathbf{A}^2 = \mathbf{I}_3$. (Hint: Modify your answer to part (a).)
- **(c)** Find a nonidentity matrix **A** such that $\mathbf{A}^3 = \mathbf{I}_3$.

**10.** Let **A** be an $m \times n$ matrix, and let **B** be an $n \times m$ matrix, with $m, n \geq 5$. Each of the following sums represents an entry of either **AB** or **BA**. Determine which product is involved and which entry of that product is represented.
- ★ **(a)** $\Sigma_{k=1}^{n} a_{3k} b_{k4}$
- **(b)** $\Sigma_{q=1}^{n} a_{4q} b_{q1}$
- ★ **(c)** $\Sigma_{k=1}^{m} a_{k2} b_{3k}$
- **(d)** $\Sigma_{q=1}^{m} b_{2q} a_{q5}$

**11.** Let **A** be an $m \times n$ matrix, and let **B** be an $n \times m$ matrix, where $m, n \geq 4$. Use sigma ($\boldsymbol{\Sigma}$) notation to express the following entries symbolically:
- ★ **(a)** The entry in the 3rd row and 2nd column of **AB**
- **(b)** The entry in the 4th row and 1st column of **BA**

★ **12.** For the matrix $\mathbf{A} = \begin{bmatrix} 4 & 7 & -2 \\ -3 & -6 & 5 \\ -9 & 2 & -8 \end{bmatrix}$, use matrix multiplication (as in Example 4) to find the following linear combinations:
- **(a)** $3\mathbf{v}_1 - 2\mathbf{v}_2 + 5\mathbf{v}_3$, where $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are the rows of **A**
- **(b)** $2\mathbf{w}_1 + 6\mathbf{w}_2 - 3\mathbf{w}_3$, where $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ are the columns of **A**

**13.** For the matrix $\mathbf{A} = \begin{bmatrix} 7 & -3 & -4 & 1 \\ -5 & 8 & 2 & -3 \\ -1 & 9 & 3 & -7 \end{bmatrix}$, use matrix multiplication (as in Example 4) to find the following linear combinations:
- **(a)** $-5\mathbf{v}_1 + 7\mathbf{v}_2 - 4\mathbf{v}_3$, where $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are the rows of **A**
- **(b)** $4\mathbf{w}_1 - 6\mathbf{w}_2 - 2\mathbf{w}_3 + 3\mathbf{w}_4$, where $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4$ are the columns of **A**

**14.**
- **(a)** Consider the unit vectors **i**, **j**, and **k** in $\mathbb{R}^3$. Show that, if **A** is an $m \times 3$ matrix, then $\mathbf{Ai} =$ first column of **A**, $\mathbf{Aj} =$ second column of **A**, and $\mathbf{Ak} =$ third column of **A**.
- **(b)** Generalize part (a) to a similar result involving an $m \times n$ matrix **A** and the standard unit vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ in $\mathbb{R}^n$.
- **(c)** Let **A** be an $m \times n$ matrix. Use part (b) to show that, if $\mathbf{Ax} = \mathbf{0}$ for all vectors $\mathbf{x} \in \mathbb{R}^n$, then $\mathbf{A} = \mathbf{O}_{mn}$.

▶ **15.** This exercise asks for proofs of various parts of Theorem 1.16.
- **(a)** Prove part (2) of Theorem 1.16.
- **(b)** Prove part (3) of Theorem 1.16.
- **(c)** Prove part (4) of Theorem 1.16.

**16.** Let **A** be an $m \times n$ matrix. Prove $\mathbf{AO}_{np} = \mathbf{O}_{mp}$.

**17.** Let **A** be an $m \times n$ matrix. Prove $\mathbf{AI}_n = \mathbf{I}_m\mathbf{A} = \mathbf{A}$.

**18.** This exercise involves products of certain types of matrices. (The proof for part (c) involves the technique discussed in Section 1.3 of reducing to a previous result.)
- **(a)** Prove that the product of two diagonal matrices is diagonal. (Hint: If $\mathbf{C} = \mathbf{AB}$ where **A** and **B** are diagonal, show that $c_{ij} = 0$ when $i \neq j$.)

(b) Prove that the product of two upper triangular matrices is upper triangular. (Hint: Let **A** and **B** be upper triangular and $\mathbf{C} = \mathbf{AB}$. Show $c_{ij} = 0$ when $i > j$ by checking that all terms $a_{ik}b_{kj}$ in the formula for $c_{ij}$ have at least one zero factor. Consider the following two cases: $i > k$ and $i \leq k$.)

(c) Prove that the product of two lower triangular matrices is lower triangular. (Hint: Use Theorem 1.18 and part (b) of this exercise.)

**19.** Suppose that $c \in \mathbb{R}$, **A** is a square matrix, and $n \geq 1$ is an integer. In each part, use a proof by induction to establish the given equation.

(a) $(c\mathbf{A})^n = c^n\mathbf{A}^n$ (b) $(\mathbf{A}^T)^n = (\mathbf{A}^n)^T$

▶ **20.** This exercise asks for proofs of various parts of Theorem 1.17.

(a) Prove part (1) of Theorem 1.17. (Hint: Use induction on $t$.)

(b) Prove part (2) of Theorem 1.17. (Hint: Use induction on $t$, along with part (1).)

**21.** Use Theorem 1.17 to prove that if **A** is a square matrix, then $\mathbf{A}^k$ commutes with $\mathbf{A}^l$, for any nonnegative integers $k$ and $l$.

**22.** Suppose **A** is an $n \times n$ matrix. (The proofs for parts (b) and (c) involve the technique discussed in Section 1.3 of reducing to a previous result.)

(a) If **A** commutes with each of the $n \times n$ matrices $\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_k$, show that **A** commutes with $c_1\mathbf{B}_1 + c_2\mathbf{B}_2 + \cdots + c_k\mathbf{B}_k$, for any scalars $c_1, c_2, \ldots, c_k$.

(b) Use part (a) and Exercise 21 to prove that $\mathbf{A}^j$ commutes with $c_o\mathbf{I}_n + c_1\mathbf{A} + c_2\mathbf{A}^2 + \cdots + c_k\mathbf{A}^k$, for any nonnegative integer $j$, and any scalars $c_o, c_1, c_2, \ldots, c_k$.

(c) Use parts (a) and (b) to prove that $(c_o\mathbf{I}_n + c_1\mathbf{A} + c_2\mathbf{A}^2 + \cdots + c_k\mathbf{A}^k)$ commutes with $(d_o\mathbf{I}_n + d_1\mathbf{A} + d_2\mathbf{A}^2 + \cdots + d_m\mathbf{A}^m)$, for any scalars $c_o, c_1, c_2, \ldots, c_k$ and $d_o, d_1, d_2, \ldots, d_m$.

**23.** This exercise gives conditions for certain matrix products to be commutative.

(a) Show $\mathbf{AB} = \mathbf{BA}$ only if **A** and **B** are square matrices of the same size.

(b) Prove two square matrices **A** and **B** of the same size commute if and only if $(\mathbf{A}+\mathbf{B})^2 = \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$.

**24.** If **A**, **B**, and **C** are all square matrices of the same size, show that **AB** commutes with **C** if **A** and **B** both commute with **C**.

**25.** Show that **A** and **B** commute if and only if $\mathbf{A}^T$ and $\mathbf{B}^T$ commute.

**26.** Let **A** be any matrix. Show that $\mathbf{AA}^T$ and $\mathbf{A}^T\mathbf{A}$ are both symmetric.

**27.** Let **A** and **B** both be $n \times n$ matrices.

(a) Show that $(\mathbf{AB})^T = \mathbf{BA}$ if **A** and **B** are both symmetric or both skew-symmetric.

(b) If **A** and **B** are both symmetric, show that **AB** is symmetric if and only if **A** and **B** commute.

**28.** Recall the definition of the **trace** of a matrix given in Exercise 13 of Section 1.4.

(a) If **A** is any matrix, prove that trace($\mathbf{AA}^T$) is the sum of the squares of all entries of **A**. (Note that for any matrix **A**, $\mathbf{AA}^T$ is a square matrix, so that its trace can be computed.) (Hint: Describe the $(i,i)$ entry of $\mathbf{AA}^T$.)

(b) Show that for every matrix **A**, if trace($\mathbf{AA}^T$) $= 0$, then $\mathbf{A} = \mathbf{O}_n$. (Hint: Use part (a) of this exercise.)

(c) If **A** and **B** are both $n \times n$ matrices, prove that trace(**AB**) = trace(**BA**). (Hint: Calculate trace(**AB**) and trace(**BA**) in the $3 \times 3$ case to discover how to prove the general $n \times n$ case.)

**29.** An **idempotent** matrix is a square matrix **A** for which $\mathbf{A}^2 = \mathbf{A}$. (Note that if **A** is idempotent, then $\mathbf{A}^n = \mathbf{A}$ for every integer $n \geq 1$.)

★ (a) Find a $2 \times 2$ idempotent matrix other than $\mathbf{I}_n$ and $\mathbf{O}_n$.

(b) Show $\begin{bmatrix} -1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$ is idempotent.

(c) If **A** is an $n \times n$ idempotent matrix, show $\mathbf{I}_n - \mathbf{A}$ is also idempotent.

(d) Use parts (b) and (c) to get another example of an idempotent matrix.

(e) Let **A** and **B** be $n \times n$ matrices. Show that **A** is idempotent if both $\mathbf{AB} = \mathbf{A}$ and $\mathbf{BA} = \mathbf{B}$.

**30.** This exercise concerns matrix products that equal the zero matrix.

(a) Let **A** be an $m \times n$ matrix, and let **B** be an $n \times p$ matrix. Prove that $\mathbf{AB} = \mathbf{O}_{mp}$ if and only if every (vector) row of **A** is orthogonal to each column of **B**.

★ (b) Find a $2 \times 3$ matrix $\mathbf{A} \neq \mathbf{O}$ and a $3 \times 2$ matrix $\mathbf{B} \neq \mathbf{O}$ such that $\mathbf{AB} = \mathbf{O}_2$.

(c) Using your answers from part (b), find a matrix $\mathbf{C} \neq \mathbf{B}$ such that $\mathbf{AB} = \mathbf{AC}$.

★ **31.** What form does a $2 \times 2$ matrix have if it commutes with every other $2 \times 2$ matrix? Prove that your answer is correct.

★ **32.** True or False:

**(a)** If $\mathbf{AB}$ is defined, the $j$th column of $\mathbf{AB} = \mathbf{A}(j\text{th column of } \mathbf{B})$.
**(b)** If $\mathbf{A}, \mathbf{B}, \mathbf{D}$ are $n \times n$ matrices, then $\mathbf{D}(\mathbf{A}+\mathbf{B}) = \mathbf{DB}+\mathbf{DA}$.
**(c)** If $t$ is a scalar, and $\mathbf{D}$ and $\mathbf{E}$ are $n \times n$ matrices, then $(t\mathbf{D})\mathbf{E} = \mathbf{D}(t\mathbf{E})$.
**(d)** If $\mathbf{D}, \mathbf{E}$ are $n \times n$ matrices, then $(\mathbf{DE})^2 = \mathbf{D}^2\mathbf{E}^2$.
**(e)** If $\mathbf{D}, \mathbf{E}$ are $n \times n$ matrices, then $(\mathbf{DE})^T = \mathbf{D}^T\mathbf{E}^T$.
**(f)** If $\mathbf{DE} = \mathbf{O}$, then $\mathbf{D} = \mathbf{O}$ or $\mathbf{E} = \mathbf{O}$.
**(g)** For any square matrix $\mathbf{A}$, $\mathbf{A}$ commutes with $\mathbf{A}^T$.

## Review Exercises for Chapter 1

**1.** Determine whether the quadrilateral $ABCD$ formed by the points $A(5,7)$, $B(10,10)$, $C(4,20)$, $D(-1,17)$ is a rectangle.

★ **2.** Find a unit vector $\mathbf{u}$ in the same direction as $\mathbf{x} = \left[\frac{1}{4}, -\frac{3}{5}, \frac{3}{4}\right]$. Is $\mathbf{u}$ shorter or longer than $\mathbf{x}$?

**3.** A motorized glider is attempting to travel 9 mi/hr southeast but the wind is pulling the glider 7 mi/hr west. What is the net velocity of the glider? What is its resultant speed?

★ **4.** Find the acceleration vector on a 7 kg object when the forces $\mathbf{f}_1$ and $\mathbf{f}_2$ are simultaneously applied, if $\mathbf{f}_1$ is a force of 133 newtons in the direction of the vector $[6, 17, -6]$ and $\mathbf{f}_2$ is a force of 168 newtons in the direction of the vector $[-8, -4, 8]$.

**5.** Verify that the Cauchy-Schwarz Inequality holds for the vectors $\mathbf{x} = [-9, -5, 3]$ and $\mathbf{y} = [7, -1, 2]$.

★ **6.** Find the angle (to the nearest degree) between $\mathbf{x} = [-4, 7, -6]$ and $\mathbf{y} = [8, -1, 5]$.

**7.** For the vectors $\mathbf{a} = [4, -7, 5, -3, 1]$ and $\mathbf{b} = [8, 1, -2, -6, 9]$, find $\mathbf{proj_a b}$ and verify that $\mathbf{b} - \mathbf{proj_a b}$ is orthogonal to $\mathbf{a}$.

**8.** Suppose $\mathbf{x}$ and $\mathbf{y}$ are nonzero vectors in $\mathbb{R}^n$, and let $\mathbf{z} = \mathbf{proj_y x}$. Show that $\mathbf{proj_x z} = \left(\cos^2\theta\right)\mathbf{x}$, where $\theta$ is the angle between $\mathbf{x}$ and $\mathbf{y}$.

★ **9.** Find the work (in joules) performed by a force of 34 newtons acting in the direction of the vector $[15, -8]$ that displaces an object 75 m in the direction of the vector $[-7, 24]$.

**10.** Use a proof by contrapositive to show that if $(\mathbf{x}+\mathbf{y})\cdot(\mathbf{x}-\mathbf{y}) \neq 0$, then $||\mathbf{x}|| \neq ||\mathbf{y}||$.

★ **11.** Suppose $\mathbf{y} \neq \mathbf{proj_x y}$. Use a proof by contradiction to show that $\mathbf{x}$ is not parallel to $\mathbf{y}$.

★ **12.** Let $\mathbf{A} = \begin{bmatrix} 5 & -2 & -1 \\ 3 & -1 & 4 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 2 & -3 & -1 \\ -4 & 5 & -2 \\ 3 & -4 & 3 \end{bmatrix}$, and $\mathbf{C} = \begin{bmatrix} 3 & 5 \\ -2 & 4 \\ -4 & 3 \end{bmatrix}$.

**(a)** Find, if possible: $3\mathbf{A} - 4\mathbf{C}^T$, $\mathbf{AB}$, $\mathbf{BA}$, $\mathbf{AC}$, $\mathbf{CA}$, $\mathbf{A}^3$, $\mathbf{B}^3$.
**(b)** Find (only) the third row of $\mathbf{BC}$.

**13.** Express $\begin{bmatrix} 8 & -2 & 3 \\ 5 & 7 & -4 \\ 6 & -9 & 1 \end{bmatrix}$ as the sum of a symmetric matrix $\mathbf{S}$ and a skew-symmetric matrix $\mathbf{V}$.

★ **14.** If $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ skew-symmetric, prove that $3(\mathbf{A}-\mathbf{B})^T$ is skew-symmetric.

**15.** If $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ lower triangular, prove $\mathbf{A}+\mathbf{B}$ is lower triangular.

★ **16.** The following matrices detail the price and shipping cost (per pound) for steel and iron, as well as the amount (in pounds) of each used by three different companies. Calculate the total price and shipping cost incurred by each company.

| | Steel | Iron |
|---|---|---|
| **Price (per lb)** | \$20 | \$15 |
| **Shipping Cost (per lb)** | \$3 | \$2 |

| | Steel (lb) | Iron (lb) |
|---|---|---|
| **Company I** | 5200 | 4300 |
| **Company II** | 6300 | 5100 |
| **Company III** | 4600 | 4200 |

★ **17.** Prove: If $\mathbf{A}^T\mathbf{B}^T = \mathbf{B}^T\mathbf{A}^T$, then $(\mathbf{AB})^2 = \mathbf{A}^2\mathbf{B}^2$.

**18.** State and disprove the negation of the following statement: For some square matrix $\mathbf{A}$, $\mathbf{A}^2 \neq \mathbf{A}$.

★ **19.** Prove: If $\mathbf{A}$ is a nonzero $2 \times 2$ matrix, then $\mathbf{A}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ or $\mathbf{A}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

**20.** Prove by induction: The product of $k$ upper triangular matrices is upper triangular for $k \geq 2$.

★ **21.** True or False:

**(a)** There exist a nonzero scalar $c$ and a nonzero matrix $\mathbf{A} \in \mathcal{M}_{mn}$ such that $c\mathbf{A} = \mathbf{O}_{mn}$.

**(b)** Every nonzero vector in $\mathbb{R}^n$ is parallel to a unit vector in $\mathbb{R}^n$.

**(c)** Every linear combination of $[1, 4, 3]$ and $[2, 5, 4]$ has all nonnegative entries.

**(d)** The angle between $[1, 0]$ and $[0, -1]$ in $\mathbb{R}^2$ is $\frac{3\pi}{2}$.

**(e)** For $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$, if $\mathbf{proj_x y} \neq \mathbf{0}$, then $\mathbf{proj_x y}$ is in the same direction as $\mathbf{x}$.

**(f)** For all $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ in $\mathbb{R}^n$, $\|\mathbf{x} + \mathbf{y} + \mathbf{z}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\| + \|\mathbf{z}\|$.

**(g)** The negation of "$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a mutually orthogonal set of vectors" is "For every pair $\mathbf{v}_i, \mathbf{v}_j$ of vectors in $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, $\mathbf{v}_i \cdot \mathbf{v}_j \neq 0$."

**(h)** Disproving a statement involving an existential quantifier involves finding a single counterexample.

**(i)** The sum of an upper triangular matrix and a lower triangular matrix is a symmetric matrix.

**(j)** The trace of a skew-symmetric matrix must equal zero.

**(k)** $\mathcal{U}_n \cap \mathcal{L}_n = \mathcal{D}_n$.

**(l)** The transpose of a linear combination of matrices equals the corresponding linear combination of the transposes of the matrices.

**(m)** If $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is an $n \times 1$ matrix, then $\mathbf{AB}$ is an $m \times 1$ matrix representing a linear combination of the columns of $\mathbf{A}$.

**(n)** If $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{D}$ is an $n \times n$ diagonal matrix, then $\mathbf{AD}$ is an $m \times n$ matrix whose $i$th row is the $i$th row of $\mathbf{A}$ multiplied by $d_{ii}$.

**(o)** If $\mathbf{A}$ and $\mathbf{B}$ are matrices such that $\mathbf{AB}$ and $\mathbf{BA}$ are both defined, then $\mathbf{A}$ and $\mathbf{B}$ are both square.

**(p)** If $\mathbf{A}$ and $\mathbf{B}$ are square matrices of the same size, then $(\mathbf{A} + \mathbf{B})^2 = \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$.

**(q)** The product of two skew-symmetric matrices of the same size is skew-symmetric.

**(r)** If $\mathbf{A}$ is a square matrix, then $\left(\mathbf{A}^4\right)^5 = \left(\mathbf{A}^5\right)^4$.

Chapter 2

# Systems of Linear Equations

**A Systematic Approach**

One important mathematical problem that arises frequently is the need to unscramble data that have been mixed together by an apparently irreversible process. A common problem of this type is the calculation of the exact ratios of chemical elements that were combined to produce a certain compound. To solve this problem, we must unscramble the given mix of elements to determine the original ratios involved. An analogous type of problem involves the deciphering of a coded message, where in order to find the answer we must recover the original message before it was scrambled into code.

We will see that whenever information is scrambled in a "linear" fashion, a system of linear equations corresponding to the scrambling process can be constructed. Unscrambling the data is then accomplished by solving that linear system. Attempts to solve such systems of linear equations inspired much of the development of linear algebra. In this chapter, we develop a systematic method for solving linear systems, and then study some of the theoretical consequences of that technique.

## 2.1 Solving Linear Systems Using Gaussian Elimination

In this section, we introduce systems of linear equations and a method for solving such systems known as Gaussian Elimination.

### Systems of Linear Equations

A **linear equation** is an equation involving one or more variables in which only the operations of multiplication by real numbers and summing of terms are allowed. For example, $6x - 3y = 4$ and $8x_1 + 3x_2 - 4x_3 = -20$ are linear equations in two and three variables, respectively.

When several linear equations involving the same variables are considered together, we have a **system of linear equations**. For example, the following system has four equations and three variables:

$$\begin{cases} 3x_1 - 2x_2 - 5x_3 = 4 \\ 2x_1 + 4x_2 - x_3 = 2 \\ 6x_1 - 4x_2 - 10x_3 = 8 \\ -4x_1 + 8x_2 + 9x_3 = -6 \end{cases}.$$

We often need to find the solutions to a given system. The ordered triple, or 3-tuple, $(x_1, x_2, x_3) = (4, -1, 2)$ is a solution to the preceding system because each equation in the system is satisfied for these values of $x_1$, $x_2$, and $x_3$. Notice that $\left(-\frac{3}{2}, \frac{3}{4}, -2\right)$ is another solution for that same system. These two particular solutions are part of the complete set of all solutions for that system.

We now formally define linear systems and their solutions.

**Definition** A **system of** $m$ (simultaneous) **linear equations** in $n$ variables

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\ \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad \ddots \quad\quad \vdots \quad\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m \end{cases}$$

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is a collection of $m$ equations, each containing a linear combination of the same $n$ variables summing to a scalar. A **particular solution** to a system of linear equations in the variables $x_1, x_2, \ldots, x_n$ is an $n$-tuple $(s_1, s_2, \ldots, s_n)$ that satisfies each equation in the system when $s_1$ is substituted for $x_1$, $s_2$ for $x_2$, and so on. The **(complete) solution set** for a system of linear equations in $n$ variables is the set of all $n$-tuples that form particular solutions to the system.

The coefficients of $x_1, x_2, \ldots, x_n$ in this definition can be collected together in an $m \times n$ **coefficient matrix**

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

If we also let

$$\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix},$$

then the linear system is equivalent to the matrix equation $\mathbf{AX} = \mathbf{B}$ (verify!).

An alternate way to express this system is to form the **augmented matrix**

$$[\mathbf{A}|\,\mathbf{B}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right].$$

Each row of $[\mathbf{A}|\,\mathbf{B}]$ represents one equation in the original system, and each column to the left of the vertical bar represents one of the variables in the system. Hence, this augmented matrix contains all the vital information from the original system.

**Example 1**

Consider the linear system

$$\begin{cases} 4w - 2x + y - 3z = 5 \\ 3w + x \qquad + 5z = 12 \end{cases}.$$

Letting

$$\mathbf{A} = \begin{bmatrix} 4 & -2 & 1 & -3 \\ 3 & 1 & 0 & 5 \end{bmatrix}, \ \mathbf{X} = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}, \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 5 \\ 12 \end{bmatrix},$$

we see that the system is equivalent to $\mathbf{AX} = \mathbf{B}$, or,

$$\begin{bmatrix} 4 & -2 & 1 & -3 \\ 3 & 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4w - 2x + y - 3z \\ 3w + x + 5z \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}.$$

This system can also be represented by the augmented matrix

$$[\mathbf{A}|\,\mathbf{B}] = \left[\begin{array}{cccc|c} 4 & -2 & 1 & -3 & 5 \\ 3 & 1 & 0 & 5 & 12 \end{array}\right].$$

## Number of Solutions to a System

There are only three possibilities for the size of the solution set of a linear system: a single solution, an infinite number of solutions, or no solutions. There are no other possibilities because if at least two solutions exist, we can show that an infinite number of solutions must exist (see Exercise 10(c)). For instance, in a system of two equations and two variables—say, $x$ and $y$—the solution set for each equation forms a line in the $xy$-plane. The solution to the system is the intersection of the lines corresponding to each equation. But any two given lines in the plane either intersect in exactly one point (unique solution), are equal (infinite number of solutions, all points on the common line), or are parallel (no solutions).

For example, the system

$$\begin{cases} 4x_1 - 3x_2 = \ \ 0 \\ 2x_1 + 3x_2 = 18 \end{cases}$$

(where $x_1$ and $x_2$ are used instead of $x$ and $y$) has the unique solution $(3, 4)$ because that is the only intersection point of the two lines. On the other hand, the system

$$\begin{cases} 4x - 6y = 10 \\ 6x - 9y = 15 \end{cases}$$

has an infinite number of solutions because the two given lines are really the same, and so every point on one line is also on the other. Finally, the system

$$\begin{cases} 2x_1 + x_2 = 3 \\ 2x_1 + x_2 = 1 \end{cases}$$

has no solutions at all because the two lines are parallel but not equal. (Both of their slopes are $-2$.) The solution set for this system is the empty set $\{\} = \emptyset$. All three systems are pictured in Fig. 2.1.

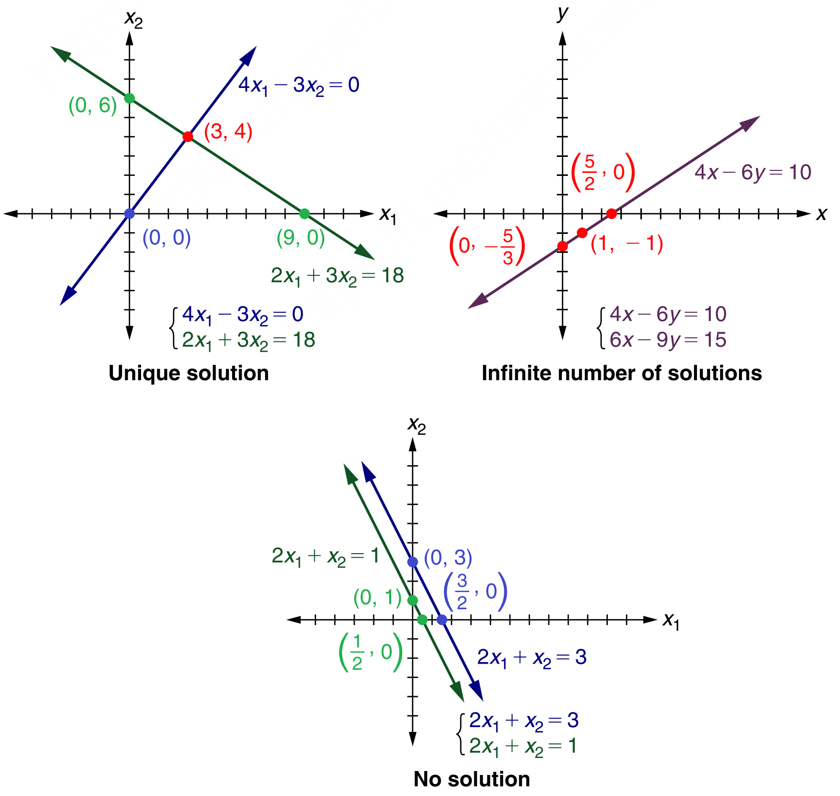


**FIGURE 2.1** Three systems: unique solution, infinite number of solutions, no solution

Any system that has at least one solution (either unique or infinitely many) is said to be **consistent**. A system whose solution set is empty is called **inconsistent**. The first two systems in Fig. 2.1 are consistent, and the last one is inconsistent.

## Gaussian Elimination

Many methods are available for finding the complete solution set for a given linear system. The first one we present, **Gaussian Elimination**, involves systematically replacing most of the coefficients in the system with simpler numbers (1's and 0's) to make the solution apparent.

In Gaussian Elimination, we begin with the augmented matrix for the given system, then examine each column in turn from left to right. In each column, if possible, we choose a special entry which is converted to "1." This converted entry is then referred to as a **pivot entry**. We then perform further operations to zero out the entries below each pivot. The pivots will be "staggered" so that as we proceed from column to column, each new pivot occurs in a lower row.

## Row Operations and Their Notation

There are three types of operations that we are allowed to use on the augmented matrix in Gaussian Elimination. These are as follows:

**Row Operations**

**(I)** Multiplying a row by a *nonzero* scalar
**(II)** Adding a scalar multiple of one row to another row
**(III)** Switching the positions of two rows in the matrix

To save space, we will use a shorthand notation for these row operations. For example, a row operation of Type (I) in which each entry of row 3 is multiplied by $\frac{1}{2}$ is represented by

$$\text{(I)}: \langle 3\rangle \leftarrow \frac{1}{2}\langle 3\rangle .$$

That is, each entry of row 3 is multiplied by $\frac{1}{2}$, and the result replaces the previous row 3. A Type (II) row operation in which $(-3) \times$ (row 4) is added to row 2 is represented by

$$\text{(II)}: \langle 2\rangle \leftarrow -3\langle 4\rangle + \langle 2\rangle .$$

That is, a multiple ($-3$, in this case) of one row (in this case, row 4) is added to row 2, and the result replaces the previous row 2. Finally, a Type (III) row operation in which the second and third rows are exchanged is represented by

$$\text{(III)}: \langle 2\rangle \leftrightarrow \langle 3\rangle .$$

(Note that a double arrow is used for Type (III) operations.)

We now illustrate the use of the first two operations with the following example:

### Example 2

Let us solve the following system of linear equations:

$$\begin{cases} 5x - 5y - 15z = 40 \\ 4x - 2y - \phantom{1}6z = 19 \\ 3x - 6y - 17z = 41 \end{cases}.$$

The augmented matrix associated with this system is

$$\left[\begin{array}{rrr|r} 5 & -5 & -15 & 40 \\ 4 & -2 & -6 & 19 \\ 3 & -6 & -17 & 41 \end{array}\right].$$

We now perform row operations on this matrix to give it a simpler form, proceeding through the columns from left to right.

**First column:** We want to place 1 in the $(1, 1)$ position—that is, the $(1, 1)$ entry will become the first pivot. The row containing the current pivot position (in this case, row 1) is usually referred to as the **pivot row**. We always use a Type (I) operation to convert a nonzero number in the pivot position to 1. This is done by multiplying the pivot row by the *reciprocal* of the value currently in that position. In this particular case, we multiply each entry of the first row by $\frac{1}{5}$.

| Row Operation | Resulting Matrix |
|---|---|
| (I): $\langle 1\rangle \leftarrow \frac{1}{5}\langle 1\rangle$ | $\left[\begin{array}{ccc\|c} 1 & -1 & -3 & 8 \\ 4 & -2 & -6 & 19 \\ 3 & -6 & -17 & 41 \end{array}\right]$ |

For reference as we proceed, we will color all pivots that are created in red.

Next we want to convert all entries below this pivot to 0. We will refer to this process as "targeting" these entries. As each such entry is changed to 0 it is called the **target**, and its row is called the **target row**. To change a target entry to 0, we always use the following Type (II) row operation:

$$\text{(II): } \langle\text{target row}\rangle \leftarrow (-\text{target value}) \times \langle\text{pivot row}\rangle + \langle\text{target row}\rangle$$

For example, to zero out (target) the $(2, 1)$ entry, we use the Type (II) operation $\langle 2\rangle \leftarrow (-4) \times \langle 1\rangle + \langle 2\rangle$. (That is, we add $(-4)$ times the pivot row to the target row.) To perform this operation, we first do the following side calculation:

| | | | | |
|---|---|---|---|---|
| $(-4)\times$ (row 1) | $-4$ | $4$ | $12$ | $-32$ |
| (row 2) | $4$ | $-2$ | $-6$ | $19$ |
| (sum) | $0$ | $2$ | $6$ | $-13$ |

The resulting sum is now substituted in place of the old row 2.

| Row Operation | Resulting Matrix |
|---|---|
| (II): $\langle 2\rangle \leftarrow (-4) \times \langle 1\rangle + \langle 2\rangle$ | $\left[\begin{array}{ccc\|c} 1 & -1 & -3 & 8 \\ 0 & 2 & 6 & -13 \\ 3 & -6 & -17 & 41 \end{array}\right]$ |

Note that even though we multiplied row 1 by $-4$ in the side calculation, row 1 itself was not changed in the matrix. Only row 2, the target row, was altered by this Type (II) row operation.

Similarly, to target the $(3, 1)$ position (that is, convert the $(3, 1)$ entry to 0), row 3 becomes the target row. We use the Type (II) operation $\langle 3\rangle \leftarrow (-3) \times \langle 1\rangle + \langle 3\rangle$. The side calculation involved is:

| | | | | |
|---|---|---|---|---|
| $(-3)\times$ (row 1) | $-3$ | $3$ | $9$ | $-24$ |
| (row 3) | $3$ | $-6$ | $-17$ | $41$ |
| (sum) | $0$ | $-3$ | $-8$ | $17$ |

The resulting sum is now substituted in place of the old row 3.

| Row Operation | Resulting Matrix |
|---|---|
| (II): $\langle 3\rangle \leftarrow (-3) \times \langle 1\rangle + \langle 3\rangle$ | $\left[\begin{array}{ccc\|c} 1 & -1 & -3 & 8 \\ 0 & 2 & 6 & -13 \\ 0 & -3 & -8 & 17 \end{array}\right]$ |

Our work on the first column is finished. The last matrix is associated with the linear system

$$\begin{cases} x - \phantom{2}y - 3z = 8 \\ \phantom{x -} 2y + 6z = -13 \\ \phantom{x} -3y - 8z = 17 \end{cases}.$$

Note that $x$ has been eliminated from the second and third equations, which makes this system simpler than the original. However, as we will prove later, this new system has the same solution set.

**Second column:** A pivot element for the second column must be chosen, if possible, in the nearest row below the most recent pivot. We therefore want the $(2, 2)$ position to become the next pivot. Thus, row 2 is now the pivot row. We first perform a Type (I) operation on the pivot row to convert the $(2, 2)$ entry to 1. Multiplying each entry of row 2 by $\frac{1}{2}$ (the reciprocal of the current $(2, 2)$ entry), we obtain

| Row Operation | Resulting Matrix |
|---|---|
| (I): $\langle 2\rangle \leftarrow \frac{1}{2}\langle 2\rangle$ | $\left[\begin{array}{ccc\|c} 1 & -1 & -3 & 8 \\ 0 & 1 & 3 & -\frac{13}{2} \\ 0 & -3 & -8 & 17 \end{array}\right]$. |

Next, we target the $(3, 2)$ entry, so row 3 becomes the target row. We use the Type (II) operation $\langle 3\rangle \leftarrow 3 \times \langle 2\rangle + \langle 3\rangle$. The side calculation is as follows:

$$\begin{array}{rrrr|r} 3\times \text{ (row 2)} & 0 & 3 & 9 & -\frac{39}{2} \\ \text{(row 3)} & 0 & -3 & -8 & 17 \\ \hline \text{(sum)} & 0 & 0 & 1 & -\frac{5}{2} \end{array}$$

The resulting sum is now substituted in place of the old row 3.

| Row Operation | Resulting Matrix |
|---|---|
| (II): $\langle 3\rangle \leftarrow 3 \times \langle 2\rangle + \langle 3\rangle$ | $\left[\begin{array}{rrr\|r} 1 & -1 & -3 & 8 \\ 0 & 1 & 3 & -\frac{13}{2} \\ 0 & 0 & 1 & -\frac{5}{2} \end{array}\right]$. |

Our work on the second column is finished. The last matrix corresponds to the linear system

$$\begin{cases} x - y - 3z = 8 \\ y + 3z = -\frac{13}{2} \\ z = -\frac{5}{2} \end{cases}.$$

Notice that $y$ has been eliminated from the third equation. Again, this new system has exactly the same solution set as the original system.

**Third column:** Since a pivot entry for the third column must be in the nearest row below the most recent pivot, we choose the $(3, 3)$ position to become the next pivot. Thus, row 3 is now the pivot row. However, the $(3, 3)$ entry already has the value 1, so no Type (I) operation is required. Also, there are no more rows below the pivot row, and so there are no remaining entries to target. Hence, we need no further row operations, and the final matrix is

$$\left[\begin{array}{rrr|r} 1 & -1 & -3 & 8 \\ 0 & 1 & 3 & -\frac{13}{2} \\ 0 & 0 & 1 & -\frac{5}{2} \end{array}\right],$$

which corresponds to the last linear system given above.

**Conclusion:** At this point, we know from the third equation that $z = -\frac{5}{2}$. Substituting this result into the second equation and solving for $y$, we obtain $y + 3(-\frac{5}{2}) = -\frac{13}{2}$, and hence, $y = 1$. Finally, substituting these values for $y$ and $z$ into the first equation, we obtain $x - 1 - 3(-\frac{5}{2}) = 8$, and hence $x = \frac{3}{2}$. This process of working backward through the set of equations to solve for each variable in turn is called **back substitution**.

Thus, the final system has a unique solution—the ordered triple $\left(\frac{3}{2}, 1, -\frac{5}{2}\right)$. However, we can check by substitution that $\left(\frac{3}{2}, 1, -\frac{5}{2}\right)$ is also a solution to the original system. In fact, Gaussian Elimination always produces the complete solution set, and so $\left(\frac{3}{2}, 1, -\frac{5}{2}\right)$ is the unique solution to the original linear system.

## The Strategy in the Simplest Case

In Gaussian Elimination, we work on one column of the augmented matrix at a time. Beginning with the first column, we choose row 1 as our initial pivot row, convert the $(1, 1)$ entry to 1 so that it becomes the first pivot, and target (zero out) the entries below that pivot. After each column is simplified, we proceed to the next column to the right. In each column, if possible, we choose a nonzero entry in the row directly below the most recent pivot, and this entry is converted to 1 to become the next *pivot entry*. The row containing this entry is referred to as the *pivot row*. Once a pivot is created, the entries below that pivot are targeted (converted to 0) before proceeding to the next column. The process advances to additional columns until we reach the augmentation bar, or until we run out of rows to use as a pivot row.

To convert a nonzero entry into a pivot (that is, to convert it to 1), we multiply the pivot row by the *reciprocal* of the current value. Then we use Type (II) operations of the form

$$\langle \text{target row}\rangle \leftarrow (-\text{target value}) \times \langle \text{pivot row}\rangle + \langle \text{target row}\rangle$$

to target (zero out) each entry below the pivot entry. In effect, this eliminates the variable corresponding to that column from each equation in the system below the pivot row. Note that in Type (II) operations, we add an appropriate multiple of the *pivot* row to the target row. (Any other Type (II) operation could destroy work done in previous columns.)

## Using Type (III) Operations

So far, we have used only Type (I) and Type (II) operations. However, when we begin work on a new column, if the logical choice for an entry to become a pivot in that column has the value 0, it is impossible to convert that value to 1 using a Type (I) operation. Frequently, this dilemma can be resolved by first using a Type (III) operation to switch the pivot row with another row *below* it. (We never switch the pivot row with a row *above* it, because such a Type (III) operation could destroy work done in previous columns.)

### Example 3

Let us solve the following system using Gaussian Elimination:

$$\begin{cases} 3x + \ \ y = -5 \\ -6x - 2y = \ 10 \\ 4x + 5y = \ \ 8 \end{cases}, \quad \text{with augmented matrix} \quad \left[\begin{array}{rr|r} 3 & 1 & -5 \\ -6 & -2 & 10 \\ 4 & 5 & 8 \end{array}\right].$$

**First column:** We establish row 1 as the pivot row, and the entry in the $(1, 1)$ position as the first pivot. This is converted to 1 by multiplying the pivot row by the reciprocal of the pivot entry.

| Row Operation | Resulting Matrix |
|---|---|
| (I): $\langle 1\rangle \leftarrow \frac{1}{3}\langle 1\rangle$ | $\left[\begin{array}{rr\|r} 1 & \frac{1}{3} & -\frac{5}{3} \\ -6 & -2 & 10 \\ 4 & 5 & 8 \end{array}\right]$ |

Next, we use Type (II) operations to target the rest of the first column by adding appropriate multiples of the pivot row (the first row) to the target rows.

| Row Operations | Resulting Matrix |
|---|---|
| (II): $\langle 2\rangle \leftarrow 6 \times \langle 1\rangle + \langle 2\rangle$<br>(II): $\langle 3\rangle \leftarrow (-4) \times \langle 1\rangle + \langle 3\rangle$ | $\left[\begin{array}{rr\|r} 1 & \frac{1}{3} & -\frac{5}{3} \\ 0 & 0 & 0 \\ 0 & \frac{11}{3} & \frac{44}{3} \end{array}\right]$ |

**Second column:** We designate row 2 as the pivot row. We would ordinarily want to convert the entry $(2, 2)$ to 1, but because this entry is 0, a Type (I) operation will not work here. Instead, we first perform a Type (III) operation, switching the pivot row with the row below it, in order to first change the $(2, 2)$ entry to a nonzero number.

| Row Operation | Resulting Matrix |
|---|---|
| (III): $\langle 2\rangle \leftrightarrow \langle 3\rangle$ | $\left[\begin{array}{rr\|r} 1 & \frac{1}{3} & -\frac{5}{3} \\ 0 & \frac{11}{3} & \frac{44}{3} \\ 0 & 0 & 0 \end{array}\right]$ |

Now, using a Type (I) operation, we can convert the $(2, 2)$ entry to 1, and this becomes the next pivot.

| Row Operation | Resulting Matrix |
|---|---|
| (I): $\langle 2\rangle \leftarrow \frac{3}{11}\langle 2\rangle$ | $\left[\begin{array}{rr\|r} 1 & \frac{1}{3} & -\frac{5}{3} \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{array}\right]$ |

Since the entry below this pivot is already 0, the second column is now simplified.

**Conclusion:** Because there are no more columns to the left of the augmentation bar, we stop. The final matrix corresponds to the following system:

$$\begin{cases} x + \frac{1}{3}y = -\frac{5}{3} \\ \qquad y = \ \ 4 \\ \qquad 0 = \ \ 0 \end{cases}.$$

The third equation is always satisfied, no matter what values $x$ and $y$ have, and provides us with no information. The second equation gives $y = 4$. Back substituting into the first equation, we obtain $x + \frac{1}{3}(4) = -\frac{5}{3}$, and so $x = -3$. Thus, the unique solution for our original system is $(-3, 4)$.

The general rule for using Type (III) operations is:

> When starting work on a new column, if the entry in the desired position for the next pivot is 0, we look for a nonzero number in this column *below* that position. If there is one, we use a Type (III) operation to switch rows so that the row containing this nonzero number moves into the desired position.

## Skipping a Column

Occasionally when we progress to a new column, the value in the desired pivot position as well as all lower entries in that column equal 0. Here, a Type (III) operation cannot help. In such cases, we skip over the current column and advance to the next column instead. That is, the next pivot position will occur horizontally to the right of the originally intended position. Of course, if there are no more columns remaining on the left of the augmentation bar to which the pivot position can move, we stop the process.

We illustrate the use of this rule in the next few examples. Example 4 involves an inconsistent system, and Examples 5, 6, and 7 involve infinitely many solutions.

## Inconsistent Systems

### Example 4

Let us solve the following system using Gaussian Elimination:

$$\begin{cases} 3x_1 - 6x_2 \phantom{+ 2x_3} + 3x_4 = 9 \\ -2x_1 + 4x_2 + 2x_3 - x_4 = -11 \\ 4x_1 - 8x_2 + 6x_3 + 7x_4 = -5 \end{cases}.$$

First, we set up the augmented matrix

$$\left[\begin{array}{rrrr|r} 3 & -6 & 0 & 3 & 9 \\ -2 & 4 & 2 & -1 & -11 \\ 4 & -8 & 6 & 7 & -5 \end{array}\right].$$

**First column:** We establish row 1 as the pivot row. We use a Type (I) operation to convert the (1, 1) entry to 1, which becomes the first pivot.

| Row Operation | Resulting Matrix |
|---|---|
| (I) : $\langle 1\rangle \leftarrow \frac{1}{3}\langle 1\rangle$ | $\left[\begin{array}{rrrr\|r} 1 & -2 & 0 & 1 & 3 \\ -2 & 4 & 2 & -1 & -11 \\ 4 & -8 & 6 & 7 & -5 \end{array}\right]$ |

Next, we target the entries below the pivot using Type (II) row operations.

| Row Operations | Resulting Matrix |
|---|---|
| (II): $\langle 2\rangle \leftarrow 2\langle 1\rangle + \langle 2\rangle$<br>(II): $\langle 3\rangle \leftarrow -4\langle 1\rangle + \langle 3\rangle$ | $\left[\begin{array}{rrrr\|r} 1 & -2 & 0 & 1 & 3 \\ 0 & 0 & 2 & 1 & -5 \\ 0 & 0 & 6 & 3 & -17 \end{array}\right]$ |

Our work on the first column is finished.

**Second and third columns:** In the second column, the desired pivot position is the (2, 2) entry, which unfortunately is 0. We search for a nonzero entry below the (2, 2) entry but do not find one. Hence, we skip over this column. We advance horizontally to the third column, so that the (2, 3) entry becomes the next pivot entry, and row 2 becomes the pivot row.

Changing the (2, 3) entry into 1 produces

| Row Operation | Resulting Matrix |
|---|---|
| (I) : $\langle 2\rangle \leftarrow \frac{1}{2}\langle 2\rangle$ | $\left[\begin{array}{rrrr\|r} 1 & -2 & 0 & 1 & 3 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{5}{2} \\ 0 & 0 & 6 & 3 & -17 \end{array}\right].$ |

Targeting the entry below this pivot, we obtain

| Row Operation | Resulting Matrix |
|---|---|
| (II) : $\langle 3\rangle \leftarrow -6\,\langle 2\rangle + \langle 3\rangle$ | $\left[\begin{array}{cccc\|c} 1 & -2 & 0 & 1 & 3 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{5}{2} \\ 0 & 0 & 0 & 0 & -2 \end{array}\right]$. |

**Fourth column:** In the fourth column, the next desired pivot position is the $(3, 4)$ entry, but this is also 0. Because there is no nonzero entry below the $(3, 4)$ entry that can be switched with it, our work on the fourth column is finished. Since we have reached the augmentation bar, we stop.

**Conclusion:** The resulting system is

$$\begin{cases} x_1 - 2x_2 \quad + \quad x_4 = \quad 3 \\ \qquad\qquad x_3 + \frac{1}{2}x_4 = -\frac{5}{2} \\ \qquad\qquad\qquad\quad 0 = \ -2 \end{cases}.$$

Regardless of the values of $x_1$, $x_2$, $x_3$, and $x_4$, the last equation, $0 = -2$, is *never* satisfied. This equation has no solutions. But any solution to the system must satisfy every equation in the system. Therefore, this system is inconsistent, as is the original system with which we started.

For inconsistent systems, the final augmented matrix always contains at least one row of the form

$$\left[\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & c \end{array}\right],$$

with all zeroes on the left of the augmentation bar and a nonzero number $c$ on the right. Such a row corresponds to the equation $0 = c$, for some $c \neq 0$, which certainly has no solutions. In fact, if we encounter such a row at *any* stage of the Gaussian Elimination process, the original system is inconsistent.

Beware! An entire row of zeroes, with *zero* on the right of the augmentation bar, does not imply the system is inconsistent. Such a row is simply ignored, as in Example 3.

## Infinite Solution Sets

### Example 5

Let us solve the following system using Gaussian Elimination:

$$\begin{cases} 3x_1 + \ \ x_2 + \ \ 7x_3 + 2x_4 = \ \ 13 \\ 2x_1 - \ 4x_2 + 14x_3 - \ \ x_4 = -10 \\ 5x_1 + 11x_2 - \ 7x_3 + 8x_4 = \ \ 59 \\ 2x_1 + \ 5x_2 - \ 4x_3 - 3x_4 = \ \ 39 \end{cases}.$$

The augmented matrix for this system is

$$\left[\begin{array}{cccc|c} 3 & 1 & 7 & 2 & 13 \\ 2 & -4 & 14 & -1 & -10 \\ 5 & 11 & -7 & 8 & 59 \\ 2 & 5 & -4 & -3 & 39 \end{array}\right].$$

After simplifying the first two columns as in earlier examples, we obtain

$$\left[\begin{array}{cccc|c} 1 & \frac{1}{3} & \frac{7}{3} & \frac{2}{3} & \frac{13}{3} \\ 0 & 1 & -2 & \frac{1}{2} & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{13}{2} & 13 \end{array}\right].$$

The entry in the desired pivot position in the third column is zero, as are all entries below it. Therefore, we advance to the fourth column and use row operation (III): ⟨3⟩ ↔ ⟨4⟩ to put a nonzero number into the (3, 4) position, obtaining

$$\left[\begin{array}{cccc|c} 1 & \frac{1}{3} & \frac{7}{3} & \frac{2}{3} & \frac{13}{3} \\ 0 & 1 & -2 & \frac{1}{2} & 4 \\ 0 & 0 & 0 & -\frac{13}{2} & 13 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Converting the (3, 4) entry to 1 leads to the final augmented matrix

$$\left[\begin{array}{cccc|c} 1 & \frac{1}{3} & \frac{7}{3} & \frac{2}{3} & \frac{13}{3} \\ 0 & 1 & -2 & \frac{1}{2} & 4 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

This matrix corresponds to

$$\begin{cases} x_1 + \frac{1}{3}x_2 + \frac{7}{3}x_3 + \frac{2}{3}x_4 = \frac{13}{3} \\ \qquad x_2 - 2x_3 + \frac{1}{2}x_4 = 4 \\ \qquad\qquad x_4 = -2 \\ \qquad\qquad 0 = 0 \end{cases}.$$

We discard the last equation, which gives no information about the solution set. The third equation gives $x_4 = -2$, but values for the other three variables are not uniquely determined—there are infinitely many solutions. We can let $x_3$ take on any value whatsoever, which then determines the values for $x_1$ and $x_2$. For example, if we let $x_3 = 5$, then back substituting into the second equation for $x_2$ yields $x_2 - 2(5) + \frac{1}{2}(-2) = 4$, which gives $x_2 = 15$. Back substituting into the first equation gives $x_1 + \frac{1}{3}(15) + \frac{7}{3}(5) + \frac{2}{3}(-2) = \frac{13}{3}$, which reduces to $x_1 = -11$. Thus, *one* solution is $(-11, 15, 5, -2)$. However, different solutions can be found by choosing alternate values for $x_3$. For example, letting $x_3 = -4$ gives the solution $x_1 = 16$, $x_2 = -3$, $x_3 = -4$, $x_4 = -2$. All such solutions satisfy the original system.

How can we express the complete solution set? Of course, $x_4 = -2$. If we use a variable, say $c$, to represent $x_3$, then from the second equation, we obtain $x_2 - 2c + \frac{1}{2}(-2) = 4$, which gives $x_2 = 5 + 2c$. Then from the first equation, we obtain $x_1 + \frac{1}{3}(5 + 2c) + \frac{7}{3}(c) + \frac{2}{3}(-2) = \frac{13}{3}$, which leads to $x_1 = 4 - 3c$. Thus, the infinite solution set can be expressed as

$$\{(4 - 3c,\ 5 + 2c,\ c, -2) \,|\, c \in \mathbb{R}\}.$$

After Gaussian Elimination, the columns having no pivot entries are often referred to as **nonpivot columns**, while those with pivots are called **pivot columns**. Recall that the columns to the left of the augmentation bar correspond to the variables $x_1, x_2$, and so on, in the system. The variables for nonpivot columns are called **independent variables**, while those for pivot columns are **dependent variables**. If a given system is consistent, solutions are found by letting each independent variable take on any real value whatsoever. The values of the dependent variables are then calculated from these choices. Thus, in Example 5, the third column is the only nonpivot column. Hence, $x_3$ is an independent variable, while $x_1$, $x_2$, and $x_4$ are dependent variables. We found a general solution by letting $x_3$ take on any value, and we determined the remaining variables from that choice.

### Example 6

Suppose that the final matrix after Gaussian Elimination is

$$\left[\begin{array}{cccccc|c} 1 & -2 & 0 & 3 & 5 & -1 & 1 \\ 0 & 0 & 1 & 4 & 23 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 & 1 & 16 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

which corresponds to the system

$$\begin{cases} x_1 - 2x_2 \quad + 3x_4 + 5x_5 - x_6 = 1 \\ \qquad\qquad x_3 + 4x_4 + 23x_5 \qquad = -9 \\ \qquad\qquad\qquad\qquad\qquad x_6 = 16 \end{cases}.$$

Note that we have ignored the row of zeroes. Because the 2nd, 4th, and 5th columns are the nonpivot columns, $x_2$, $x_4$, and $x_5$ are the independent variables. Therefore, we can let $x_2$, $x_4$, and $x_5$ take on any real values—say, $x_2 = b$, $x_4 = d$, and $x_5 = e$. We know $x_6 = 16$. We now use back substitution to solve the remaining equations in the system for the dependent variables $x_1$ and $x_3$, yielding $x_3 = -9 - 4d - 23e$ and $x_1 = 17 + 2b - 3d - 5e$. Hence, the complete solution set is

$$\{(17 + 2b - 3d - 5e,\ b,\ -9 - 4d - 23e,\ d,\ e,\ 16) \mid b,\ d,\ e \in \mathbb{R}\}.$$

Particular solutions can be found by choosing values for $b$, $d$, and $e$. For example, choosing $b = 1$, $d = -1$, and $e = 0$ yields $(22, 1, -5, -1, 0, 16)$.

### Example 7

Suppose that the final matrix after Gaussian Elimination is

$$\left[\begin{array}{ccccc|c} 1 & 4 & -1 & 2 & 1 & 8 \\ 0 & 1 & 3 & -2 & 6 & -11 \\ 0 & 0 & 0 & 1 & -3 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Because the 3rd and 5th columns are nonpivot columns, $x_3$ and $x_5$ are the independent variables. Therefore, we can let $x_3$ and $x_5$ take on any real values—say, $x_3 = c$ and $x_5 = e$. We now use back substitution to solve the remaining equations in the system for the dependent variables $x_1$, $x_2$, and $x_4$, yielding $x_4 = 9 + 3e$, $x_2 = -11 - 3c + 2(9 + 3e) - 6e = 7 - 3c$, and $x_1 = 8 - 4(7 - 3c) + c - 2(9 + 3e) - e = -38 + 13c - 7e$. Hence, the complete solution set is

$$\{(-38 + 13c - 7e,\ 7 - 3c,\ c,\ 9 + 3e,\ e) \mid c,\ e \in \mathbb{R}\}.$$

Particular solutions can be found by choosing values for $c$ and $e$. For example, choosing $c = -1$ and $e = 2$ yields $(-65, 10, -1, 15, 2)$.

## Application: Curve Fitting

We now apply linear systems to the problem of finding a polynomial that passes through a given set of points.

### Example 8

Let us find the unique quadratic equation of the form $y = ax^2 + bx + c$ that goes through the points $(-2, 20)$, $(1, 5)$, and $(3, 25)$ in the $xy$-plane. By substituting each of the $(x, y)$ pairs in turn into the equation, we get

$$\begin{cases} a(-2)^2 + b(-2) + c = 20 \\ a(1)^2 + \ \ b(1) + c = \ 5 \\ a(3)^2 + \ \ b(3) + c = 25 \end{cases}, \quad \text{which is} \quad \begin{cases} 4a - 2b + c = 20 \\ \ \ a + \ \ b + c = \ 5 \\ 9a + 3b + c = 25 \end{cases}.$$

Using Gaussian Elimination on this system leads to the final augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{4} & 5 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 4 \end{array}\right].$$

Thus, $c = 4$, and after back substituting, we find $b = -2$, and $a = 3$, and so the desired quadratic equation is $y = 3x^2 - 2x + 4$.

## The Effect of Row Operations on Matrix Multiplication

We conclude this section with a property involving row operations and matrix multiplication that will be useful later. The following notation is helpful: if a row operation $R$ is performed on a matrix $\mathbf{A}$, we represent the resulting matrix by $R(\mathbf{A})$.

**Theorem 2.1** *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be matrices for which the product* $\mathbf{AB}$ *is defined.*

**(1)** *If $R$ is any row operation, then* $R(\mathbf{AB}) = (R(\mathbf{A}))\mathbf{B}$.

**(2)** *If $R_1, \ldots, R_n$ are row operations, then* $R_n(\cdots(R_2(R_1(\mathbf{AB})))\cdots) = (R_n(\cdots(R_2(R_1(\mathbf{A})))\cdots))\mathbf{B}$.

Part (1) of this theorem asserts that whenever a row operation is performed on the product of two matrices, the same answer is obtained by performing the row operation on the first matrix alone before multiplying. Part (1) is proved by considering each type of row operation in turn. Part (2) generalizes this result to any finite number of row operations, and is proved by using part (1) and induction. We leave the proof of Theorem 2.1 for you to do in Exercise 8.

**Example 9**

Let

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 1 \\ 3 & 4 & 2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 3 & 7 \\ 0 & -1 \\ 5 & 2 \end{bmatrix},$$

and let $R$ be the row operation $\langle 2 \rangle \leftarrow -2\langle 1 \rangle + \langle 2 \rangle$. Then

$$R(\mathbf{AB}) = R\left(\begin{bmatrix} 8 & 11 \\ 19 & 21 \end{bmatrix}\right) = \begin{bmatrix} 8 & 11 \\ 3 & -1 \end{bmatrix}, \quad \text{and}$$

$$(R(\mathbf{A}))\mathbf{B} = \left(R\left(\begin{bmatrix} 1 & -2 & 1 \\ 3 & 4 & 2 \end{bmatrix}\right)\right)\begin{bmatrix} 3 & 7 \\ 0 & -1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 8 & 0 \end{bmatrix}\begin{bmatrix} 3 & 7 \\ 0 & -1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 11 \\ 3 & -1 \end{bmatrix}.$$

Similarly, with $R_1$: $\langle 1 \rangle \leftrightarrow \langle 2 \rangle$, $R_2$: $\langle 1 \rangle \leftarrow -3\langle 2 \rangle + \langle 1 \rangle$, and $R_3$: $\langle 1 \rangle \leftarrow 4\langle 1 \rangle$, you can verify that

$$R_3(R_2(R_1(\mathbf{AB}))) = R_3\left(R_2\left(R_1\left(\begin{bmatrix} 8 & 11 \\ 19 & 21 \end{bmatrix}\right)\right)\right) = \begin{bmatrix} -20 & -48 \\ 8 & 11 \end{bmatrix}, \quad \text{and}$$

$$(R_3(R_2(R_1(\mathbf{A}))))\mathbf{B} = \left(R_3\left(R_2\left(R_1\left(\begin{bmatrix} 1 & -2 & 1 \\ 3 & 4 & 2 \end{bmatrix}\right)\right)\right)\right)\begin{bmatrix} 3 & 7 \\ 0 & -1 \\ 5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 40 & -4 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 7 \\ 0 & -1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} -20 & -48 \\ 8 & 11 \end{bmatrix} \quad \text{also.}$$

## New Vocabulary

augmented matrix (for a system)
back substitution
coefficient matrix (for a system)
complete solution set (for a system)
consistent system
dependent variable
Gaussian Elimination
inconsistent system
independent variable
nonpivot column
particular solution (to a system)
pivot column
pivot (entry)
pivot row
row operations
system of (simultaneous) linear equations
target (entry)
target row
target (*verb*)
Type (I), (II), (III) row operations

## Highlights

- A system of linear equations can be represented either as a matrix equation of the form $\mathbf{AX} = \mathbf{B}$, or as an augmented matrix of the form $[\mathbf{A}|\ \mathbf{B}]$. In both cases, the matrix $\mathbf{A}$ contains the coefficients of the variables in the system, with each column representing a different variable. The column matrix $\mathbf{B}$ contains the constants for the equations in the system. The column matrix $\mathbf{X}$ contains the variables in the system. Each row of $[\mathbf{A}|\ \mathbf{B}]$ represents one equation in the linear system.
- A system of linear equations has either no solutions (inconsistent), one solution, or an infinite number of solutions.
- Performing any of the following three row operations on the augmented matrix for a linear system does not alter the solution set of the system: Type (I): multiplying a row by a nonzero scalar, Type (II): adding a multiple of one row to another, and Type (III): switching two rows.

- When performing Gaussian Elimination on an augmented matrix, we proceed through the columns from left to right in an attempt to create a pivot in each column, if possible. Each new pivot is created in the row immediately following the rows that already contain pivots.
- If the next logical choice for a pivot position contains a *nonzero* entry $c$, and that entry is in row $i$, then we convert that entry to 1 (using the Type (I) row operation: $\langle i \rangle \leftarrow \frac{1}{c}\langle i \rangle$). Then, for each nonzero entry $d$ below this pivot in row $j$, we zero out this entry (using the Type (II) row operation: $\langle j \rangle \leftarrow -d\langle i \rangle + \langle j \rangle$).
- If the next logical choice for a pivot position contains a *zero* entry, and that entry is in row $i$, and if in the same column a *nonzero* value exists below this entry in some row $j$, we switch these rows (using the Type (III) row operation: $\langle i \rangle \leftrightarrow \langle j \rangle$).
- If the next logical choice for a pivot position contains a *zero* entry, and all entries below this value are also *zero*, then we skip over the current column and proceed to the next column.
- The Gaussian Elimination process stops when there are either no more columns before the augmentation bar to use as potential pivot columns, or no more rows to use as potential pivot rows.
- If the final matrix includes a row of all zeroes to the left of the augmentation bar and a nonzero number to the right of the bar, then the system is inconsistent. (In fact, the system is inconsistent if such a row is encountered at any step in the Gaussian Elimination process, at which point we can stop.)
- At the conclusion of the Gaussian Elimination process, if the system is consistent, each nonpivot column represents an independent variable that can take on any value, and the values of all other, dependent, variables are determined from the independent variables, using back substitution.

## Exercises for Section 2.1

1. Use Gaussian Elimination to solve each of the following systems of linear equations. In each case, indicate whether the system is consistent or inconsistent. Give the complete solution set, and if the solution set is infinite, specify three particular solutions.

★ (a) $$\begin{cases} -5x_1 - 2x_2 + 2x_3 = 16 \\ 3x_1 + x_2 - x_3 = -9 \\ 2x_1 + 2x_2 - x_3 = -4 \end{cases}$$

(b) $$\begin{cases} 3x_1 - 3x_2 - 2x_3 = 19 \\ -6x_1 + 4x_2 + 3x_3 = -36 \\ -2x_1 + x_2 + x_3 = -11 \end{cases}$$

★ (c) $$\begin{cases} 3x_1 - 2x_2 + 4x_3 = -54 \\ -x_1 + x_2 - 2x_3 = 20 \\ 5x_1 - 4x_2 + 8x_3 = -83 \end{cases}$$

(d) $$\begin{cases} 2x_1 + 3x_2 + 4x_3 - x_4 = 62 \\ -4x_1 + x_2 + 6x_3 + 2x_4 = 9 \\ 5x_1 + 9x_2 + 13x_3 - 3x_4 = 179 \\ 4x_1 + 3x_2 + 2x_3 - 2x_4 = 67 \end{cases}$$

★ (e) $$\begin{cases} 6x_1 - 12x_2 - 5x_3 + 16x_4 - 2x_5 = -53 \\ -3x_1 + 6x_2 + 3x_3 - 9x_4 + x_5 = 29 \\ -4x_1 + 8x_2 + 3x_3 - 10x_4 + x_5 = 33 \end{cases}$$

(f) $$\begin{cases} 5x_1 - 5x_2 - 15x_3 - 3x_4 = -34 \\ -2x_1 + 2x_2 + 6x_3 + x_4 = 12 \\ x_1 - x_2 - 3x_3 + x_4 = 8 \end{cases}$$

★ (g) $$\begin{cases} 4x_1 - 2x_2 - 7x_3 = 5 \\ -6x_1 + 5x_2 + 10x_3 = -11 \\ -2x_1 + 3x_2 + 4x_3 = -3 \\ -3x_1 + 2x_2 + 5x_3 = -5 \end{cases}$$

(h) $$\begin{cases} x_1 + 3x_2 - 13x_3 + 19x_4 = 78 \\ -3x_1 - 4x_2 + 14x_3 - 37x_4 = -139 \\ 2x_1 + 10x_2 - 46x_3 + 54x_4 = 232 \\ 5x_1 + 17x_2 - 75x_3 + 103x_4 = 428 \end{cases}$$

2. Suppose that each of the following is the final augmented matrix obtained after Gaussian Elimination. In each case, give the complete solution set for the corresponding system of linear equations.

★ (a) $$\left[\begin{array}{ccccc|c} 1 & -5 & 2 & 3 & -2 & -4 \\ 0 & 1 & -1 & -3 & -7 & -2 \\ 0 & 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(b) $$\left[\begin{array}{cccccc|c} 1 & -5 & 3 & 1 & -1 & 2 & 23 \\ 0 & 0 & 1 & 1 & 7 & -9 & 15 \\ 0 & 0 & 0 & 0 & 1 & 17 & 42 \\ 0 & 0 & 0 & 0 & 0 & 0 & 11 \end{array}\right]$$

★ (c) $\left[\begin{array}{cccccc|c} 1 & 4 & -8 & -1 & 2 & -3 & -4 \\ 0 & 1 & -7 & 2 & -9 & -1 & -3 \\ 0 & 0 & 0 & 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

(d) $\left[\begin{array}{cccccc|c} 1 & -2 & 6 & 3 & -5 & 4 & 41 \\ 0 & 0 & 1 & 1 & -1 & -1 & 31 \\ 0 & 0 & 0 & 0 & 1 & 9 & 24 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

★ **3.** Solve the following problem by using a linear system: A certain number of nickels, dimes, and quarters totals \$17. There are twice as many dimes as quarters, and the total number of nickels and quarters is twenty more than the number of dimes. Find the correct number of each type of coin.

★ **4.** Find the quadratic equation $y = ax^2 + bx + c$ that goes through the points $(3, 20)$, $(2, 11)$, and $(-2, 15)$.

**5.** Find the cubic equation $y = ax^3 + bx^2 + cx + d$ that goes through the points $(1, 10)$, $(2, 22)$, $(-2, 10)$, and $(3, 60)$.

★ **6.** The general equation of a circle is $x^2 + y^2 + ax + by = c$. Find the equation of the circle that goes through the points $(6, 8)$, $(8, 4)$, and $(3, 9)$.

**7.** Let $\mathbf{A} = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 1 \\ -2 & 1 & 5 \\ 3 & 0 & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 2 & 1 & -5 \\ 2 & 3 & 0 \\ 4 & 1 & 1 \end{bmatrix}$. Compute $R(\mathbf{AB})$ and $(R(\mathbf{A}))\mathbf{B}$ to verify that they are equal, if

★ **(a)** $R$: $\langle 3\rangle \leftarrow -3\,\langle 2\rangle + \langle 3\rangle$.

**(b)** $R$: $\langle 2\rangle \leftrightarrow \langle 3\rangle$.

**8.** This exercise asks for proofs for parts of Theorem 2.1.

► **(a)** Prove part (1) of Theorem 2.1 by showing that $R(\mathbf{AB}) = (R(\mathbf{A}))\mathbf{B}$ for each type of row operation ((I), (II), (III)) in turn. (Hint: Use the fact from Section 1.5 that the $k$th row of $(\mathbf{AB}) = (k\text{th row of } \mathbf{A})\mathbf{B}$.)

**(b)** Use part (a) and induction to prove part (2) of Theorem 2.1.

**9.** Explain why the scalar used in a Type (I) row operation must be nonzero.

**10.** Suppose that $\mathbf{A}$ is an $m \times n$ matrix, $\mathbf{B}$ is in $\mathbb{R}^m$, and $\mathbf{X}_1$ and $\mathbf{X}_2$ are two different solutions to the linear system $\mathbf{AX} = \mathbf{B}$.

**(a)** Prove that if $c$ is a scalar, then $\mathbf{X}_1 + c(\mathbf{X}_2 - \mathbf{X}_1)$ is also a solution to $\mathbf{AX} = \mathbf{B}$.

**(b)** Prove that if $\mathbf{X}_1 + c(\mathbf{X}_2 - \mathbf{X}_1) = \mathbf{X}_1 + d(\mathbf{X}_2 - \mathbf{X}_1)$, then $c = d$.

**(c)** Explain how parts (a) and (b) show that if a linear system has two different solutions, then it has an infinite number of solutions.

★ **11.** True or False:

**(a)** The augmented matrix for a linear system contains all the essential information from the system.

**(b)** It is possible for a linear system of equations to have exactly three solutions.

**(c)** A consistent system must have exactly one solution.

**(d)** When creating pivots, Type (II) row operations are typically used to convert nonzero entries to 1.

**(e)** In order to create a pivot in a position where an entry is zero, a Type (III) row operation is used to replace that entry, if possible, with a nonzero entry below it.

**(f)** Multiplying matrices and then performing a row operation on the product has the same effect as performing the row operation on the first matrix and then calculating the product.

**(g)** After Gaussian Elimination, if an augmented matrix has a row of all zeroes, then the corresponding linear system is inconsistent.

## 2.2 Gauss-Jordan Row Reduction and Reduced Row Echelon Form

In this section, we introduce the Gauss-Jordan Method, an extension of Gaussian Elimination. We also examine homogeneous linear systems and their solutions.

### Introduction to Gauss-Jordan Row Reduction

In Gaussian Elimination, we created the augmented matrix for a given linear system and systematically proceeded through the columns from left to right, creating pivots and targeting (zeroing out) entries below the pivots. Although we occasionally skipped over a column, we placed pivots into successive rows, and so the overall effect was to create a **staircase pattern**

**of pivots**, as in

$$\left[\begin{array}{ccc|c} 1 & 3 & -7 & 3 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 1 & 8 \end{array}\right], \quad \left[\begin{array}{cccc|c} 1 & 2 & 17 & 9 & 6 \\ 0 & 0 & 1 & 3 & 9 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

$$\text{and} \quad \left[\begin{array}{cccccc|c} 1 & -3 & 6 & -2 & 4 & -5 & -3 \\ 0 & 0 & 1 & -5 & 2 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Such matrices are said to be in **row echelon form**. However, we can extend Gaussian Elimination further to target (zero out) the entries *above* each pivot as well, as we proceed from column to column. This extension is called the **Gauss-Jordan Method**, or, **Gauss-Jordan row reduction**, and is sometimes simply referred to as "row reduction."

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### Example 1

We will solve the following system of equations using the Gauss-Jordan Method:

$$\begin{cases} 2x_1 + x_2 + 3x_3 = 16 \\ 3x_1 + 2x_2 + x_4 = 16 \\ 2x_1 + 12x_3 - 5x_4 = 5 \end{cases}.$$

This system has the corresponding augmented matrix

$$\left[\begin{array}{cccc|c} 2 & 1 & 3 & 0 & 16 \\ 3 & 2 & 0 & 1 & 16 \\ 2 & 0 & 12 & -5 & 5 \end{array}\right].$$

**First column:** As in Gaussian Elimination, the $(1,1)$ entry will become the first pivot, and row 1 becomes the pivot row. The following operation places 1 in the $(1,1)$ entry:

| Row Operation | Resulting Matrix |
|---|---|
| (I): $\langle 1\rangle \leftarrow \frac{1}{2}\langle 1\rangle$ | $\left[\begin{array}{cccc\|c} 1 & \frac{1}{2} & \frac{3}{2} & 0 & 8 \\ 3 & 2 & 0 & 1 & 16 \\ 2 & 0 & 12 & -5 & 5 \end{array}\right].$ |

The next operations target (zero out) the entries below the $(1,1)$ pivot.

| Row Operations | Resulting Matrix |
|---|---|
| (II): $\langle 2\rangle \leftarrow (-3)\langle 1\rangle + \langle 2\rangle$<br>(II): $\langle 3\rangle \leftarrow (-2)\langle 1\rangle + \langle 3\rangle$ | $\left[\begin{array}{cccc\|c} 1 & \frac{1}{2} & \frac{3}{2} & 0 & 8 \\ 0 & \frac{1}{2} & -\frac{9}{2} & 1 & -8 \\ 0 & -1 & 9 & -5 & -11 \end{array}\right]$ |

**Second column:** The $(2,2)$ entry will become the next pivot, and row 2 becomes the new pivot row. The following operation places 1 in the $(2,2)$ entry.

| Row Operation | Resulting Matrix |
|---|---|
| (I): $\langle 2\rangle \leftarrow 2\langle 2\rangle$ | $\left[\begin{array}{cccc\|c} 1 & \frac{1}{2} & \frac{3}{2} & 0 & 8 \\ 0 & 1 & -9 & 2 & -16 \\ 0 & -1 & 9 & -5 & -11 \end{array}\right]$ |

The next operations target the entries *above* and *below* the $(2,2)$ pivot.

| Row Operations | Resulting Matrix |
|---|---|
| (II): $\langle 1\rangle \leftarrow -\frac{1}{2}\langle 2\rangle + \langle 1\rangle$<br>(II): $\langle 3\rangle \leftarrow 1\langle 2\rangle + \langle 3\rangle$ | $\left[\begin{array}{cccc\|c} 1 & 0 & 6 & -1 & 16 \\ 0 & 1 & -9 & 2 & -16 \\ 0 & 0 & 0 & -3 & -27 \end{array}\right]$ |

**Third and fourth columns:** In the third column, the (3, 3) entry would normally become the next pivot, so we set row 3 as the pivot row. But we cannot place a pivot in the third column, so we proceed to the fourth column, keeping row 3 as the pivot row. The following operation places 1 in the (3, 4) entry:

| Row Operation | Resulting Matrix |
|---|---|
| (I): $\langle 3\rangle \leftarrow -\frac{1}{3}\langle 3\rangle$ | $\left[\begin{array}{cccc\|c} 1 & 0 & 6 & -1 & 16 \\ 0 & 1 & -9 & 2 & -16 \\ 0 & 0 & 0 & 1 & 9 \end{array}\right]$ |

The next operations target the entries *above* the (3, 4) pivot.

| Row Operations | Resulting Matrix |
|---|---|
| (II): $\langle 1\rangle \leftarrow 1\langle 3\rangle + \langle 1\rangle$<br>(II): $\langle 2\rangle \leftarrow -2\langle 3\rangle + \langle 2\rangle$ | $\left[\begin{array}{cccc\|c} 1 & 0 & 6 & 0 & 25 \\ 0 & 1 & -9 & 0 & -34 \\ 0 & 0 & 0 & 1 & 9 \end{array}\right]$ |

**Conclusion:** Since we have reached the augmentation bar, we stop. (Notice the staircase pattern of pivots in the final augmented matrix.) The corresponding system for this final matrix is

$$\begin{cases} x_1 \quad + 6x_3 \quad = \ 25 \\ \quad x_2 - 9x_3 \quad = -34 \\ \qquad\qquad x_4 = \ 9 \end{cases}.$$

The third equation gives $x_4 = 9$. Since the third column is not a pivot column, the independent variable $x_3$ can take on any real value, say $c$. The other variables $x_1$ and $x_2$ are now determined to be $x_1 = 25 - 6c$ and $x_2 = -7 + 9c$. Then the complete solution set is $\{(25 - 6c, 9c - 34, c, 9) \mid c \in \mathbb{R}\}$.

One disadvantage of the Gauss-Jordan Method is that more Type (II) operations generally need to be performed on the augmented matrix in order to target the entries *above* the pivots. Hence, Gaussian Elimination is faster. It is also more accurate when using a calculator or computer because there is less opportunity for the compounding of roundoff errors during the process. On the other hand, with the Gauss-Jordan Method there are fewer nonzero numbers in the final augmented matrix, which makes the solution set more apparent.

## Reduced Row Echelon Form

In the final augmented matrix in Example 1, each step on the staircase begins with a pivot, although the steps are not uniform in width. As in row echelon form, all entries below the staircase are 0, but now all entries above each pivot are 0 as well. When a matrix satisfies these conditions, it is said to be in **reduced row echelon form**. The following definition[1] states these conditions more formally:

**Definition** A matrix is in **reduced row echelon form** if and only if all the following conditions hold:

**(1)** The first nonzero entry in each row is 1.
**(2)** Each successive row has its first nonzero entry in a later column.
**(3)** All entries above and below the first nonzero entry of each row are zero.
**(4)** All full rows of zeroes are the final rows of the matrix.

Technically speaking, to put an augmented matrix into reduced row echelon form, this definition requires us to row reduce *all* columns. Therefore, putting an augmented matrix into reduced row echelon form may require proceeding to the column beyond the augmentation bar. However, we have seen that the solution set of a linear system can actually be determined without simplifying the column to the right of the augmentation bar. But, whenever we use a calculator or computer to perform row reduction, the process generally attempts to place a pivot after the augmentation bar as well. However, continuing the process beyond the augmentation bar will not change the solution set for the associated linear system.

[1] The formal definition for a matrix to be in **row echelon form** is identical except that in condition (3) the entries *above* the first nonzero entry of each row are not necessarily zero.

**Example 2**

The following augmented matrices are all in reduced row echelon form:

$$\mathbf{A}=\left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right], \quad \mathbf{B}=\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & 3 & 0 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

$$\text{and} \quad \mathbf{C}=\left[\begin{array}{cccc|c} 1 & 4 & 0 & -3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

Notice the staircase pattern of pivots in each matrix, with 1 as the first nonzero entry in each row. The linear system corresponding to **A** has a unique solution $(6, -2, 3)$. The system corresponding to **B** has an infinite number of solutions since the third column has no pivot entry, and its corresponding variable can take on any real value. (The complete solution set for this system is $\{(-1-2c,\ 4-3c,\ c,\ 2) \mid c \in \mathbb{R}\}$.) However, the system corresponding to **C** has no solutions, since the third row is equivalent to the equation $0 = 1$.

## Number of Solutions

The Gauss-Jordan Method also implies the following:

**Number of Solutions of a Linear System**

Let $\mathbf{AX} = \mathbf{B}$ be a system of linear equations. Let **C** be the reduced row echelon form augmented matrix obtained by row reducing $[\mathbf{A} | \mathbf{B}]$.

- If there is a row of **C** having all zeroes to the left of the augmentation bar but with its last entry nonzero, then $\mathbf{AX} = \mathbf{B}$ has no solution.
- If not, and if one of the columns of **C** to the left of the augmentation bar has no pivot entry, then $\mathbf{AX} = \mathbf{B}$ has an infinite number of solutions. The nonpivot columns correspond to (independent) variables that can take on any value, and the values of the remaining (dependent) variables are determined from those.
- Otherwise, $\mathbf{AX} = \mathbf{B}$ has a unique solution.

## Homogeneous Systems

**Definition** A system of linear equations having matrix form $\mathbf{AX} = \mathbf{0}$, where $\mathbf{0}$ represents a zero column matrix, is called a **homogeneous system**.

For example, the following are homogeneous systems:

$$\begin{cases} 2x - 3y = 0 \\ -4x + 6y = 0 \end{cases} \quad \text{and} \quad \begin{cases} 5x_1 - 2x_2 + 3x_3 = 0 \\ 6x_1 + x_2 - 7x_3 = 0 \\ -x_1 + 3x_2 + x_3 = 0 \end{cases}.$$

Notice that homogeneous systems are always consistent. This is because all of the variables can be set equal to zero to satisfy all of the equations. This special solution, $(0, 0, \ldots, 0)$, is called the **trivial solution**. Any other solution of a homogeneous system is called a **nontrivial solution**. For example, for the first homogeneous system shown, $(0, 0)$ is the trivial solution, but $(9, 6)$ is a nontrivial solution. Whenever a homogeneous system has a nontrivial solution, it actually has infinitely many solutions (why?).

An important result about homogeneous systems is the following:

**Theorem 2.2** *Let* $\mathbf{AX} = \mathbf{0}$ *be a homogeneous system in* $n$ *variables.*

**(1)** *If the reduced row echelon form for* **A** *has fewer than* $n$ *pivot entries, then the system has a nontrivial solution (and hence an infinite number of solutions).*

**(2)** *If the reduced row echelon form for* **A** *has exactly* $n$ *pivot entries, then the system has only the trivial solution.*

*Proof.* Note that the augmented matrix for $\mathbf{AX} = \mathbf{0}$ is $[\mathbf{A}|\mathbf{0}]$. When row reducing this augmented matrix, the column of zeroes beyond the augmentation bar never changes. Hence, the pivots will be the same whether we are row reducing $\mathbf{A}$ or row reducing $[\mathbf{A}|\mathbf{0}]$. Now, if there are fewer than $n$ pivots for $\mathbf{A}$, then some variable of the system is an independent variable, and so there are infinitely many solutions. That is, there are nontrivial solutions in addition to the trivial solution, which proves part (1) of the theorem. If there is a pivot for every column of $\mathbf{A}$, then there is a unique solution—that is, the trivial solution—which proves part (2) of the theorem. □

### Example 3

Consider the following $3 \times 3$ homogeneous systems:

$$\begin{cases} 4x_1 - 8x_2 - 2x_3 = 0 \\ 3x_1 - 5x_2 - 2x_3 = 0 \\ 2x_1 - 8x_2 + \phantom{2}x_3 = 0 \end{cases} \quad \text{and} \quad \begin{cases} 2x_1 + \phantom{2}x_2 + 4x_3 = 0 \\ 3x_1 + 2x_2 + 5x_3 = 0 \\ \phantom{3x_1} -x_2 + \phantom{5}x_3 = 0 \end{cases}.$$

Applying Gauss-Jordan row reduction to the coefficient matrices for these systems, we obtain, respectively,

$$\begin{bmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

By part (1) of Theorem 2.2, the first system has a nontrivial solution because only 2 of the 3 columns of the coefficient matrix are pivot columns (that is, the coefficient matrix has at least one nonpivot column). Since the column of zeroes in the augmented matrix for this system is not affected by row reduction, the complete solution set for the first system is

$$\left\{ \left( \frac{3}{2}c,\ \frac{1}{2}c,\ c \right) \middle|\, c \in \mathbb{R} \right\} = \left\{ c\left( \frac{3}{2}, \frac{1}{2}, 1 \right) \middle|\, c \in \mathbb{R} \right\}.$$

On the other hand, by part (2) of Theorem 2.2, the second system has only the trivial solution because all 3 columns of the coefficient matrix are pivot columns.

Notice that if there are fewer equations than variables in a homogeneous system, we are bound to get at least one nonpivot column. Therefore, we have

**Corollary 2.3** *Let* $\mathbf{AX} = \mathbf{0}$ *be a homogeneous system of* $m$ *linear equations in* $n$ *variables. If* $m < n$*, then the system has a nontrivial solution.*

## Fundamental Solutions for a Homogeneous System

### Example 4

Consider the following homogeneous system:

$$\begin{cases} \phantom{-2}x_1 - 3x_2 + 2x_3 - \phantom{1}4x_4 + \phantom{2}8x_5 + 17x_6 = 0 \\ \phantom{-}3x_1 - 9x_2 + 6x_3 - 12x_4 + 24x_5 + 49x_6 = 0 \\ -2x_1 + 6x_2 - 5x_3 + 11x_4 - 18x_5 - 40x_6 = 0 \end{cases}.$$

Because this homogeneous system has fewer equations than variables, Corollary 2.3 tells us that the system has a nontrivial solution. To find all the solutions, we row reduce to obtain the final augmented matrix

$$\left[\begin{array}{cccccc|c} 1 & -3 & 0 & 2 & 4 & 0 & 0 \\ 0 & 0 & 1 & -3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

The 2nd, 4th, and 5th columns are nonpivot columns, so we can let $x_2$, $x_4$, and $x_5$ take on any real values—say, $b$, $d$, and $e$, respectively. The values of the remaining variables are then determined by solving the equations $x_1 - 3b + 2d + 4e = 0$, $x_3 - 3d + 2e = 0$, and $x_6 = 0$. The complete solution set is

$$\{(3b - 2d - 4e,\ b,\ 3d - 2e,\ d,\ e,\ 0) \mid b,\ d,\ e \in \mathbb{R}\}.$$

Notice that the solutions for the homogeneous system in Example 4 can be expressed as linear combinations of three particular solutions as follows:

$$(3b-2d-4e,\ b,\ 3d-2e,\ d,\ e,\ 0) = b(3,1,0,0,0,0)+d(-2,0,3,1,0,0)+e(-4,0,-2,0,1,0).$$

Each particular solution was found in turn by setting one of the independent variables equal to 1 and the others equal to 0. We refer to particular solutions obtained in this way as **fundamental solutions** for the homogeneous system. It is frequently useful to express solutions to homogeneous systems in this way, as in the following example.

### Example 5

Suppose we have the following reduced row echelon form augmented matrix for a homogeneous system:

$$\left[\begin{array}{cccccc|c} 1 & 0 & \frac{5}{3} & -2 & 0 & 7 & 0 \\ 0 & 1 & -\frac{1}{3} & 8 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 1 & 6 & 0 \end{array}\right].$$

Our usual method for finding the complete solution set produces

$$\left\{ \left(-\frac{5}{3}c+2d-7f,\ \frac{1}{3}c-8d+4f,\ c,\ d,\ -6f,\ f\right) \,\middle|\, c,d,f\in\mathbb{R}\right\}.$$

By setting each of the independent variables in turn equal to 1 and the others equal to 0, this general solution can be expressed as a linear combination of fundamental solutions as follows:

$$c\left(-\frac{5}{3},\frac{1}{3},1,0,0,0\right)+d\,(2,-8,0,1,0,0)+f\,(-7,4,0,0,-6,1)\,.$$

That is, the fundamental solutions are $\left(-\frac{5}{3},\frac{1}{3},1,0,0,0\right)$, $(2,-8,0,1,0,0)$, and $(-7,4,0,0,-6,1)$. Every solution can be expressed as a linear combination of these.

It is often convenient to eliminate fractions in fundamental solutions by multiplying by an appropriate nonzero scalar. For example, we can replace the fundamental solution $\left(-\frac{5}{3},\frac{1}{3},1,0,0,0\right)$ with $(-5,1,3,0,0,0)$, where we multiplied by 3 to eliminate the denominators. A little thought will convince you that every solution to the homogeneous system can still be expressed as a linear combination of the form

$$c(-5,1,3,0,0,0)+d\,(2,-8,0,1,0,0)+f\,(-7,4,0,0,-6,1)\,.$$

Even though we have used the same variable $c$ for convenience in both of these linear combinations, it should be understood that any particular value of $c$ in the second linear combination would equal $\frac{1}{3}$ of the corresponding value of $c$ in the first linear combination.

Notice that there is a quick method to read off the fundamental solutions from the reduced row echelon form matrix. In Example 5, note that the 3rd, 4th, and 6th columns do not have pivots, and so represent independent variables. We get one fundamental solution corresponding to each of these columns, each having 1 in one of those three positions, while the other two positions have 0's. This gives three fundamental solutions, with forms $(*,*,1,0,*,0)$, $(*,*,0,1,*,0)$, and $(*,*,0,0,*,1)$, for which we still need to find the values of the $*$'s. The next step is to put in "0" for each $*$ that comes *after* "1" in any solution. This yields $(*,*,1,0,0,0)$ and $(*,*,0,1,0,0)$ for the forms of the first two fundamental solutions. The third fundamental solution, $(*,*,0,0,*,1)$, remains unchanged at this point.

Finally, to fill in the $*$'s *before* a "1", examine the nonpivot column in the reduced row echelon form matrix corresponding to its position. Then, enter the *negations* of the numbers above the staircase in that column into the solution. Here, the "1" in the first fundamental solution is in column 3, so for this solution we fill in the negations of the numbers above the staircase in column 3, namely $-\frac{5}{3}$ and $\frac{1}{3}$, giving $\left(-\frac{5}{3},\frac{1}{3},1,0,0,0\right)$. The "1" in the second fundamental solution is in column 4, so we fill in the negations of $-2$ and 8 (values above the staircase), producing $(2,-8,0,1,0,0)$. Finally, the "1" in the third fundamental solution is in column 6, so we fill in the negations of 7, $-4$, and 6, yielding $(-7,4,0,0,-6,1)$.

## Application: Balancing Chemical Equations

Homogeneous systems frequently occur when balancing chemical equations. In chemical reactions, we often know the **reactants** (initial substances) and **products** (results of the reaction). For example, it is known that the reactants phosphoric acid and calcium hydroxide produce calcium phosphate and water. This reaction can be symbolized as

$$\underset{\text{Phosphoric acid}}{H_3PO_4} + \underset{\text{Calcium hydroxide}}{Ca(OH)_2} \rightarrow \underset{\text{Calcium phosphate}}{Ca_3(PO_4)_2} + \underset{\text{Water}}{H_2O}.$$

An **empirical formula** for this reaction is an equation containing the minimal integer multiples of the reactants and products so that the number of atoms of each element agrees on both sides. (Finding the empirical formula is called **balancing** the equation.)

## Example 6

To find the empirical formula for the preceding chemical equation, we look for minimal positive integer values of $a$, $b$, $c$, and $d$ such that

$$aH_3PO_4 + bCa(OH)_2 \rightarrow cCa_3(PO_4)_2 + dH_2O$$

balances the number of hydrogen (H), phosphorus (P), oxygen (O), and calcium (Ca) atoms on both sides.[2] Considering each element in turn, we get

$$\begin{cases} 3a + 2b = \qquad 2d & \textbf{(H)} \\ a \qquad = 2c & \textbf{(P)} \\ 4a + 2b = 8c + \ d & \textbf{(O)} \\ \qquad b = 3c & \textbf{(Ca)} \end{cases}.$$

Bringing the $c$ and $d$ terms to the left side of each equation, we get the following augmented matrix for this system:

$$\left[\begin{array}{cccc|c} 3 & 2 & 0 & -2 & 0 \\ 1 & 0 & -2 & 0 & 0 \\ 4 & 2 & -8 & -1 & 0 \\ 0 & 1 & -3 & 0 & 0 \end{array}\right], \quad \text{which row reduces to} \quad \left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{6} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

For this homogeneous system, the only nonpivot column is column 4, giving a single fundamental solution of the form $(*, *, *, 1)$, where the $*$'s are filled in with the negations of the values in the first three rows of column 4. Thus, the solution set has the form $\left\{d\left(\frac{1}{3}, \frac{1}{2}, \frac{1}{6}, 1\right) \mid d \in \mathbb{R}\right\}$. Letting $d = 6$ gives the particular solution having minimal positive integer values for every variable, producing $(2, 3, 1, 6)$. Therefore, with $a = 2$, $b = 3$, $c = 1$, and $d = 6$, the empirical formula for this reaction is

$$2H_3PO_4 + 3Ca(OH)_2 \rightarrow Ca_3(PO_4)_2 + 6H_2O.$$

## Solving Several Systems Simultaneously

In many cases, we need to solve two or more systems having the same coefficient matrix.

## Example 7

Suppose we want to solve both of the systems

$$\begin{cases} 3x_1 + \ x_2 - 2x_3 = 1 \\ 4x_1 \qquad - \ x_3 = 7 \\ 2x_1 - 3x_2 + 5x_3 = 18 \end{cases} \quad \text{and} \quad \begin{cases} 3x_1 + \ x_2 - 2x_3 = 8 \\ 4x_1 \qquad - \ x_3 = -1 \\ 2x_1 - 3x_2 + 5x_3 = -32 \end{cases}.$$

It is wasteful to do two almost identical row reductions on the augmented matrices

$$\left[\begin{array}{ccc|c} 3 & 1 & -2 & 1 \\ 4 & 0 & -1 & 7 \\ 2 & -3 & 5 & 18 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{ccc|c} 3 & 1 & -2 & 8 \\ 4 & 0 & -1 & -1 \\ 2 & -3 & 5 & -32 \end{array}\right].$$

[2] In expressions like $(OH)_2$ and $(PO_4)_2$, the number immediately following the parentheses indicates that every term in the unit should be considered to appear that many times. Hence, $(PO_4)_2$ is equivalent to $PO_4PO_4$ for our purposes.

Instead, we can create the following "simultaneous" matrix containing the information from both systems:

$$\left[\begin{array}{rrr|rr} 3 & 1 & -2 & 1 & 8 \\ 4 & 0 & -1 & 7 & -1 \\ 2 & -3 & 5 & 18 & -32 \end{array}\right].$$

Row reducing this matrix completely yields

$$\left[\begin{array}{rrr|rr} 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & -3 & 5 \\ 0 & 0 & 1 & 1 & -3 \end{array}\right].$$

By considering both of the right-hand columns separately, we discover that the unique solution of the first system is $x_1 = 2$, $x_2 = -3$, and $x_3 = 1$ and that the unique solution of the second system is $x_1 = -1$, $x_2 = 5$, and $x_3 = -3$.

Any number of systems with the same coefficient matrix can be handled similarly, with one column on the right side of the augmented matrix for each system.

♦ **Applications**: You now have covered the prerequisites for Section 8.2, "Ohm's Law," Section 8.3, "Least-Squares Polynomials," and Section 8.4, "Markov Chains."

## New Vocabulary

fundamental solution for a homogeneous system
Gauss-Jordan Method
homogeneous system
nontrivial solution
reduced row echelon form
row echelon form
staircase pattern (of pivots)
trivial solution

## Highlights

- The Gauss-Jordan Method is similar to Gaussian Elimination, except that the entries both above and below each pivot are targeted (zeroed out).
- After performing Gaussian Elimination on a matrix, the result is in row echelon form. After the Gauss-Jordan Method, the result is in reduced row echelon form.
- Every homogeneous linear system is consistent because it must have at least the trivial solution.
- A homogeneous system of linear equations has a nontrivial solution if and only if it has an infinite number of solutions.
- If a homogeneous linear system has at least one nonpivot column, then the system has an infinite number of solutions.
- If a homogeneous linear system has more variables than equations, then the system has an infinite number of solutions.
- If a homogeneous system has nontrivial solutions, then the solution set can be expressed as a linear combination of fundamental solutions for the system. These fundamental solutions can be found quickly by using the negations of the entries above the staircase in the nonpivot columns of the reduced row echelon form of the coefficient matrix.
- Several linear systems having the same coefficient matrix can be solved simultaneously by using a corresponding column for each system after the (common) augmentation bar.

## Exercises for Section 2.2

★ **1.** Which of these matrices are not in reduced row echelon form? Why?

**(a)** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

**(b)** $\begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

**(c)** $\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 3 & 0 \end{bmatrix}$

**(d)** $\begin{bmatrix} 1 & -4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & -2 & 0 & -2 & 3 \\ 0 & 0 & 1 & 5 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

2. Use the Gauss-Jordan Method to convert each matrix to reduced row echelon form, and draw in the correct staircase pattern.

★ (a) $\left[\begin{array}{rrr|r} 5 & 20 & -18 & -11 \\ 3 & 12 & -14 & 3 \\ -4 & -16 & 13 & 13 \end{array}\right]$

★ (b) $\begin{bmatrix} -2 & 1 & 1 & 15 \\ 6 & -1 & -2 & -36 \\ 1 & -1 & -1 & -11 \\ -5 & -5 & -5 & -14 \end{bmatrix}$

★ (c) $\begin{bmatrix} -5 & 10 & -19 & -17 & 20 \\ -3 & 6 & -11 & -11 & 14 \\ -7 & 14 & -26 & -25 & 31 \\ 9 & -18 & 34 & 31 & -37 \end{bmatrix}$

(d) $\left[\begin{array}{rrr|r} 1 & 4 & 7 & -13 \\ 5 & 17 & 29 & -53 \\ -3 & -11 & -19 & 4 \\ 6 & 27 & 48 & -85 \\ 2 & 6 & 10 & -16 \end{array}\right]$

★ (e) $\left[\begin{array}{rrrrr|r} -3 & 6 & -1 & -5 & 0 & -5 \\ -1 & 2 & 3 & -5 & 10 & 5 \end{array}\right]$

(f) $\begin{bmatrix} 5 & 2 & 3 & 4 & 5 \\ 6 & 2 & -4 & 1 & 6 \\ 7 & 5 & 46 & -3 & 7 \\ 4 & 2 & 1 & -2 & 4 \\ 2 & 3 & 52 & 2 & 2 \end{bmatrix}$

★ 3. In parts (a), (e), and (g) of Exercise 1 in Section 2.1, take the final row echelon form matrix that you obtained from Gaussian Elimination and convert it to reduced row echelon form. Then check that the reduced row echelon form leads to the same solution set that you obtained using Gaussian Elimination.

4. By Corollary 2.3, each of the following homogeneous systems has a nontrivial solution. Use the Gauss-Jordan Method to determine the complete solution set for each system, and give one particular nontrivial solution.

★ (a) $\begin{cases} -2x_1 - 3x_2 + 2x_3 - 13x_4 = 0 \\ -4x_1 - 7x_2 + 4x_3 - 29x_4 = 0 \\ x_1 + 2x_2 - x_3 + 8x_4 = 0 \end{cases}$

(b) $\begin{cases} 2x_1 + x_2 - 8x_3 - 23x_4 - 43x_5 = 0 \\ -4x_1 + x_2 + 25x_3 + 67x_4 + 155x_5 = 0 \\ 2x_1 + 3x_2 - 2x_3 + 39x_5 = 0 \end{cases}$

★ (c) $\begin{cases} 7x_1 + 28x_2 + 4x_3 - 2x_4 + 10x_5 + 19x_6 = 0 \\ -9x_1 - 36x_2 - 5x_3 + 3x_4 - 15x_5 - 29x_6 = 0 \\ 3x_1 + 12x_2 + 2x_3 + 6x_5 + 11x_6 = 0 \\ 6x_1 + 24x_2 + 3x_3 - 3x_4 + 10x_5 + 20x_6 = 0 \end{cases}$

5. Use the Gauss-Jordan Method to find the complete solution set for each of the following homogeneous systems, and express each solution set as a linear combination of fundamental solutions.

★ (a) $\begin{cases} -2x_1 + x_2 + 8x_3 = 0 \\ 7x_1 - 2x_2 - 22x_3 = 0 \\ 3x_1 - x_2 - 10x_3 = 0 \end{cases}$

(b) $\begin{cases} 2x_1 + 4x_2 + 27x_3 = 0 \\ 6x_1 + 13x_2 + 89x_3 = 0 \\ 2x_1 - 2x_2 - 21x_3 = 0 \end{cases}$

★ (c) $\begin{cases} 2x_1 + 6x_2 + 13x_3 + x_4 = 0 \\ x_1 + 4x_2 + 10x_3 + x_4 = 0 \\ 2x_1 + 8x_2 + 20x_3 + x_4 = 0 \\ 3x_1 + 10x_2 + 21x_3 + 2x_4 = 0 \end{cases}$

(d) $\begin{cases} x_1 + 2x_2 - 5x_3 + 3x_4 = 0 \\ -2x_1 + x_2 + 17x_3 - 15x_4 = 0 \\ 3x_1 + 21x_2 + 6x_3 - 18x_4 = 0 \\ 5x_1 + 5x_2 - 32x_3 + 24x_4 = 0 \end{cases}$

(e) $\begin{cases} 8x_1 + 3x_2 - 3x_3 = 0 \\ 3x_1 + x_2 - 2x_3 = 0 \\ 17x_1 + 5x_2 - 13x_3 = 0 \\ 2x_1 + x_2 = 0 \end{cases}$

**6.** Use the Gauss-Jordan Method to find the minimal integer values for the variables that will balance each of the following chemical equations[3]:

★ **(a)** $aC_6H_6 + bO_2 \rightarrow cCO_2 + dH_2O$

**(b)** $aC_8H_{18} + bO_2 \rightarrow cCO_2 + dH_2O$

★ **(c)** $a\mathrm{AgNO_3} + bH_2O \rightarrow c\mathrm{Ag} + dO_2 + e\mathrm{HNO_3}$

**(d)** $a\mathrm{HNO_3} + b\mathrm{HCl} + c\mathrm{Au} \rightarrow d\mathrm{NOCl} + e\mathrm{HAuCl_4} + fH_2O$

**7.** Use the Gauss-Jordan Method to find the values of $A$, $B$, $C$ (and $D$ in part (b)) in the following partial fractions problems:

★ **(a)** $\dfrac{5x^2 + 23x - 58}{(x-1)(x-3)(x+4)} = \dfrac{A}{x-1} + \dfrac{B}{x-3} + \dfrac{C}{x+4}$

**(b)** $\dfrac{-3x^3 + 30x^2 - 97x + 103}{(x-2)^2(x-3)^2} = \dfrac{A}{(x-2)^2} + \dfrac{B}{x-2} + \dfrac{C}{(x-3)^2} + \dfrac{D}{x-3}$

★ **8.** Solve the systems $\mathbf{AX} = \mathbf{B}_1$ and $\mathbf{AX} = \mathbf{B}_2$ simultaneously, as illustrated in Example 7, where

$$\mathbf{A} = \begin{bmatrix} 9 & 2 & 2 \\ 3 & 2 & 4 \\ 27 & 12 & 22 \end{bmatrix}, \quad \mathbf{B}_1 = \begin{bmatrix} -6 \\ 0 \\ 12 \end{bmatrix}, \quad \text{and} \quad \mathbf{B}_2 = \begin{bmatrix} -12 \\ -3 \\ 8 \end{bmatrix}.$$

**9.** Solve the systems $\mathbf{AX} = \mathbf{B}_1$ and $\mathbf{AX} = \mathbf{B}_2$ simultaneously, as illustrated in Example 7, where

$$\mathbf{A} = \begin{bmatrix} 4 & 10 & 1 & 0 \\ 3 & -7 & -3 & -5 \\ 4 & 9 & 1 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix}, \quad \mathbf{B}_1 = \begin{bmatrix} 123 \\ -77 \\ 114 \\ 25 \end{bmatrix}, \quad \text{and} \quad \mathbf{B}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

**10.** Let $\mathbf{A} = \begin{bmatrix} 0 & 4 & 8 \\ 1 & -3 & -1 \\ 3 & -4 & 7 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 2 & 1 \\ 3 & -3 \\ -4 & 1 \end{bmatrix}$.

**(a)** Find row operations $R_1, \ldots, R_n$ such that $R_n(R_{n-1}(\cdots(R_2(R_1(\mathbf{A})))\cdots))$ is in reduced row echelon form.

**(b)** Verify part (2) of Theorem 2.1 using $\mathbf{A}$, $\mathbf{B}$, and the row operations from part (a).

**11.** This exercise relates conic sections to homogeneous systems.

**(a)** Show that, if five distinct points in the plane are given, then they must lie on a nontrivial conic section: an equation of the form $ax^2 + bxy + cy^2 + dx + ey + f = 0$ having at least one nonzero coefficient. (Hint: Create a corresponding homogeneous system of five equations with variables $a$, $b$, $c$, $d$, $e$, and $f$, and use Corollary 2.3.)

**(b)** Is this result also true when fewer than five points are given? Why or why not?

**12.** Consider the homogeneous system $\mathbf{AX} = \mathbf{0}$ having $m$ equations and $n$ variables.

**(a)** Prove that, if $\mathbf{X}_1$ and $\mathbf{X}_2$ are both solutions to this system, then $\mathbf{X}_1 + \mathbf{X}_2$ and any scalar multiple $c\mathbf{X}_1$ are also solutions.

★ **(b)** Give a counterexample to show that the results of part (a) do not necessarily hold if the system is nonhomogeneous.

**(c)** Consider a nonhomogeneous system $\mathbf{AX} = \mathbf{B}$ having the same coefficient matrix as the homogeneous system $\mathbf{AX} = \mathbf{0}$. Prove that, if $\mathbf{X}_1$ is a solution of $\mathbf{AX} = \mathbf{B}$ and if $\mathbf{X}_2$ is a solution of $\mathbf{AX} = \mathbf{0}$, then $\mathbf{X}_1 + \mathbf{X}_2$ is also a solution of $\mathbf{AX} = \mathbf{B}$.

**(d)** Show that if $\mathbf{AX} = \mathbf{B}$ has a unique solution, with $\mathbf{B} \neq \mathbf{0}$, then the corresponding homogeneous system $\mathbf{AX} = \mathbf{0}$ can have only the trivial solution. (Hint: Use part (c) and proof by contradiction.)

**13.** Prove that the following homogeneous system has a nontrivial solution if and only if $ad - bc = 0$:

$$\begin{cases} ax_1 + bx_2 = 0 \\ cx_1 + dx_2 = 0 \end{cases}.$$

[3] The chemical elements used in these equations are silver (Ag), gold (Au), carbon (C), chlorine (Cl), hydrogen (H), nitrogen (N), and oxygen (O). The compounds are water ($H_2O$), carbon dioxide ($CO_2$), benzene ($C_6H_6$), octane ($C_8H_{18}$), silver nitrate ($\mathrm{AgNO_3}$), nitric acid ($\mathrm{HNO_3}$), hydrochloric acid (HCl), nitrous chloride (NOCl), and hydrogen tetrachloroaurate (III) ($\mathrm{HAuCl_4}$).

(Hint: First, suppose that $a \neq 0$, and show that under the Gauss-Jordan Method, the second column has a zero in the desired pivot position if and only if $ad - bc = 0$. Next, consider the case $a = 0$ and $c \neq 0$. Finally, prove the result when $a = c = 0$.)

**14.** Suppose that $\mathbf{AX} = \mathbf{0}$ is a homogeneous system of $n$ equations in $n$ variables.
 - **(a)** If the system $\mathbf{A}^2\mathbf{X} = \mathbf{0}$ has a nontrivial solution, show that $\mathbf{AX} = \mathbf{0}$ also has a nontrivial solution. (Hint: Prove the contrapositive.)
 - **(b)** Generalize the result of part (a) to show that, if the system $\mathbf{A}^k\mathbf{X} = \mathbf{0}$ has a nontrivial solution for some positive integer $k$, then $\mathbf{AX} = \mathbf{0}$ has a nontrivial solution. (Hint: Use a proof by induction.)

**15.** Let $\mathbf{x}_1, \ldots, \mathbf{x}_{n+1}$ be vectors in $\mathbb{R}^n$. Show that there exist real numbers $a_1, \ldots, a_{n+1}$, not all zero, such that the linear combination $a_1\mathbf{x}_1 + \cdots + a_{n+1}\mathbf{x}_{n+1}$ equals $\mathbf{0}$. (Hint: Create an appropriate homogeneous system. See the second half of Example 4 in Section 1.5 for inspiration in creating this system. Then apply Corollary 2.3.)

**16.** Use Exercise 15 from this section and Result 7 from Section 1.3 to show that there does not exist a set of $n+1$ mutually orthogonal nonzero vectors in $\mathbb{R}^n$. (Hint: Use a proof by contradiction.)

★ **17.** True or False:
 - **(a)** In Gaussian Elimination, a descending "staircase" pattern of pivots is created, in which each step starts with 1 and the entries below the staircase are all 0.
 - **(b)** The Gauss-Jordan Method differs from Gaussian Elimination by targeting (zeroing out) entries above each pivot as well as those below the pivot.
 - **(c)** In a reduced row echelon form matrix, the pivot entries are always located in successive rows and columns.
 - **(d)** No homogeneous system is inconsistent.
 - **(e)** Nontrivial solutions to a homogeneous system are found by setting the dependent (pivot column) variables equal to any real number and then determining the independent (nonpivot column) variables from those choices.
 - **(f)** If a homogeneous system has more equations than variables, then the system must have a nontrivial solution.

## 2.3 Equivalent Systems, Rank, and Row Space

In this section, we introduce equivalent linear systems and row equivalent matrices, and use these concepts to prove that the Gaussian Elimination and Gauss-Jordan Methods actually do produce the complete solution set for a given linear system. We also note that the reduced row echelon form of any given matrix is unique, which allows us to define the rank of a matrix. Finally, we introduce the row space of a matrix and illustrate how the row space is unchanged by row operations.

### Equivalent Systems and Row Equivalence of Matrices

The first two definitions below involve related concepts. The connection between them will be shown in Theorem 2.5.

**Definition** Two systems of $m$ linear equations in $n$ variables are **equivalent** if and only if they have exactly the same solution set.

For example, the systems

$$\begin{cases} 2x - y = 1 \\ 3x + y = 9 \end{cases} \quad \text{and} \quad \begin{cases} x + 4y = 14 \\ 5x - 2y = 4 \end{cases}$$

are equivalent, because the solution set of both is exactly $\{(2, 3)\}$.

**Definition** An (augmented) matrix $\mathbf{C}$ is **row equivalent** to a matrix $\mathbf{D}$ if and only if $\mathbf{D}$ is obtained from $\mathbf{C}$ by a finite number of row operations of Types (I), (II), and (III).

For example, given any matrix, either Gaussian Elimination or the Gauss-Jordan Method produces a matrix that is row equivalent to the original.

Now, if $\mathbf{C}$ is row equivalent to $\mathbf{D}$, then $\mathbf{D}$ is also row equivalent to $\mathbf{C}$. The reason is that each row operation is reversible; that is, the effect of any row operation can be undone by performing another row operation. These **reverse**, or **inverse**, row operations are shown in Table 2.1. Notice a row operation of Type (I) is reversed by using the reciprocal $1/c$ and an operation of Type (II) is reversed by using the additive inverse $-c$. (Do you see why?)

**TABLE 2.1** Row operations and their inverses

| Type of operation | Operation | Reverse operation |
|---|---|---|
| (I) | $\langle i\rangle \leftarrow c\langle i\rangle$ | $\langle i\rangle \leftarrow \frac{1}{c}\langle i\rangle$ |
| (II) | $\langle j\rangle \leftarrow c\langle i\rangle + \langle j\rangle$ | $\langle j\rangle \leftarrow -c\langle i\rangle + \langle j\rangle$ |
| (III) | $\langle i\rangle \leftrightarrow \langle j\rangle$ | $\langle i\rangle \leftrightarrow \langle j\rangle$ |

Thus, if **D** is obtained from **C** by the sequence

$$\mathbf{C} \stackrel{R_1}{\rightarrow} \mathbf{A}_1 \stackrel{R_2}{\rightarrow} \mathbf{A}_2 \stackrel{R_3}{\rightarrow} \cdots \stackrel{R_n}{\rightarrow} \mathbf{A}_n \stackrel{R_{n+1}}{\rightarrow} \mathbf{D},$$

then **C** can be obtained from **D** using the reverse operations in reverse order:

$$\mathbf{D} \stackrel{R_{n+1}^{-1}}{\rightarrow} \mathbf{A}_n \stackrel{R_n^{-1}}{\rightarrow} \mathbf{A}_{n-1} \stackrel{R_{n-1}^{-1}}{\rightarrow} \cdots \stackrel{R_2^{-1}}{\rightarrow} \mathbf{A}_1 \stackrel{R_1^{-1}}{\rightarrow} \mathbf{C}.$$

($R_i^{-1}$ represents the inverse operation of $R_i$, as indicated in Table 2.1.) These comments provide a sketch for the proof of part (1) of the following theorem. You are asked to fill in the details for part (1) in Exercise 13(a), and to prove part (2) in Exercise 13(b).

**Theorem 2.4** *Let* **C**, **D**, *and* **E** *be matrices of the same size.*

**(1)** *If* **C** *is row equivalent to* **D**, *then* **D** *is row equivalent to* **C**.

**(2)** *If* **C** *is row equivalent to* **D**, *and* **D** *is row equivalent to* **E**, *then* **C** *is row equivalent to* **E**.

The next theorem asserts that if one augmented matrix is obtained from another using a finite sequence of row operations, then their corresponding linear systems have identical solution sets. This result guarantees that Gaussian Elimination and the Gauss-Jordan Method (as given in Sections 2.1 and 2.2) are valid because the only steps allowed in those procedures are the three familiar row operations. Therefore, a final augmented matrix produced by either method represents a system equivalent to the original—that is, a system with precisely the same solution set. This, of course, is not a surprise, since we are merely formally confirming what we have been assuming throughout Sections 2.1 and 2.2.

**Theorem 2.5** *Let* $\mathbf{AX} = \mathbf{B}$ *be a system of linear equations. If* $[\,\mathbf{C}|\,\mathbf{D}]$ *is row equivalent to* $[\,\mathbf{A}|\,\mathbf{B}]$, *then the system* $\mathbf{CX} = \mathbf{D}$ *is equivalent to* $\mathbf{AX} = \mathbf{B}$.

The proof of Theorem 2.5 is outlined in Exercise 15.

The converse of Theorem 2.5 is false because it is possible to have two inconsistent systems whose corresponding augmented matrices are the same size but are not row equivalent. However, since both are inconsistent, they have the same solution set (the empty set), and therefore, are equivalent systems. Exercise 16 asks you to provide a specific counterexample for the converse.

## Rank of a Matrix

When the Gauss-Jordan Method is performed on a matrix, only one final augmented matrix can result. This fact is stated in the following theorem, the proof of which appears in Appendix A:

**Theorem 2.6** *Every matrix is row equivalent to a unique matrix in reduced row echelon form.*

While each matrix is row equivalent to exactly one matrix in *reduced* row echelon form, there may be many matrices in row echelon form to which it is row equivalent. This is one of the advantages of Gauss-Jordan row reduction over Gaussian Elimination.

Using Theorems 2.4 and 2.6, we can show that two matrices have the same reduced row echelon form if and only if they are row equivalent. You are asked to prove one half of this assertion in Exercise 3(a) and the other half in Exercise 4(a).

Because each matrix has a *unique* corresponding reduced row echelon form matrix, we can make the following definition:

**Definition** Let $\mathbf{A}$ be a matrix. Then the **rank** of $\mathbf{A}$ is the number of nonzero rows (that is, rows with pivot entries) in the unique reduced row echelon form matrix that is row equivalent to $\mathbf{A}$.

To determine the rank of an augmented matrix we must row reduce past the augmentation bar.

### Example 1

Consider the following matrices:

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & 0 & 1 & -9 \\ 0 & -2 & 12 & -8 & -6 \\ 2 & -3 & 22 & -14 & -17 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \left[\begin{array}{cc|c} 2 & 1 & 4 \\ 3 & 2 & 5 \\ 0 & -1 & 1 \end{array}\right].$$

The unique reduced row echelon form matrices for $\mathbf{A}$ and $\mathbf{B}$ are, respectively:

$$\begin{bmatrix} 1 & 0 & 2 & -1 & -4 \\ 0 & 1 & -6 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right].$$

Therefore, the rank of $\mathbf{A}$ is 2, since the reduced row echelon form of $\mathbf{A}$ has 2 nonzero rows (and hence, 2 pivot entries). On the other hand, the rank of $\mathbf{B}$ is 3 since the reduced row echelon form of $\mathbf{B}$ has 3 nonzero rows (and hence, 3 pivot entries).

In Exercise 17 you are asked to prove that two row equivalent matrices have the same rank.

We can now restate Theorem 2.2 about homogeneous systems in terms of rank.

**Theorem 2.7** *Let $\mathbf{AX} = \mathbf{0}$ be a homogeneous system in n variables.*

**(1)** *If* $\text{rank}(\mathbf{A}) < n$*, then the system has a nontrivial solution (and hence an infinite number of solutions).*

**(2)** *If* $\text{rank}(\mathbf{A}) = n$*, then the system has only the trivial solution.*

## Linear Combinations of Vectors

In Section 1.1, we introduced linear combinations of vectors. Recall that a linear combination of vectors is a sum of scalar multiples of the vectors.

### Example 2

Let $\mathbf{a}_1 = [-4, 1, 2]$, $\mathbf{a}_2 = [2, 1, 0]$, and $\mathbf{a}_3 = [6, -3, -4]$ in $\mathbb{R}^3$. Consider the vector $[-18, 15, 16]$. Because

$$[-18, 15, 16] = 2[-4, 1, 2] + 4[2, 1, 0] - 3[6, -3, -4],$$

the vector $[-18, 15, 16]$ is a linear combination of the vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$.

Recall the Generalized Etch A Sketch® (GEaS) discussion from Section 1.1. The previous linear combination indicates how to turn the dials on a GEaS that is programmed with the vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ in order to move a reference point on the GEaS along the vector $[-18, 15, 16]$: we turn the first dial 2 times clockwise, the second dial 4 times clockwise, and the third dial 3 times counterclockwise.

Now consider the vector $[16, -3, 8]$. This vector is not a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$. For if it were, the equation

$$[16, -3, 8] = c_1[-4, 1, 2] + c_2[2, 1, 0] + c_3[6, -3, -4]$$

would have a solution. But, equating coordinates, we get the following system:

$$\begin{cases} -4c_1 + 2c_2 + 6c_3 = 16 & \text{first coordinates} \\ c_1 + c_2 - 3c_3 = -3 & \text{second coordinates} \\ 2c_1 \qquad - 4c_3 = 8 & \text{third coordinates.} \end{cases}$$

We solve this system by row reducing

$$\left[\begin{array}{ccc|c} -4 & 2 & 6 & 16 \\ 1 & 1 & -3 & -3 \\ 2 & 0 & -4 & 8 \end{array}\right] \text{ to obtain } \left[\begin{array}{ccc|c} 1 & 0 & -2 & -\frac{11}{3} \\ 0 & 1 & -1 & \frac{2}{3} \\ 0 & 0 & 0 & \frac{46}{3} \end{array}\right].$$

The third row of this final matrix indicates that the system has no solutions, which means there are no possible values for $c_1$, $c_2$, and $c_3$. Therefore, $[16, -3, 8]$ is not a linear combination of the vectors $[-4, 1, 2]$, $[2, 1, 0]$, and $[6, -3, -4]$. In other words, using a GEaS, it is impossible to move a reference point along the vector $[16, -3, 8]$ by turning dials programmed with the vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$. That type of movement cannot be accomplished by this particular GEaS.

The next example shows that a vector $\mathbf{x}$ can sometimes be expressed as a linear combination of vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ in more than one way.

### Example 3

To determine whether $[14, -21, 7]$ is a linear combination of $[2, -3, 1]$ and $[-4, 6, -2]$, we need to find scalars $c_1$ and $c_2$ such that

$$[14, -21, 7] = c_1[2, -3, 1] + c_2[-4, 6, -2].$$

This is equivalent to solving the system

$$\begin{cases} 2c_1 - 4c_2 = 14 \\ -3c_1 + 6c_2 = -21 \\ c_1 - 2c_2 = 7 \end{cases}.$$

We solve this system by row reducing

$$\left[\begin{array}{cc|c} 2 & -4 & 14 \\ -3 & 6 & -21 \\ 1 & -2 & 7 \end{array}\right] \text{ to obtain } \left[\begin{array}{cc|c} 1 & -2 & 7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Because $c_2$ is an independent variable, we may take $c_2$ to be any real value. Then $c_1 = 2c_2 + 7$. Hence, the number of solutions to the system is infinite.

For example, we could let $c_2 = 1$, which forces $c_1 = 2(1) + 7 = 9$, yielding

$$[14, -21, 7] = 9[2, -3, 1] + 1[-4, 6, -2].$$

On the other hand, we could let $c_2 = 0$, which forces $c_1 = 7$, yielding

$$[14, -21, 7] = 7[2, -3, 1] + 0[-4, 6, -2].$$

Thus, we have expressed $[14, -21, 7]$ as a linear combination of $[2, -3, 1]$ and $[-4, 6, -2]$ in more than one way.

In the context of a GEaS with dials programmed for $[2, -3, 1]$ and $[-4, 6, -2]$, this means that there is more than one way to turn the dials to move the reference point along the vector $[14, -21, 7]$.

In Examples 2 and 3 we saw that to find the coefficients to express a given vector $\mathbf{a}$ as a linear combination of other vectors, we row reduce an augmented matrix whose rightmost column is $\mathbf{a}$, and whose remaining *columns* are the other vectors.

It is possible to have a linear combination of a single vector: any scalar multiple of $\mathbf{a}$ is considered a linear combination of $\mathbf{a}$. For example, if $\mathbf{a} = [3, -1, 5]$, then $-2\mathbf{a} = [-6, 2, -10]$ is a linear combination of $\mathbf{a}$. (Think of a GEaS with only one dial!)

## The Row Space of a Matrix

Suppose $\mathbf{A}$ is an $m \times n$ matrix. Recall that each of the $m$ rows of $\mathbf{A}$ is a vector with $n$ entries—that is, a vector in $\mathbb{R}^n$.

**Definition** Let $\mathbf{A}$ be an $m \times n$ matrix. The subset of $\mathbb{R}^n$ consisting of all vectors that are linear combinations of the rows of $\mathbf{A}$ is called the **row space** of $\mathbf{A}$.

We can envision the row space of a matrix $\mathbf{A}$ as the set of all possible displacements of a reference point on a GEaS having a dial programmed for each row of $\mathbf{A}$. That is, the row space of $\mathbf{A}$ is the set of all possible movements obtained by turning the dials on that particular GEaS.

### Example 4

Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & 1 \\ -2 & 4 & -3 \end{bmatrix}.$$

We want to determine whether $[5, 17, -20]$ is in the row space of $\mathbf{A}$. If so, $[5, 17, -20]$ can be expressed as a linear combination of the rows of $\mathbf{A}$, as follows:

$$[5, 17, -20] = c_1[3, 1, -2] + c_2[4, 0, 1] + c_3[-2, 4, -3].$$

Equating the coordinates on each side leads to the following system:

$$\begin{cases} 3c_1 + 4c_2 - 2c_3 = 5 \\ c_1 \qquad + 4c_3 = 17 \\ -2c_1 + c_2 - 3c_3 = -20 \end{cases}, \quad \text{whose matrix}$$

$$\left[\begin{array}{ccc|c} 3 & 4 & -2 & 5 \\ 1 & 0 & 4 & 17 \\ -2 & 1 & -3 & -20 \end{array}\right] \text{ row reduces to } \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right].$$

Hence, $c_1 = 5$, $c_2 = -1$, and $c_3 = 3$, and

$$[5, 17, -20] = 5[3, 1, -2] - 1[4, 0, 1] + 3[-2, 4, -3].$$

Therefore, $[5, 17, -20]$ is in the row space of $\mathbf{A}$.

Example 4 shows that to check whether a vector $\mathbf{X}$ is in the row space of $\mathbf{A}$, we row reduce the augmented matrix $\left[\mathbf{A}^T \middle| \mathbf{X}\right]$ to determine whether its corresponding system has a solution.

### Example 5

The vector $\mathbf{X} = [3, 5]$ is not in the row space of $\mathbf{B} = \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}$ because there is no way to express $[3, 5]$ as a linear combination of the rows $[2, -4]$ and $[-1, 2]$ of $\mathbf{B}$. That is, row reducing

$$\left[\mathbf{B}^T \middle| \mathbf{X}\right] = \left[\begin{array}{cc|c} 2 & -1 & 3 \\ -4 & 2 & 5 \end{array}\right] \quad \text{yields} \quad \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 11 \end{array}\right],$$

thus showing that the corresponding linear system is inconsistent.

If $\mathbf{A}$ is any $m \times n$ matrix with rows $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$, then the zero vector $[0, 0, \ldots, 0]$ in $\mathbb{R}^n$ is always in the row space of $\mathbf{A}$ because we can express it as a linear combination of the rows of $\mathbf{A}$ as follows:

$$[0, 0, \ldots, 0] = 0\mathbf{a}_1 + 0\mathbf{a}_2 + \cdots + 0\mathbf{a}_m.$$

Similarly, each individual row of $\mathbf{A}$ is in the row space of $\mathbf{A}$. For example, the $i$th row of $\mathbf{A}$ is a linear combination of the rows of $\mathbf{A}$ because

$$\mathbf{a}_i = 0\mathbf{a}_1 + 0\mathbf{a}_2 + \cdots + 1\mathbf{a}_i + \cdots + 0\mathbf{a}_m.$$

## Row Equivalence Determines the Row Space

The following lemma is used in the proof of Theorem 2.9:

**Lemma 2.8** *Suppose that* $\mathbf{x}$ *is a linear combination of* $\mathbf{q}_1, \ldots, \mathbf{q}_k$*, and suppose also that each of* $\mathbf{q}_1, \ldots, \mathbf{q}_k$ *is itself a linear combination of* $\mathbf{r}_1, \ldots, \mathbf{r}_l$*. Then* $\mathbf{x}$ *is a linear combination of* $\mathbf{r}_1, \ldots, \mathbf{r}_l$.

If we create a matrix $\mathbf{Q}$ whose rows are the vectors $\mathbf{q}_1, \ldots, \mathbf{q}_k$ and a matrix $\mathbf{R}$ whose rows are the vectors $\mathbf{r}_1, \ldots, \mathbf{r}_l$, then Lemma 2.8 can be rephrased as:

*If* $\mathbf{x}$ *is in the row space of* $\mathbf{Q}$ *and each row of* $\mathbf{Q}$ *is in the row space of* $\mathbf{R}$*, then* $\mathbf{x}$ *is in the row space of* $\mathbf{R}$.

The proof of Lemma 2.8 is outlined in Exercise 19.

### Example 6

Suppose $\mathbf{x}$ is a linear combination of $\mathbf{q}_1$ and $\mathbf{q}_2$ with $\mathbf{x} = 2\mathbf{q}_1 - 3\mathbf{q}_2$, and suppose that $\mathbf{q}_1$ and $\mathbf{q}_2$ are both linear combinations of $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$, with $\mathbf{q}_1 = 4\mathbf{r}_1 + 5\mathbf{r}_2 + \mathbf{r}_3$, and $\mathbf{q}_2 = 2\mathbf{r}_1 - \mathbf{r}_2 + 5\mathbf{r}_3$. Then,

$$\begin{aligned}\mathbf{x} &= 2\mathbf{q}_1 - 3\mathbf{q}_2\\ &= 2\,(4\mathbf{r}_1 + 5\mathbf{r}_2 + \mathbf{r}_3) - 3\,(2\mathbf{r}_1 - \mathbf{r}_2 + 5\mathbf{r}_3)\\ &= 8\mathbf{r}_1 + 10\mathbf{r}_2 + 2\mathbf{r}_3 - 6\mathbf{r}_1 + 3\mathbf{r}_2 - 15\mathbf{r}_3\\ &= 2\mathbf{r}_1 + 13\mathbf{r}_2 - 13\mathbf{r}_3.\end{aligned}$$

Hence, $\mathbf{x}$ is a linear combination of $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$, as Lemma 2.8 asserts.

The next theorem illustrates an important connection between row equivalence and row space.

**Theorem 2.9** *Suppose that* $\mathbf{A}$ *and* $\mathbf{B}$ *are row equivalent matrices. Then the row space of* $\mathbf{A}$ *equals the row space of* $\mathbf{B}$.

In other words, if $\mathbf{A}$ and $\mathbf{B}$ are row equivalent, then any vector that is a linear combination of the rows of $\mathbf{A}$ must be a linear combination of the rows of $\mathbf{B}$, and vice versa. Theorem 2.9 assures us that we do not gain or lose any linear combinations of the rows when we perform row operations. That is, a GEaS with dials programmed for the rows of $\mathbf{A}$ can move the reference point to the same new positions as a GEaS with dials programmed for the rows of $\mathbf{B}$. However, the dials on each GEaS may have to be turned by different amounts in order to reach the same destination.

*Proof.* (Abridged) Let $\mathbf{A}$ and $\mathbf{B}$ be row equivalent $m \times n$ matrices. We will show that if $\mathbf{x}$ is a vector in the row space of $\mathbf{B}$, then $\mathbf{x}$ is in the row space of $\mathbf{A}$. (A similar argument can then be used to show that if $\mathbf{x}$ is in the row space of $\mathbf{A}$, then $\mathbf{x}$ is in the row space of $\mathbf{B}$.)

First consider the case in which $\mathbf{B}$ is obtained from $\mathbf{A}$ by performing a single row operation. In this case, the definition for each type of row operation implies that each row of $\mathbf{B}$ is a linear combination of the rows of $\mathbf{A}$ (see Exercise 20). Now, suppose $\mathbf{x}$ is in the row space of $\mathbf{B}$. Then $\mathbf{x}$ is a linear combination of the rows of $\mathbf{B}$. But since each of the rows of $\mathbf{B}$ is a linear combination of the rows of $\mathbf{A}$, Lemma 2.8 indicates that $\mathbf{x}$ is in the row space of $\mathbf{A}$. By induction, this argument can be extended to the case where $\mathbf{B}$ is obtained from $\mathbf{A}$ by any (finite) sequence of row operations (see Exercise 21). □

### Example 7

Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 5 & 10 & 12 & 33 & 19\\ 3 & 6 & -4 & -25 & -11\\ 1 & 2 & -2 & -11 & -5\\ 2 & 4 & -1 & -10 & -4 \end{bmatrix}.$$

The reduced row echelon form matrix for **A** is

$$\mathbf{B} = \begin{bmatrix} 1 & 2 & 0 & -3 & -1 \\ 0 & 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, **A** and **B** are row equivalent. Theorem 2.9 then asserts that the row spaces of **A** and **B** are equal. Hence, the linear combinations that can be created from the rows of **A** are identical to those that can be created from **B**. For example, the vector $\mathbf{x} = [4, 8, -30, -132, -64]$ is in both row spaces:

$$\mathbf{x} = -1[5, 10, 12, 33, 19] + 3[3, 6, -4, -25, -11] + 4[1, 2, -2, -11, -5] - 2[2, 4, -1, -10, -4],$$

which shows **x** is in the row space of **A**. But **x** is also in the row space of **B**, since

$$\mathbf{x} = 4[1, 2, 0, -3, -1] - 30[0, 0, 1, 4, 2].$$

The matrix **A** in Example 7 essentially has two unneeded, or "redundant" rows because its reduced row echelon form matrix **B** has two fewer rows (those that are nonzero in **B**) producing the same row space. In other words, the reference point of a GEaS that is programmed for the two nonzero rows of **B** reaches the same destinations using just 2 dials as a GEaS with 4 dials programmed for the four rows of **A**. We do not need a GEaS with 4 dials! In fact, we will prove in Chapter 4 that the rank of **A** gives precisely the minimal number of rows of **A** needed to produce the same set of linear combinations.

♦ **Numerical Method:** You have now covered the prerequisites for Section 9.1, "Numerical Methods for Solving Systems."

## New Vocabulary

equivalent systems of linear equations
rank (of a matrix)
reverse (inverse) row operations
row equivalent matrices
row space (of a matrix)

## Highlights

- Two matrices are row equivalent if one can be produced from the other using some finite sequence of Type (I), Type (II), and Type (III) row operations.
- If the augmented matrices for two linear systems are row equivalent, then the systems have precisely the same solution set (that is, the systems are equivalent).
- Every matrix is row equivalent to a unique reduced row echelon form matrix.
- The rank of a matrix is the number of nonzero rows (= number of columns with pivots) in its corresponding reduced row echelon form matrix.
- For a homogeneous linear system $\mathbf{AX} = \mathbf{0}$, if the rank of **A** is less than the number of variables (= the number of columns of **A**), then the system has an infinite number of solutions.
- For a homogeneous linear system $\mathbf{AX} = \mathbf{0}$, if the rank of **A** equals the number of variables (= the number of columns of **A**), then the system has only the trivial solution.
- A given vector **x** is a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ if the linear system whose augmented matrix consists of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ as its leftmost columns and **x** as the rightmost column is consistent.
- The row space of a matrix **A** is the set of all possible linear combinations of the rows of **A**.
- If two matrices are row equivalent, then their row spaces are identical (that is, each linear combination of the rows that can be produced using one matrix can also be produced from the other).

# Exercises for Section 2.3

**Note**: To save time, you should use a calculator or an appropriate software package to perform nontrivial row reductions.

**1.** For each of the following pairs of matrices **A** and **B**, give a reason why **A** and **B** are row equivalent:

★ **(a)** $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

**(b)** $\mathbf{A} = \begin{bmatrix} 12 & 9 & -5 \\ 4 & 6 & -2 \\ 0 & 1 & 3 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 & 1 & 3 \\ 4 & 6 & -2 \\ 12 & 9 & -5 \end{bmatrix}$

★ **(c)** $\mathbf{A} = \begin{bmatrix} 3 & 2 & 7 \\ -4 & 1 & 6 \\ 2 & 5 & 4 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 3 & 2 & 7 \\ -2 & 6 & 10 \\ 2 & 5 & 4 \end{bmatrix}$

**(d)** $\mathbf{A} = \begin{bmatrix} -3 & 2 & 5 \\ 6 & -4 & 1 \\ 2 & -1 & 4 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} -3 & 2 & 5 \\ 0 & 0 & 11 \\ -6 & 3 & -12 \end{bmatrix}$

**2.** This exercise relates row operations to reduced row echelon form.

**(a)** Find the reduced row echelon form **B** of the following matrix **A**, keeping track of the row operations used:

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & -20 \\ -2 & 0 & 11 \\ 3 & 1 & -15 \end{bmatrix}.$$

★ **(b)** Use your answer to part (a) to give a sequence of row operations that converts **B** back to **A**. Check your answer. (Hint: Use the inverses of the row operations from part (a), but in reverse order.)

**3.** This exercise involves matrices having a common reduced row echelon form.

**(a)** Prove that if two matrices **A** and **B** have the same reduced row echelon form matrix, then **A** and **B** are row equivalent. (Hint: Use both parts of Theorem 2.4.)

★ **(b)** Verify that the following matrices are row equivalent by showing they have the same reduced row echelon form:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 9 \\ 0 & 1 & -3 \\ 0 & -2 & 5 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} -5 & 3 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}.$$

★ **(c)** Find a sequence of row operations that converts **A** into **B**. (Hint: Let **C** be the common matrix in reduced row echelon form corresponding to **A** and **B**. In part (b) you found a sequence of row operations that converts **A** to **C** and another sequence that converts **B** to **C**. Reverse the operations in the second sequence to obtain a sequence that converts **C** to **B**. Finally, combine the first sequence with these "reversed" operations to create a sequence from **A** to **B**.)

**4.** This exercise involves the concept of row equivalent matrices.

**(a)** Prove that if two matrices **A** and **B** are row equivalent, then **A** and **B** have the same reduced row echelon form. (Hint: Use Theorem 2.6 and part (2) of Theorem 2.4.)

**(b)** Use the contrapositive of part (a) to verify that the following matrices are not row equivalent by showing that their corresponding matrices in reduced row echelon form are distinct:

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 3 & 2 & 0 \\ -1 & 2 & 0 & 0 & -3 \end{bmatrix}.$$

**5.** Find the rank of each of the following matrices:

★ **(a)** $\begin{bmatrix} 1 & -1 & 3 \\ 2 & 0 & 4 \\ -1 & -3 & 1 \end{bmatrix}$

**(b)** $\begin{bmatrix} 4 & -2 & 1 & 5 \\ -1 & 1 & 0 & 7 \\ 1 & 0 & 1 & -8 \end{bmatrix}$

★ **(c)** $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

**(d)** $\begin{bmatrix} 6 & -9 & 3 \\ 4 & -6 & 2 \\ -10 & 15 & -5 \end{bmatrix}$

★ (e) $\left[\begin{array}{ccc|c} -1 & -1 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 4 & 0 & -2 & 1 \\ 3 & -1 & 0 & 4 \end{array}\right]$

(f) $\left[\begin{array}{cccc|c} 1 & -1 & -5 & 6 & 4 \\ -2 & 3 & 13 & -14 & -7 \\ 1 & -2 & -8 & 8 & 3 \end{array}\right]$

**6.** Find the rank of the coefficient matrix for each of these homogeneous systems. From the rank, what does Theorem 2.7 predict about the solution set? (Note that Corollary 2.3 does not apply here.) Find the complete solution set to verify this prediction.

★ (a) $\begin{cases} -2x_1 + 6x_2 + 3x_3 = 0 \\ 5x_1 - 9x_2 - 4x_3 = 0 \\ 4x_1 - 8x_2 - 3x_3 = 0 \\ 6x_1 - 11x_2 - 5x_3 = 0 \end{cases}$

(b) $\begin{cases} 2x_1 + 6x_2 - 8x_3 = 0 \\ 2x_1 - x_2 + 13x_3 = 0 \\ -x_1 \quad - 5x_3 = 0 \\ \quad - 3x_2 + 9x_3 = 0 \\ x_1 - 2x_2 + 11x_3 = 0 \end{cases}$

**7.** Assume that for each type of system described below there is at least one variable having a nonzero coefficient in at least one equation. Find the smallest and largest rank possible for the corresponding augmented matrix in each case.

★ (a) Four equations, three variables, nonhomogeneous
(b) Three equations, five variables, homogeneous
★ (c) Three equations, four variables, inconsistent
(d) Five equations, four variables, nonhomogeneous, consistent
(e) Three equations, three variables, homogeneous, inconsistent

**8.** In each of the following cases, express the vector $\mathbf{x}$ as a linear combination of the other vectors, if possible:

★ (a) $\mathbf{x} = [-3, -6]$, $\mathbf{a}_1 = [1, 4]$, $\mathbf{a}_2 = [-2, 3]$
(b) $\mathbf{x} = [8, -5, 2]$, $\mathbf{a}_1 = [4, 4, 19]$, $\mathbf{a}_2 = [9, 10, 46]$, $\mathbf{a}_3 = [5, -2, 1]$
★ (c) $\mathbf{x} = [2, -1, 4]$, $\mathbf{a}_1 = [3, 6, 2]$, $\mathbf{a}_2 = [2, 10, -4]$
(d) $\mathbf{x} = [-3, 5, 1]$, $\mathbf{a}_1 = [5, 4, 2]$, $\mathbf{a}_2 = [9, 12, 5]$, $\mathbf{a}_3 = [11, 14, 6]$
★ (e) $\mathbf{x} = [7, 2, 3]$, $\mathbf{a}_1 = [1, -2, 3]$, $\mathbf{a}_2 = [5, -2, 6]$, $\mathbf{a}_3 = [4, 0, 3]$
(f) $\mathbf{x} = [-10, 19, 8, -12]$, $\mathbf{a}_1 = [-3, 1, 0, -1]$, $\mathbf{a}_2 = [6, 1, 2, 0]$, $\mathbf{a}_3 = [2, -4, -1, 2]$
★ (g) $\mathbf{x} = [2, 3, -7, 3]$, $\mathbf{a}_1 = [3, 2, -2, 4]$, $\mathbf{a}_2 = [-2, 0, 1, -3]$, $\mathbf{a}_3 = [6, 1, 2, 8]$
(h) $\mathbf{x} = [-3, 1, 2, 0, 1]$, $\mathbf{a}_1 = [-6, 2, 4, -1, 7]$, $\mathbf{a}_2 = [0, 0, 1, -1, 0]$

**9.** In each of the following cases, determine whether the given vector is in the row space of the given matrix:

★ (a) $[7, 1, 18]$, with $\begin{bmatrix} 3 & 6 & 2 \\ 2 & 10 & -4 \\ 2 & -1 & 4 \end{bmatrix}$

(b) $[-1, 10, -4]$, with $\begin{bmatrix} -2 & -1 & 1 \\ 2 & 3 & -2 \\ -7 & 0 & 2 \end{bmatrix}$

★ (c) $[2, 2, -3]$, with $\begin{bmatrix} 4 & -1 & 2 \\ -2 & 3 & 5 \\ 6 & 1 & 9 \end{bmatrix}$

(d) $[-2, 0, 1, 2]$, with $\begin{bmatrix} -1 & 0 & 2 & 5 \\ -5 & 1 & 2 & 4 \\ 6 & -2 & -1 & -1 \end{bmatrix}$

★ (e) $[1, 11, -4, 11]$, with $\begin{bmatrix} 2 & -4 & 1 & -3 \\ 7 & -1 & -1 & 2 \\ 3 & 7 & -3 & 8 \end{bmatrix}$

★ **10.** This exercise illustrates Lemma 2.8 with a specific example.

(a) Express the vector $[13, -23, 60]$ as a linear combination of the vectors

$$\mathbf{q}_1 = [-1, -5, 11], \ \mathbf{q}_2 = [-10, 3, -8], \text{ and } \mathbf{q}_3 = [7, -12, 30].$$

(b) Express each of the vectors $\mathbf{q}_1$, $\mathbf{q}_2$, and $\mathbf{q}_3$ in turn as a linear combination of the vectors $\mathbf{r}_1 = [3, -2, 4]$, $\mathbf{r}_2 = [2, 1, -3]$, and $\mathbf{r}_3 = [4, -1, 2]$.
(c) Use the results of parts (a) and (b) to express the vector $[13, -23, 60]$ as a linear combination of the vectors $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$. (Hint: Follow the method of Example 6.)

**11.** For each given matrix $\mathbf{A}$, perform the following steps:

(i) Find $\mathbf{B}$, the reduced row echelon form of $\mathbf{A}$.
(ii) Show that every nonzero row of $\mathbf{B}$ is in the row space of $\mathbf{A}$ by solving for an appropriate linear combination.
(iii) Show that every row of $\mathbf{A}$ is in the row space of $\mathbf{B}$ by solving for an appropriate linear combination.

★ **(a)** $\begin{bmatrix} 0 & 4 & 12 & 8 \\ 2 & 7 & 19 & 18 \\ 1 & 2 & 5 & 6 \end{bmatrix}$

**(b)** $\begin{bmatrix} 2 & 4 & -8 & 5 & 26 \\ 1 & 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & 2 & 9 \\ 2 & 1 & 1 & 1 & 4 \end{bmatrix}$

**12.** Let $\mathbf{A}$ be a diagonal $n \times n$ matrix. Prove that $\mathbf{A}$ is row equivalent to $\mathbf{I}_n$ if and only if $a_{ii} \neq 0$, for all $i$, $1 \le i \le n$.

▶ **13.** This exercise asks you to complete the proof of Theorem 2.4.

**(a)** Finish the proof of part (1) of Theorem 2.4 by showing that the three inverse row operations given in Table 2.1 correctly reverse their corresponding Type (I), (II), and (III) row operations.

**(b)** Prove part (2) of Theorem 2.4.

★ **14.** Let $\mathbf{A}$ be an $m \times n$ matrix. If $\mathbf{B}$ is a nonzero $m$-vector, explain why the systems $\mathbf{AX} = \mathbf{B}$ and $\mathbf{AX} = \mathbf{0}$ are not equivalent.

**15.** The purpose of this exercise is to outline a proof of Theorem 2.5. Let $S_A$ represent the complete solution set of the system $\mathbf{AX} = \mathbf{B}$, and let $S_C$ be the complete solution set of $\mathbf{CX} = \mathbf{D}$. Our goal is to prove that if $[\mathbf{C}|\mathbf{D}]$ is row equivalent to $[\mathbf{A}|\mathbf{B}]$, then $S_A = S_C$.

▶ **(a)** Prove that if $[\mathbf{C}|\mathbf{D}] = R([\mathbf{A}|\mathbf{B}])$ for a single row operation $R$, then $S_A \subseteq S_C$. (Hint: Let $\mathbf{X}$ be a solution for the system $[\mathbf{A}|\mathbf{B}]$. We want to show that $\mathbf{X}$ is also a solution for the system $[\mathbf{C}|\mathbf{D}]$. Note that $\mathbf{AX} = \mathbf{B}$. Multiply both sides of this equation by $R(\mathbf{I})$. Apply Theorem 2.1 to get that $\mathbf{CX} = \mathbf{D}$.)

▶ **(b)** Use part (a) and an induction argument to prove that if $[\mathbf{C}|\mathbf{D}]$ is row equivalent to $[\mathbf{A}|\mathbf{B}]$, then $S_A \subseteq S_C$.

▶ **(c)** Use part (b) to prove that if $[\mathbf{C}|\mathbf{D}]$ is row equivalent to $[\mathbf{A}|\mathbf{B}]$, then $S_C \subseteq S_A$. (Hint: Use Theorem 2.4.)

**(d)** Use parts (b) and (c) to complete the proof of Theorem 2.5.

★ **16.** Show that the converse to Theorem 2.5 is not true by exhibiting two inconsistent systems (with the same number of equations and variables) whose corresponding augmented matrices are not row equivalent.

★ **17.** Prove that if $\mathbf{A}$ is row equivalent to $\mathbf{B}$, then $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{B})$. (Hint: Let $\mathbf{D}$ be the unique reduced row echelon form matrix for $\mathbf{A}$. Show that $\mathbf{B}$ is row equivalent to $\mathbf{D}$ using an argument analogous to that in Exercise 4(a).)

**18.** Let $\mathbf{A}$ and $\mathbf{B}$ be $m \times n$ and $n \times p$ matrices, respectively.

**(a)** Let $R_1, \ldots, R_k$ be row operations. Prove that $\text{rank}(R_k(\cdots(R_1(\mathbf{A}))\cdots)) = \text{rank}(\mathbf{A})$. (Hint: Use Exercise 17.)

**(b)** Show that if $\mathbf{A}$ has $k$ rows of all zeroes, then $\text{rank}(\mathbf{A}) \le m - k$. (Hint: Argue that the row reduction process never places a nonzero entry into a row of all zeroes.)

**(c)** Show that if $\mathbf{A}$ is in reduced row echelon form, then $\text{rank}(\mathbf{AB}) \le \text{rank}(\mathbf{A})$. (Hint: If $\mathbf{A}$ has $k$ rows of all zeroes, show that the last $k$ rows of $\mathbf{AB}$ also have all zeroes. Then use part (b).)

**(d)** Use parts (a) and (c) to prove that for a general matrix $\mathbf{A}$, $\text{rank}(\mathbf{AB}) \le \text{rank}(\mathbf{A})$. (Hint: Also use Theorem 2.1.)

**19.** The purpose of this exercise is to outline a proof of Lemma 2.8. Suppose that $\mathbf{x}$ is a linear combination of $\mathbf{q}_1, \ldots, \mathbf{q}_k$, and each of $\mathbf{q}_1, \ldots, \mathbf{q}_k$ is itself a linear combination of $\mathbf{r}_1, \ldots, \mathbf{r}_l$. We want to show $\mathbf{x}$ is a linear combination of $\mathbf{r}_1, \ldots, \mathbf{r}_l$.

**(a)** Using scalars $c_1, \ldots, c_k$, write an equation that expresses $\mathbf{x}$ as a linear combination of $\mathbf{q}_1, \ldots, \mathbf{q}_k$.

**(b)** Using scalars $d_{ij}$, for $1 \le i \le k$ and $1 \le j \le l$, write a system of $k$ equations that expresses each of $\mathbf{q}_1, \ldots, \mathbf{q}_k$ as a linear combination of $\mathbf{r}_1, \ldots, \mathbf{r}_l$.

**(c)** In the linear combination for $\mathbf{x}$ created in part (a), for each $\mathbf{q}_i$, substitute the corresponding linear combination of $\mathbf{r}_1, \ldots, \mathbf{r}_l$ found in part (b).

**(d)** Use the distributive law to expand the expression for $\mathbf{x}$ obtained in part (c). Then, combine like terms in order to get $\mathbf{x}$ as a linear combination of $\mathbf{r}_1, \ldots, \mathbf{r}_l$.

▶ **20.** Suppose a matrix $\mathbf{B}$ is created from a matrix $\mathbf{A}$ by a single row operation (of Type (I), (II), or (III)). Verify the assertion in the proof of Theorem 2.9 that each row of $\mathbf{B}$ is a linear combination of the rows of $\mathbf{A}$.

▶ **21.** Complete the proof of Theorem 2.9 by showing that if a matrix $\mathbf{B}$ is obtained from a matrix $\mathbf{A}$ by any finite sequence of row operations, then the row space of $\mathbf{B}$ is contained in the row space of $\mathbf{A}$. (Hint: The case for a single row operation follows from Exercise 20. Use induction and Lemma 2.8 to extend this result to the case of more than one row operation.)

★ **22.** True or False:

**(a)** Two linear systems are equivalent if their corresponding augmented matrices are row equivalent.

**(b)** If $\mathbf{A}$ is row equivalent to $\mathbf{B}$, and $\mathbf{B}$ has rank 3, then $\mathbf{A}$ has rank 3.

**(c)** The inverse of a Type (I) row operation is a Type (II) row operation.

**(d)** If the matrix for a linear system with $n$ variables has rank $< n$, then the system must have a nontrivial solution.

**(e)** If the matrix for a homogeneous system with $n$ variables has rank $n$, then the system has a nontrivial solution.

**(f)** If $\mathbf{x}$ is a linear combination of the rows of $\mathbf{A}$, and $\mathbf{B}$ is row equivalent to $\mathbf{A}$, then $\mathbf{x}$ is in the row space of $\mathbf{B}$.

## 2.4 Inverses of Matrices

In this section, we discover that most square matrices have a *multiplicative* inverse. We examine some properties of multiplicative inverses and illustrate methods for finding these inverses when they exist.

### Multiplicative Inverse of a Matrix

When the word "inverse" is used with matrices, it usually refers to the *multiplicative* inverse in the next definition, rather than the additive inverse of Theorem 1.12, part (4).

**Definition** Let $\mathbf{A}$ be an $n \times n$ matrix. Then an $n \times n$ matrix $\mathbf{B}$ is a **(multiplicative) inverse** of $\mathbf{A}$ if and only if $\mathbf{AB} = \mathbf{BA} = \mathbf{I}_n$.

Note that if $\mathbf{B}$ is an inverse of $\mathbf{A}$, then $\mathbf{A}$ is also an inverse of $\mathbf{B}$, as can be seen by switching the roles of $\mathbf{A}$ and $\mathbf{B}$ in the definition.

### Example 1

The matrices

$$\mathbf{A} = \begin{bmatrix} 1 & -4 & 1 \\ 1 & 1 & -2 \\ -1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 3 & 5 & 7 \\ 1 & 2 & 3 \\ 2 & 3 & 5 \end{bmatrix}$$

are inverses of each other because

$$\underbrace{\begin{bmatrix} 1 & -4 & 1 \\ 1 & 1 & -2 \\ -1 & 1 & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} 3 & 5 & 7 \\ 1 & 2 & 3 \\ 2 & 3 & 5 \end{bmatrix}}_{\mathbf{B}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{I}_3} = \underbrace{\begin{bmatrix} 3 & 5 & 7 \\ 1 & 2 & 3 \\ 2 & 3 & 5 \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} 1 & -4 & 1 \\ 1 & 1 & -2 \\ -1 & 1 & 1 \end{bmatrix}}_{\mathbf{A}}.$$

However, $\mathbf{C} = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$ has no inverse. For, if

$$\begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{for some } \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then multiplying out the left side would give

$$\begin{bmatrix} 2a + c & 2b + d \\ 6a + 3c & 6b + 3d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This would force $2a + c = 1$ and $6a + 3c = 0$, but these are contradictory equations, since $6a + 3c = 3(2a + c)$.

When checking whether two given square matrices $\mathbf{A}$ and $\mathbf{B}$ are inverses, we do not need to multiply both products $\mathbf{AB}$ and $\mathbf{BA}$, as the next theorem asserts.

**Theorem 2.10** *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be* $n \times n$ *matrices. If either product* $\mathbf{AB}$ *or* $\mathbf{BA}$ *equals* $\mathbf{I}_n$*, then the other product also equals* $\mathbf{I}_n$*, and* $\mathbf{A}$ *and* $\mathbf{B}$ *are inverses of each other.*

The proof is tedious and is in Appendix A for the interested reader.

**Definition** A matrix is **singular** if and only if it is square and does not have an inverse.
A matrix is **nonsingular** if and only if it is square and has an inverse.

For example, the $2 \times 2$ matrix $\mathbf{C}$ from Example 1 is a singular matrix since we proved that it does not have an inverse. Another example of a singular matrix is the $n \times n$ zero matrix $\mathbf{O}_n$ (why?). On the other hand, the $3 \times 3$ matrix $\mathbf{A}$ from Example 1 is nonsingular, because we found an inverse $\mathbf{B}$ for $\mathbf{A}$.

## Properties of the Matrix Inverse

The next theorem shows that the inverse of a matrix must be unique (when it exists).

**Theorem 2.11 (Uniqueness of Inverse Matrix)** *If* $\mathbf{B}$ *and* $\mathbf{C}$ *are both inverses of an* $n \times n$ *matrix* $\mathbf{A}$*, then* $\mathbf{B} = \mathbf{C}$.

*Proof.* $\mathbf{B} = \mathbf{B}\mathbf{I}_n = \mathbf{B}(\mathbf{AC}) = (\mathbf{BA})\mathbf{C} = \mathbf{I}_n\mathbf{C} = \mathbf{C}$. □

Because Theorem 2.11 asserts that a nonsingular matrix $\mathbf{A}$ can have exactly one inverse, we denote the unique inverse of $\mathbf{A}$ by $\mathbf{A}^{-1}$.

For a nonsingular matrix $\mathbf{A}$, we can use the inverse to define negative integral powers of $\mathbf{A}$.

**Definition** Let $\mathbf{A}$ be a nonsingular matrix. Then the negative integral powers of $\mathbf{A}$ are given as follows: $\mathbf{A}^{-1}$ is the (unique) inverse of $\mathbf{A}$, and for $k \geq 2$, $\mathbf{A}^{-k} = \left(\mathbf{A}^{-1}\right)^k$.

### Example 2

We know from Example 1 that

$$\mathbf{A} = \begin{bmatrix} 1 & -4 & 1 \\ 1 & 1 & -2 \\ -1 & 1 & 1 \end{bmatrix} \quad \text{has} \quad \mathbf{A}^{-1} = \begin{bmatrix} 3 & 5 & 7 \\ 1 & 2 & 3 \\ 2 & 3 & 5 \end{bmatrix}$$

as its unique inverse. Since $\mathbf{A}^{-3} = \left(\mathbf{A}^{-1}\right)^3$, we have

$$\mathbf{A}^{-3} = \begin{bmatrix} 3 & 5 & 7 \\ 1 & 2 & 3 \\ 2 & 3 & 5 \end{bmatrix}^3 = \begin{bmatrix} 272 & 445 & 689 \\ 107 & 175 & 271 \\ 184 & 301 & 466 \end{bmatrix}.$$

**Theorem 2.12** *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be nonsingular* $n \times n$ *matrices. Then,*

**(1)** $\mathbf{A}^{-1}$ *is nonsingular, and* $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.
**(2)** $\mathbf{A}^k$ *is nonsingular, and* $(\mathbf{A}^k)^{-1} = (\mathbf{A}^{-1})^k = \mathbf{A}^{-k}$*, for any integer* $k$.
**(3)** $\mathbf{AB}$ *is nonsingular, and* $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.
**(4)** $\mathbf{A}^T$ *is nonsingular, and* $\left(\mathbf{A}^T\right)^{-1} = (\mathbf{A}^{-1})^T$.

Part (3) says that the inverse of a product equals the product of the inverses in *reverse* order. To prove each part of this theorem, show that the right side of each equation is the inverse of the term in parentheses on the left side. This is done by simply multiplying them together and observing that their product is $\mathbf{I}_n$. We prove parts (3) and (4) here and leave the others as Exercise 15(a).

*Proof.* (Abridged)

**Part (3)**: We must show that $\mathbf{B}^{-1}\mathbf{A}^{-1}$ (right side) is the inverse of $\mathbf{AB}$ (in parentheses on the left side). Multiplying them together gives

$$(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) = \mathbf{A}\left(\mathbf{B}\mathbf{B}^{-1}\right)\mathbf{A}^{-1} = \mathbf{A}\mathbf{I}_n\mathbf{A}^{-1} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}_n.$$

**Part (4):** We must show that $\left(\mathbf{A}^{-1}\right)^T$ (right side) is the inverse of $\mathbf{A}^T$ (in parentheses on the left side). Multiplying them together gives $\mathbf{A}^T\left(\mathbf{A}^{-1}\right)^T = \left(\mathbf{A}^{-1}\mathbf{A}\right)^T$ (by Theorem 1.18) $= (\mathbf{I}_n)^T = \mathbf{I}_n$, since $\mathbf{I}_n$ is symmetric. □

Using a proof by induction, part (3) of Theorem 2.12 generalizes as follows: if $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_k$ are nonsingular matrices of the same size, then

$$(\mathbf{A}_1\mathbf{A}_2 \cdots \mathbf{A}_k)^{-1} = \mathbf{A}_k^{-1} \cdots \mathbf{A}_2^{-1}\mathbf{A}_1^{-1}$$

(see Exercise 15(b)). Notice that the order of the matrices on the right side is reversed.

Theorem 1.17 can also be generalized to show that the laws of exponents hold for negative integer powers, as follows:

**Theorem 2.13 (Expanded Version of Theorem 1.17)** *If* $\mathbf{A}$ *is a nonsingular matrix and if* $s$ *and* $t$ *are integers, then*

(1) $\mathbf{A}^{s+t} = (\mathbf{A}^s)(\mathbf{A}^t)$

(2) $(\mathbf{A}^s)^t = \mathbf{A}^{st} = \left(\mathbf{A}^t\right)^s$

The proof of this theorem is a bit tedious. Some special cases are considered in Exercise 17.

Recall that in Section 1.5 we observed that if $\mathbf{AB} = \mathbf{AC}$ for three matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, it does not necessarily follow that $\mathbf{B} = \mathbf{C}$. However, if $\mathbf{A}$ is a nonsingular matrix, then $\mathbf{B} = \mathbf{C}$ because we can multiply both sides of $\mathbf{AB} = \mathbf{AC}$ by $\mathbf{A}^{-1}$ on the left to effectively cancel out the $\mathbf{A}$'s.

## Inverses for 2 × 2 Matrices

So far, we have studied many properties of the matrix inverse, but we have not discussed methods for finding inverses. In fact, there is an immediate way to find the inverse (if it exists) of a $2 \times 2$ matrix. Note that if we let $\delta = ad - bc$, then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \delta & 0 \\ 0 & \delta \end{bmatrix} = \delta\mathbf{I}_n.$$

Hence, if $\delta \neq 0$, we can divide this equation by $\delta$ to prove one half of the following theorem:

**Theorem 2.14** *The matrix* $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *has an inverse if and only if* $\delta = ad - bc \neq 0$. *In that case,*

$$\mathbf{A}^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\delta}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

To finish the proof of Theorem 2.14, we use a proof by contradiction to show that if $\mathbf{A}$ has an inverse then $\delta = ad - bc \neq 0$. Suppose that $\mathbf{A}^{-1}$ exists and $\delta = ad - bc = 0$. Then $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \mathbf{O}_2$. Multiplying both sides of this equation by $\mathbf{A}^{-1}$ on the left shows that $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \mathbf{A}^{-1}\mathbf{O}_2 = \mathbf{O}_2$. Hence, $a = b = c = d = 0$, implying that $\mathbf{A} = \mathbf{O}_2$, which does not have an inverse, thus giving a contradiction.

The quantity $\delta = ad - bc$ is called the **determinant** of $\mathbf{A}$. We will discuss determinants in more detail in Chapter 3.

### Example 3

There is no inverse for $\begin{bmatrix} 12 & -4 \\ 9 & -3 \end{bmatrix}$, since $\delta = (12)(-3) - (-4)(9) = 0$. On the other hand, $\mathbf{M} = \begin{bmatrix} -5 & 2 \\ 9 & -4 \end{bmatrix}$ does have an inverse because $\delta = (-5)(-4) - (2)(9) = 2 \neq 0$. This inverse is

$$\mathbf{M}^{-1} = \frac{1}{2}\begin{bmatrix} -4 & -2 \\ -9 & -5 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ -\frac{9}{2} & -\frac{5}{2} \end{bmatrix}.$$

(Verify this by checking that $\mathbf{MM}^{-1} = \mathbf{I}_2$.)

## Inverses of Larger Matrices

Let $\mathbf{A}$ be an $n \times n$ matrix. We now describe a process for calculating $\mathbf{A}^{-1}$, if it exists.

**Method for Finding the Inverse of a Matrix (if it exists) (Inverse Method)**
Suppose that $\mathbf{A}$ is a given $n \times n$ matrix.

**Step 1:** Augment $\mathbf{A}$ to a $n \times 2n$ matrix, whose first $n$ columns form $\mathbf{A}$ itself and whose last $n$ columns form $\mathbf{I}_n$.

**Step 2:** Convert $[\mathbf{A}|\,\mathbf{I}_n]$ into reduced row echelon form.

**Step 3:** If the first $n$ columns of $[\mathbf{A}|\,\mathbf{I}_n]$ have not been converted into $\mathbf{I}_n$, then $\mathbf{A}$ is singular. Stop.

**Step 4:** Otherwise, $\mathbf{A}$ is nonsingular, and the last $n$ columns of the augmented matrix in reduced row echelon form represent $\mathbf{A}^{-1}$. That is, $[\mathbf{A}|\,\mathbf{I}_n]$ row reduces to $\left[\mathbf{I}_n|\,\mathbf{A}^{-1}\right]$.

Before proving that this procedure is valid, we consider some examples.

### Example 4

To find the inverse of the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & -6 & 5 \\ -4 & 12 & -9 \\ 2 & -9 & 8 \end{bmatrix},$$

we first enlarge this to a $3 \times 6$ matrix by adjoining the identity matrix $\mathbf{I}_3$:

$$\left[\begin{array}{rrr|rrr} 2 & -6 & 5 & 1 & 0 & 0 \\ -4 & 12 & -9 & 0 & 1 & 0 \\ 2 & -9 & 8 & 0 & 0 & 1 \end{array}\right].$$

Row reduction yields

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & \frac{5}{2} & \frac{1}{2} & -1 \\ 0 & 1 & 0 & \frac{7}{3} & 1 & -\frac{1}{3} \\ 0 & 0 & 1 & 2 & 1 & 0 \end{array}\right].$$

The last three columns give the inverse of the original matrix $\mathbf{A}$. This is

$$\mathbf{A}^{-1} = \begin{bmatrix} \frac{5}{2} & \frac{1}{2} & -1 \\ \frac{7}{3} & 1 & -\frac{1}{3} \\ 2 & 1 & 0 \end{bmatrix}.$$

You should check that this matrix really is the inverse of $\mathbf{A}$ by showing that its product with $\mathbf{A}$ is equal to $\mathbf{I}_3$.

As we have seen, not every square matrix has an inverse. In the next example, we apply the Inverse Method to a singular matrix to see what occurs.

### Example 5

We attempt to find an inverse for the singular matrix

$$\mathbf{A} = \begin{bmatrix} 4 & 2 & 8 & 1 \\ -2 & 0 & -4 & 1 \\ 1 & 4 & 2 & 0 \\ 3 & -1 & 6 & -2 \end{bmatrix}.$$

We calculate the reduced row echelon form for $[\mathbf{A}|\mathbf{I}_4]$, obtaining

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & 0 & -\frac{8}{3} & -\frac{1}{3} & -\frac{4}{3} \\ 0 & 1 & 0 & 0 & 0 & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 & -\frac{13}{3} & -\frac{2}{3} & -\frac{8}{3} \\ 0 & 0 & 0 & 0 & 1 & \frac{41}{3} & \frac{4}{3} & \frac{22}{3} \end{array}\right].$$

Since the first four columns of the reduced row echelon form matrix are not converted into the identity matrix $\mathbf{I}_4$, the original matrix $\mathbf{A}$ has no inverse by Step 3 of the Inverse Method.

## Justification of the Inverse Method

To verify that the Inverse Method is valid, we must prove that for a given square matrix $\mathbf{A}$, the algorithm correctly predicts whether $\mathbf{A}$ has an inverse and, if it does, calculates its (unique) inverse.

Now, from the technique of solving simultaneous systems in Section 2.2, we know that row reduction of

$$[\mathbf{A}|\mathbf{I}_n] = \left[\begin{array}{c|ccccc} \mathbf{A} & \begin{array}{c}\text{1st}\\\text{column}\\\text{of } \mathbf{I}_n\end{array} & \begin{array}{c}\text{2nd}\\\text{column}\\\text{of } \mathbf{I}_n\end{array} & \begin{array}{c}\text{3rd}\\\text{column}\\\text{of } \mathbf{I}_n\end{array} & \cdots & \begin{array}{c}n\text{th}\\\text{column}\\\text{of } \mathbf{I}_n\end{array} \end{array}\right]$$

is equivalent to performing $n$ separate row reductions to solve the $n$ linear systems having augmented matrices

$$\left[\begin{array}{c|c} \mathbf{A} & \begin{array}{c}\text{1st}\\\text{column}\\\text{of } \mathbf{I}_n\end{array} \end{array}\right], \left[\begin{array}{c|c} \mathbf{A} & \begin{array}{c}\text{2nd}\\\text{column}\\\text{of } \mathbf{I}_n\end{array} \end{array}\right], \ldots, \left[\begin{array}{c|c} \mathbf{A} & \begin{array}{c}n\text{th}\\\text{column}\\\text{of } \mathbf{I}_n\end{array} \end{array}\right].$$

First, suppose $\mathbf{A}$ is a nonsingular $n \times n$ matrix (that is, $\mathbf{A}^{-1}$ exists). Now, because $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}_n$, we know $\mathbf{A}\left[\begin{array}{c}i\text{th}\\\text{column}\\\text{of } \mathbf{A}^{-1}\end{array}\right] = \left[\begin{array}{c}i\text{th}\\\text{column}\\\text{of } \mathbf{I}_n\end{array}\right]$. Therefore, the columns of $\mathbf{A}^{-1}$ are respective solutions of the $n$ systems above. Thus, these systems are all consistent. Now, if any one of these systems has more than one solution, then a second solution for that system can be used to replace the corresponding column in $\mathbf{A}^{-1}$ to give a second inverse for $\mathbf{A}$. But by Theorem 2.11, the inverse of $\mathbf{A}$ is unique, and so each of these systems must have a unique solution. Therefore, each column to the left of the augmentation bar must be a pivot column, or else there would be independent variables, giving an infinite number of solutions for these systems. Thus, $[\mathbf{A}|\mathbf{I}_n]$ must row reduce to $\left[\mathbf{I}_n|\mathbf{A}^{-1}\right]$, since the columns of $\mathbf{A}^{-1}$ are the unique solutions for these simultaneous systems.

Now consider the case where $\mathbf{A}$ is singular. Because an inverse for $\mathbf{A}$ cannot be found, at least one of the original $n$ systems, such as

$$\left[\begin{array}{c|c} \mathbf{A} & \begin{array}{c}k\text{th}\\\text{column}\\\text{of } \mathbf{I}_n\end{array} \end{array}\right],$$

has no solutions. But this occurs only if the final augmented matrix after row reduction contains a row of the form

$$\left[\begin{array}{ccccc|c} 0 & 0 & 0 & \cdots & 0 & r \end{array}\right],$$

where $r \neq 0$. Hence, there is a row that contains no pivot entry in the first $n$ columns, and so we do not obtain $\mathbf{I}_n$ to the left of the augmentation bar. Step 3 of the formal algorithm correctly concludes that $\mathbf{A}$ is singular.

Recall that $\mathbf{A}$ row reduces to $\mathbf{I}_n$ if and only if $\text{rank}(\mathbf{A}) = n$. Now, since the inverse algorithm is valid, we have the following:

**Theorem 2.15** *An* $n \times n$ *matrix* $\mathbf{A}$ *is nonsingular if and only if* $\text{rank}(\mathbf{A}) = n$.

## Solving a System Using the Inverse of the Coefficient Matrix

The following result gives another method for solving certain linear systems:

**Theorem 2.16** *Let* $\mathbf{AX} = \mathbf{B}$ *represent a system where the coefficient matrix* $\mathbf{A}$ *is square.*
**(1)** *If* $\mathbf{A}$ *is nonsingular, then the system has a unique solution* $\left(\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}\right)$.
**(2)** *If* $\mathbf{A}$ *is singular, then the system either has no solutions or an infinite number of solutions.*
*Hence,* $\mathbf{AX} = \mathbf{B}$ *has a unique solution* if and only if $\mathbf{A}$ *is nonsingular.*

*Proof.* If $\mathbf{A}$ is nonsingular, then $\mathbf{A}^{-1}\mathbf{B}$ is a solution for the system $\mathbf{AX} = \mathbf{B}$ because $\mathbf{A}(\mathbf{A}^{-1}\mathbf{B}) = (\mathbf{AA}^{-1})\mathbf{B} = \mathbf{I}_n\mathbf{B} = \mathbf{B}$. To show that this solution is unique, suppose $\mathbf{Y}$ is any solution to the system; that is, suppose that $\mathbf{AY} = \mathbf{B}$. Then we can multiply both sides of $\mathbf{AY} = \mathbf{B}$ on the left by $\mathbf{A}^{-1}$ to get

$$\begin{aligned}
\mathbf{A}^{-1}(\mathbf{AY}) = \mathbf{A}^{-1}\mathbf{B} \quad &\Longrightarrow \quad \left(\mathbf{A}^{-1}\mathbf{A}\right)\mathbf{Y} = \mathbf{A}^{-1}\mathbf{B} \\
&\Longrightarrow \quad \mathbf{I}_n\mathbf{Y} = \mathbf{A}^{-1}\mathbf{B} \\
&\Longrightarrow \quad \mathbf{Y} = \mathbf{A}^{-1}\mathbf{B}.
\end{aligned}$$

Therefore, $\mathbf{A}^{-1}\mathbf{B}$ is the only solution of $\mathbf{AX} = \mathbf{B}$.

On the other hand, if $\mathbf{A}$ is a singular $n \times n$ matrix, then by Theorem 2.15, $\text{rank}(\mathbf{A}) < n$, so not every column of $\mathbf{A}$ becomes a pivot column when we row reduce the augmented matrix $[\mathbf{A}|\mathbf{B}]$. Now, suppose $\mathbf{AX} = \mathbf{B}$ has at least one solution. Then this system has at least one independent variable (which can take on any real value), and hence, the system has an infinite number of solutions. □

Theorem 2.16 indicates that when $\mathbf{A}^{-1}$ is known, the matrix $\mathbf{X}$ of variables can be found by a simple matrix multiplication of $\mathbf{A}^{-1}$ and $\mathbf{B}$.

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### Example 6

Consider the $3 \times 3$ system

$$\begin{cases} -7x_1 + 5x_2 + 3x_3 = \phantom{-}6 \\ \phantom{-}3x_1 - 2x_2 - 2x_3 = -3 \\ \phantom{-}3x_1 - 2x_2 - \phantom{2}x_3 = \phantom{-}2 \end{cases}; \quad \text{that is,} \quad \underbrace{\begin{bmatrix} -7 & 5 & 3 \\ 3 & -2 & -2 \\ 3 & -2 & -1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{X}} = \underbrace{\begin{bmatrix} 6 \\ -3 \\ 2 \end{bmatrix}}_{\mathbf{B}}.$$

We will solve this system using the inverse

$$\mathbf{A}^{-1} = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 2 & 5 \\ 0 & -1 & 1 \end{bmatrix}$$

of the coefficient matrix. By Theorem 2.16, $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$, and so

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 2 & 5 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 17 \\ 22 \\ 5 \end{bmatrix}.$$

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This method for solving an $n \times n$ system is not as efficient as the Gauss-Jordan Method because it involves finding an inverse as well as performing a matrix multiplication. It is sometimes used when many systems, all having the same nonsingular coefficient matrix, must be solved. In that case, the inverse of the coefficient matrix can be calculated first, and then each system can be solved with a single matrix multiplication.

- ♦ **Application**: You have now covered the prerequisites for Section 8.5, "Hill Substitution: An Introduction to Coding Theory."
- ♦ **Supplemental Material:** You have now covered the prerequisites for Appendix D, "Elementary Matrices."
- ♦ **Numerical Method:** You have now covered the prerequisites for Section 9.2, "**LDU** Decomposition."

## New Vocabulary

determinant (of a $2 \times 2$ matrix)
Inverse Method
inverse (multiplicative) of a matrix
nonsingular matrix
singular matrix

## Highlights

- If a (square) matrix has a (multiplicative) inverse (that is, if the matrix is nonsingular), then that inverse is unique.
- If $\mathbf{A}$ is a nonsingular matrix, $\mathbf{A}^{-k} = \left(\mathbf{A}^k\right)^{-1} = \left(\mathbf{A}^{-1}\right)^k$. That is, the $(-k)$th power of $\mathbf{A}$ is the inverse of the $k$th power of $\mathbf{A}$ and also the $k$th power of the inverse of $\mathbf{A}$.
- If $\mathbf{A}$ and $\mathbf{B}$ are nonsingular $n \times n$ matrices, then $(\mathbf{AB})$ is nonsingular and $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.
- If $\mathbf{A}$ is a nonsingular matrix, then $\mathbf{A}^T$ is nonsingular and $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$.
- The inverse of a $2 \times 2$ matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\frac{1}{(ad-bc)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
- If an $n \times n$ matrix $\mathbf{A}$ has an inverse, it can be found by row reducing $[\mathbf{A} \mid \mathbf{I}_n]$ to obtain $\left[\mathbf{I}_n \mid \mathbf{A}^{-1}\right]$. If $[\mathbf{A} \mid \mathbf{I}_n]$ cannot be row reduced to $\left[\mathbf{I}_n \mid \mathbf{A}^{-1}\right]$, then $\mathbf{A}$ has no inverse (that is, $\mathbf{A}$ is singular).
- An $n \times n$ matrix $\mathbf{A}$ is nonsingular if and only if $\text{rank}(\mathbf{A}) = n$.
- If $\mathbf{A}$ is nonsingular, then $\mathbf{AX} = \mathbf{B}$ has the unique solution $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$. If $\mathbf{A}$ is singular, then $\mathbf{AX} = \mathbf{B}$ has either no solution or infinitely many solutions.

## Exercises for Section 2.4

**Note**: You should be using a calculator or appropriate computer software to perform nontrivial row reductions.

1. Verify that the following pairs of matrices are inverses of each other:

   (a) $\begin{bmatrix} 10 & 41 & -5 \\ -1 & -12 & 1 \\ 3 & 20 & -2 \end{bmatrix}, \begin{bmatrix} 4 & -18 & -19 \\ 1 & -5 & -5 \\ 16 & -77 & -79 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & 0 & -1 & 5 \\ -1 & 1 & 0 & -3 \\ 0 & 2 & -3 & 7 \\ 2 & -1 & -2 & 12 \end{bmatrix}, \begin{bmatrix} 1 & -4 & 1 & -2 \\ 4 & 6 & -2 & 1 \\ 5 & 11 & -4 & 3 \\ 1 & 3 & -1 & 1 \end{bmatrix}$

2. Determine whether each of the following matrices is nonsingular by calculating its rank:

   ★ (a) $\begin{bmatrix} 4 & -9 \\ -2 & 3 \end{bmatrix}$

   (b) $\begin{bmatrix} 5 & 2 & 1 \\ 4 & -7 & 3 \\ 6 & 1 & 2 \end{bmatrix}$

   ★ (c) $\begin{bmatrix} -6 & -6 & 1 \\ 2 & 3 & -1 \\ 8 & 6 & -1 \end{bmatrix}$

   (d) $\begin{bmatrix} -1 & 1 & 2 & -7 \\ 0 & -1 & 1 & -4 \\ 1 & -3 & 0 & -1 \\ -1 & 1 & 1 & -4 \end{bmatrix}$

   ★ (e) $\begin{bmatrix} 2 & 1 & -7 & 14 \\ -6 & -3 & 19 & -38 \\ 1 & 0 & -3 & 6 \\ 2 & 1 & -6 & 12 \end{bmatrix}$

3. Find the inverse, if it exists, for each of the following $2 \times 2$ matrices:

   ★ (a) $\begin{bmatrix} 4 & 2 \\ 9 & -3 \end{bmatrix}$

   (b) $\begin{bmatrix} 9 & -6 \\ 6 & -4 \end{bmatrix}$

   ★ (c) $\begin{bmatrix} -3 & 5 \\ -12 & -8 \end{bmatrix}$

   (d) $\begin{bmatrix} 2 & 3 \\ 7 & -5 \end{bmatrix}$

★ **(e)** $\begin{bmatrix} -6 & 12 \\ 4 & -8 \end{bmatrix}$

**(f)** $\begin{bmatrix} -\frac{3}{5} & -\frac{1}{10} \\ \frac{3}{2} & \frac{1}{4} \end{bmatrix}$

**4.** Use row reduction to find the inverse, if it exists, for each of the following:

★ **(a)** $\begin{bmatrix} -4 & 7 & 6 \\ 3 & -5 & -4 \\ -2 & 4 & 3 \end{bmatrix}$

**(b)** $\begin{bmatrix} 5 & 7 & -6 \\ 3 & 1 & -2 \\ 1 & -5 & 2 \end{bmatrix}$

★ **(c)** $\begin{bmatrix} 2 & -2 & 3 \\ 8 & -4 & 9 \\ -4 & 6 & -9 \end{bmatrix}$

**(d)** $\begin{bmatrix} 1 & -1 & 1 & 2 \\ -7 & 5 & -10 & -19 \\ -2 & 1 & -2 & -4 \\ 3 & -2 & 4 & 8 \end{bmatrix}$

★ **(e)** $\begin{bmatrix} 2 & 0 & -1 & 3 \\ 1 & -2 & 3 & 1 \\ 4 & 1 & 0 & -1 \\ 1 & 3 & -2 & -5 \end{bmatrix}$

**(f)** $\begin{bmatrix} 4 & -1 & 3 & -6 \\ -3 & 1 & -5 & 10 \\ 10 & -2 & 0 & 1 \\ 1 & 0 & -3 & 6 \end{bmatrix}$

**5.** Assuming that all main diagonal entries are nonzero, find the inverse of each of the following:

**(a)** $\begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}$

**(b)** $\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$

★ **(c)** $\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$

★ **6.** The following matrices are useful in computer graphics for rotating vectors (see Section 5.1). Find the inverse of each matrix, and then state what the matrix and its inverse are when $\theta = \frac{\pi}{6}, \frac{\pi}{4}$, and $\frac{\pi}{2}$.

**(a)** $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

**(b)** $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(Hint: Modify your answer from part (a).)

**7.** In each case, find the inverse of the coefficient matrix and use it to solve the system by matrix multiplication.

★ **(a)** $\begin{cases} 5x_1 - x_2 = 20 \\ -7x_1 + 2x_2 = -31 \end{cases}$

**(b)** $\begin{cases} 2x_1 + 3x_2 - 5x_3 = 27 \\ 4x_1 + x_2 + 6x_3 = 41 \\ 8x_1 + 7x_2 - x_3 = 101 \end{cases}$

★ **(c)** $\begin{cases} -2x_2 + 5x_3 + x_4 = 25 \\ -7x_1 - 4x_2 + 5x_3 + 22x_4 = -15 \\ 5x_1 + 3x_2 - 4x_3 - 16x_4 = 9 \\ -3x_1 - x_2 + 9x_4 = -16 \end{cases}$

★ **8.** A matrix with the property $\mathbf{A}^2 = \mathbf{I}_n$ is called an **involutory** matrix.
**(a)** Find an example of a $2 \times 2$ involutory matrix other than $\mathbf{I}_2$.
**(b)** Find an example of a $3 \times 3$ involutory matrix other than $\mathbf{I}_3$.
**(c)** What is $\mathbf{A}^{-1}$ if $\mathbf{A}$ is involutory?

**9.** This exercise explores properties of singular and nonsingular matrices.
**(a)** Give an example to show that $\mathbf{A} + \mathbf{B}$ can be singular if $\mathbf{A}$ and $\mathbf{B}$ are both nonsingular.
**(b)** Give an example to show that $\mathbf{A} + \mathbf{B}$ can be nonsingular if $\mathbf{A}$ and $\mathbf{B}$ are both singular.
**(c)** Give an example to show that even when $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{A} + \mathbf{B}$ are all nonsingular, $(\mathbf{A} + \mathbf{B})^{-1}$ is not necessarily equal to $\mathbf{A}^{-1} + \mathbf{B}^{-1}$.

★ **10.** Let $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ be $n \times n$ matrices.
**(a)** Suppose that $\mathbf{AB} = \mathbf{O}_n$, and $\mathbf{A}$ is nonsingular. What must $\mathbf{B}$ be?
**(b)** If $\mathbf{AB} = \mathbf{I}_n$, is it possible for $\mathbf{AC}$ to equal $\mathbf{O}_n$ if $\mathbf{C} \neq \mathbf{O}_n$? Why or why not?

★ **11.** If $\mathbf{A}^4 = \mathbf{I}_n$, but $\mathbf{A} \neq \mathbf{I}_n$, $\mathbf{A}^2 \neq \mathbf{I}_n$, and $\mathbf{A}^3 \neq \mathbf{I}_n$, which integral powers of $\mathbf{A}$ are equal to $\mathbf{A}^{-1}$?

★ **12.** For $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$, if the matrix product $\mathbf{A}^{-1}\mathbf{B}$ is known, how can we calculate $\mathbf{B}^{-1}\mathbf{A}$ without necessarily knowing $\mathbf{A}$ and $\mathbf{B}$?

**13.** Let $\mathbf{A}$ be a symmetric nonsingular matrix. Prove that $\mathbf{A}^{-1}$ is symmetric.

**14.** This exercise explores properties of certain special types of matrices.

★ **(a)** Prove that every $n \times n$ matrix containing a row of zeroes or a column of zeroes is singular. (Hint: First prove this result for a column of zeroes. Then for a row of zeroes, use a proof by contradiction together with part (4) of Theorem 2.12.)

**(b)** Why must every diagonal matrix with at least one zero main diagonal entry be singular?

**(c)** Why must every upper triangular matrix with no zero entries on the main diagonal be nonsingular?

**(d)** Use part (c) and the transpose to show that every lower triangular matrix with no zero entries on the main diagonal must be nonsingular.

**(e)** Prove that if $\mathbf{A}$ is an upper triangular matrix with no zero entries on the main diagonal, then $\mathbf{A}^{-1}$ is upper triangular. (Hint: As $[\mathbf{A}|\,\mathbf{I}_n]$ is row reduced to $\left[\mathbf{I}_n|\,\mathbf{A}^{-1}\right]$, consider the effect on the entries in the rightmost columns.)

**15.** This exercise asks for proofs for parts of Theorem 2.12.

▶ **(a)** Prove parts (1) and (2) of Theorem 2.12. (Hint: In proving part (2), consider the cases $k > 0$ and $k \leq 0$ separately.)

**(b)** Use the method of induction to prove the following generalization of part (3) of Theorem 2.12: if $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_m$ are nonsingular matrices of the same size, then $(\mathbf{A}_1\mathbf{A}_2\cdots\mathbf{A}_m)^{-1} = \mathbf{A}_m^{-1}\cdots\mathbf{A}_2^{-1}\mathbf{A}_1^{-1}$.

**16.** If $\mathbf{A}$ is a nonsingular matrix and $c \in \mathbb{R}$ with $c \neq 0$, prove that $(c\mathbf{A})^{-1} = \left(\frac{1}{c}\right)\mathbf{A}^{-1}$.

**17.** This exercise asks for proofs for parts of Theorem 2.13.

▶ **(a)** Prove part (1) of Theorem 2.13 if $s < 0$ and $t < 0$.

**(b)** Prove part (2) of Theorem 2.13 if $s \geq 0$ and $t < 0$.

**18.** Assume that $\mathbf{A}$ and $\mathbf{B}$ are nonsingular $n \times n$ matrices. Prove that $\mathbf{A}$ and $\mathbf{B}$ commute (that is, $\mathbf{AB} = \mathbf{BA}$) if and only if $(\mathbf{AB})^2 = \mathbf{A}^2\mathbf{B}^2$.

**19.** Prove that if $\mathbf{A}$ and $\mathbf{B}$ are nonsingular matrices of the same size, then $\mathbf{AB} = \mathbf{BA}$ if and only if $(\mathbf{AB})^q = \mathbf{A}^q\mathbf{B}^q$ for every positive integer $q \geq 2$. (Hint: To prove the "if" part, let $q = 2$. For the "only if" part, first show by induction that if $\mathbf{AB} = \mathbf{BA}$, then $\mathbf{AB}^q = \mathbf{B}^q\mathbf{A}$, for any positive integer $q \geq 2$. Finish the proof with a second induction argument to show $(\mathbf{AB})^q = \mathbf{A}^q\mathbf{B}^q$.)

**20.** Prove that if $\mathbf{A}$ is an $n \times n$ matrix and $\mathbf{A} - \mathbf{I}_n$ is nonsingular, then for every integer $k \geq 0$,

$$\mathbf{I}_n + \mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 + \cdots + \mathbf{A}^k = \left(\mathbf{A}^{k+1} - \mathbf{I}_n\right)(\mathbf{A} - \mathbf{I}_n)^{-1}.$$

★ **21.** Let $\mathbf{A}$ be an $n \times k$ matrix and $\mathbf{B}$ be a $k \times n$ matrix. Prove that if $\mathbf{AB} = \mathbf{I}_n$, then $n \leq k$. (Hint: Use a proof by contradiction. Suppose $\mathbf{AB} = \mathbf{I}_n$ and $n > k$. Use Corollary 2.3 to show that there is a nontrivial $\mathbf{X}$ such that $\mathbf{BX} = \mathbf{0}$. Then compute $\mathbf{ABX}$ two different ways to obtain a contradiction.)

★ **22.** True or False:

**(a)** Every $n \times n$ matrix $\mathbf{A}$ has a unique inverse.

**(b)** If $\mathbf{A}, \mathbf{B}$ are $n \times n$ matrices, and $\mathbf{BA} = \mathbf{I}_n$, then $\mathbf{A}$ and $\mathbf{B}$ are inverses of each other.

**(c)** If $\mathbf{A}, \mathbf{B}$ are nonsingular $n \times n$ matrices, then $((\mathbf{AB})^T)^{-1} = (\mathbf{A}^{-1})^T(\mathbf{B}^{-1})^T$.

**(d)** If $\mathbf{A}$ is a singular $2 \times 2$ matrix, then $a_{11}a_{22} - a_{12}a_{21} \neq 0$.

**(e)** If $\mathbf{A}$ is an $n \times n$ matrix, then $\mathbf{A}$ is nonsingular if and only if $[\mathbf{A}|\,\mathbf{I}_n]$ has fewer than $n$ pivots before the augmentation bar after row reduction.

**(f)** If $\mathbf{A}$ is an $n \times n$ matrix, then $\text{rank}(\mathbf{A}) = n$ if and only if any system of the form $\mathbf{AX} = \mathbf{B}$ has a unique solution for $\mathbf{X}$.

## Review Exercises for Chapter 2

**1.** For each of the following linear systems,

★ **(i)** Use Gaussian Elimination to give the complete solution set.

**(ii)** Use the Gauss-Jordan Method to give the complete solution set and the correct staircase pattern for the row reduced echelon form of the augmented matrix for the system.

**(a)** $\begin{cases} 2x_1 + 5x_2 - 4x_3 = 48 \\ x_1 - 3x_2 + 2x_3 = -40 \\ -3x_1 + 4x_2 + 7x_3 = 15 \\ -2x_1 + 3x_2 - x_3 = 41 \end{cases}$

**(b)** $\begin{cases} 4x_1 + 3x_2 - 7x_3 + 5x_4 = 31 \\ -2x_1 - 3x_2 + 5x_3 - x_4 = -5 \\ 2x_1 - 6x_2 - 2x_3 + 3x_4 = 52 \\ 6x_1 - 21x_2 - 3x_3 + 12x_4 = 16 \end{cases}$

**(c)** $\begin{cases} 6x_1 - 2x_2 + 2x_3 - x_4 - 6x_5 = -33 \\ -2x_1 + x_2 + 2x_4 - x_5 = 13 \\ 4x_1 - x_2 + 2x_3 - 3x_4 + x_5 = -24 \end{cases}$

★ **2.** Find the cubic equation that goes through the points $(-3, 120)$, $(-2, 51)$, $(3, -24)$, and $(4, -69)$.

**3.** Are the following matrices in reduced row echelon form? If not, explain why not.

**(a)** $\begin{bmatrix} 1 & -5 & 2 & -4 & -2 \\ 0 & 1 & -3 & 4 & -1 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 \end{bmatrix}$

★ **4.** Find minimal integer values for the variables that will satisfy the following chemical equation: $a\ NH_3 + b\ O_2 \longrightarrow c\ NO_2 + d\ H_2O$. ($NH_3$ = ammonia; $NO_2$ = nitrogen dioxide).

**5.** In each part, solve the two linear systems simultaneously:

**(a)** **(i)** $\begin{cases} 2x_1 + 7x_2 - 5x_3 = 3 \\ 4x_1 + 13x_2 - 11x_3 = -5 \\ 3x_1 - 2x_2 = 27 \end{cases}$; **(ii)** $\begin{cases} 2x_1 + 7x_2 - 5x_3 = 148 \\ 4x_1 + 13x_2 - 11x_3 = 278 \\ 3x_1 - 2x_2 = 97 \end{cases}$

**(b)** **(i)** $\begin{cases} 4x_1 + x_2 + 29x_3 = 175 \\ 2x_1 + x_2 + 17x_3 = 103 \\ x_1 + x_2 + 11x_3 = 67 \end{cases}$; **(ii)** $\begin{cases} 4x_1 + x_2 + 29x_3 = 83 \\ 2x_1 + x_2 + 17x_3 = 51 \\ x_1 + x_2 + 11x_3 = 37 \end{cases}$

**6.** Without row reducing, explain why the following homogeneous system has an infinite number of solutions:

$$\begin{cases} 2x_1 + x_2 - 3x_3 + x_4 = 0 \\ x_1 - 3x_2 - 2x_3 - 2x_4 = 0 \\ -3x_1 + 4x_2 + x_3 - 3x_4 = 0 \end{cases}$$

**7.** What is the inverse of each of the following row operations?

**(a)** (I): $\langle 3 \rangle \leftarrow -\frac{5}{6}\langle 3 \rangle$

**(b)** (II): $\langle 2 \rangle \leftarrow 7\langle 4 \rangle + \langle 2 \rangle$

**(c)** (III): $\langle 1 \rangle \leftrightarrow \langle 3 \rangle$

★ **8.** This exercise illustrates the relationship between the number of solutions of a homogeneous system and the rank of its corresponding matrix.

**(a)** Find the rank of each of the following matrices:

$$\mathbf{A} = \begin{bmatrix} 2 & -5 & 3 \\ -1 & -3 & 4 \\ 7 & -12 & 5 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 & 0 & -2 & -1 \\ -2 & 0 & -1 & 0 \\ -1 & -2 & -1 & -5 \\ 0 & 1 & 1 & 3 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 3 & -1 & -5 & -6 \\ 0 & 4 & 8 & -2 \\ -2 & -3 & -4 & 0 \end{bmatrix}$$

**(b)** Using the ranks of the matrices in part (a), determine the number of solutions for each of the systems $\mathbf{AX} = \mathbf{0}$, $\mathbf{BX} = \mathbf{0}$, $\mathbf{CX} = \mathbf{0}$.

**9.** Determine whether the following matrices **A** and **B** are row equivalent. (Hint: Do they have the same row reduced echelon form matrix?)

$$\mathbf{A} = \begin{bmatrix} 5 & 3 & 5 & 1 & 24 \\ 0 & 1 & -5 & 1 & 7 \\ 2 & -1 & 13 & -2 & -7 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 & 1 & -1 & -1 & -2 \\ 3 & 1 & 7 & -2 & -2 \\ 3 & 0 & 12 & -1 & 3 \end{bmatrix}$$

★ **10.** This exercise explores the definition of row space using a specific example.

**(a)** Determine whether $[-34, 29, -21]$ is a linear combination of $\mathbf{x}_1 = [2, -3, -1]$, $\mathbf{x}_2 = [5, -2, 1]$, and $\mathbf{x}_3 = [9, -8, 3]$.

(b) From your answer to part (a), is $[-34, 29, -21]$ in the row space of $\mathbf{A} = \begin{bmatrix} 2 & -3 & -1 \\ 5 & -2 & 1 \\ 9 & -8 & 3 \end{bmatrix}$?

11. Without using row reduction, state the inverse of the matrix $\mathbf{A} = \begin{bmatrix} 7 & -2 \\ 3 & 1 \end{bmatrix}$.

12. Find the inverse (if it exists) for each of the following matrices, and indicate whether the matrix is nonsingular.

(a) $\mathbf{A} = \begin{bmatrix} 5 & 4 & 2 \\ -3 & 4 & 6 \\ 2 & -5 & -6 \end{bmatrix}$

★ (b) $\mathbf{B} = \begin{bmatrix} 3 & 4 & 3 & 5 \\ 4 & 5 & 5 & 8 \\ 7 & 9 & 8 & 13 \\ 2 & 3 & 2 & 3 \end{bmatrix}$

13. Prove that an $n \times n$ matrix $\mathbf{A}$ is nonsingular if and only if $\mathbf{A}$ is row equivalent to $\mathbf{I}_n$.

14. If $\mathbf{A}^{-1}$ exists, does the homogeneous system $\mathbf{AX} = \mathbf{0}$ have a nontrivial solution? Why or why not?

★ 15. Find the solution set for the following linear system by calculating the inverse of the coefficient matrix and then using matrix multiplication:

$$\begin{cases} 4x_1 - 6x_2 + x_3 = \phantom{-}17 \\ -x_1 + 2x_2 - x_3 = -14 \\ 3x_1 - 5x_2 + x_3 = \phantom{-}23 \end{cases}.$$

16. Let $\mathbf{A}$ be an $m \times n$ matrix, let $\mathbf{B}$ be a nonsingular $m \times m$ matrix, and $\mathbf{C}$ be a nonsingular $n \times n$ matrix.

(a) Use Theorem 2.1 and Exercise 17 in Section 2.3 to show that $\text{rank}(\mathbf{BA}) = \text{rank}(\mathbf{A})$.

(b) Use part (d) of Exercise 18 in Section 2.3 to prove that $\text{rank}(\mathbf{AC}) = \text{rank}(\mathbf{A})$. (Hint: Set up two inequalities. One uses $(\mathbf{AC})\mathbf{C}^{-1} = \mathbf{A}$.)

★ 17. True or False:

(a) The Gauss-Jordan Method can produce *extraneous* (that is, extra) “solutions” that are not actually solutions to the original system.

(b) If $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices, and $R$ is a row operation, then $R(\mathbf{A})\mathbf{B} = \mathbf{A}R(\mathbf{B})$.

(c) If the augmented matrix $[\mathbf{A}\,|\,\mathbf{B}]$ row reduces to a matrix having a row of zeroes, then the linear system $\mathbf{AX} = \mathbf{B}$ is consistent.

(d) If $c$ is a nonzero scalar, then the linear systems $(c\mathbf{A})\mathbf{X} = c\mathbf{B}$ and $\mathbf{AX} = \mathbf{B}$ have the same solution set.

(e) If $\mathbf{A}$ is an upper triangular matrix, then $\mathbf{A}$ can be transformed into row echelon form using only Type (I) row operations.

(f) Every square reduced row echelon form matrix is upper triangular.

(g) The exact same row operations that produce the solution set for the homogeneous system $\mathbf{AX} = \mathbf{0}$ also produce the solution set for the related linear system $\mathbf{AX} = \mathbf{B}$.

(h) If a linear system has the trivial solution, then it must be a homogeneous system.

(i) It is possible for the homogeneous linear system $\mathbf{AX} = \mathbf{0}$ to have a nontrivial solution *and* the related linear system $\mathbf{AX} = \mathbf{B}$ to have a unique solution.

(j) Every row operation has a corresponding inverse row operation that “undoes” the original row operation.

(k) If the two $m \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ have the same rank, then the homogeneous linear systems $\mathbf{AX} = \mathbf{0}$ and $\mathbf{BX} = \mathbf{0}$ have the same nonempty solution set.

(l) The rank of a matrix $\mathbf{A}$ equals the number of vectors in the row space of $\mathbf{A}$.

(m) If $\mathbf{A}$ is a nonsingular matrix, then $\text{rank}(\mathbf{A}) = \text{rank}\left(\mathbf{A}^{-1}\right)$.

(n) If $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices such that $\mathbf{AB}$ is nonsingular, then $\mathbf{A}$ is nonsingular.

(o) If $\mathbf{A}$ and $\mathbf{B}$ are matrices such that $\mathbf{AB} = \mathbf{I}_n$, then $\mathbf{A}$ and $\mathbf{B}$ are square matrices.

(p) If $\mathbf{A}$ and $\mathbf{B}$ are $2 \times 2$ matrices with equal determinants, then the linear systems $\mathbf{AX} = \mathbf{0}$ and $\mathbf{BX} = \mathbf{0}$ have the same number of solutions.

(q) If $\mathbf{A}$ is an $n \times n$ nonsingular matrix, then $[\mathbf{A}\,|\,\mathbf{I}_n]$ is row equivalent to $\left[\mathbf{I}_n\,\middle|\,\mathbf{A}^{-1}\right]$.

(r) If $\mathbf{A}$ is a nonsingular matrix, then $\left(\mathbf{A}^3\right)^{-5} = \left(\mathbf{A}^{-3}\right)^5 = \left(\left(\mathbf{A}^5\right)^3\right)^{-1}$.

(s) If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are nonsingular $n \times n$ matrices, $(\mathbf{A}^{-1}\mathbf{B}^T\mathbf{C})^{-1} = \mathbf{C}^{-1}(\mathbf{B}^{-1})^T\mathbf{A}$.

Chapter 3

# Determinants and Eigenvalues

**The Determining Factor**

Amazingly, many important geometric and algebraic properties of a square matrix are revealed by a single real number associated with the matrix, known as its determinant. For example, the areas and volumes of certain figures can be found by creating a matrix based on the figure's edges and then calculating the determinant of that matrix. The determinant also provides a quick method for discovering whether certain linear systems have a unique solution.

In this chapter, we also use determinants to introduce the concept of eigenvectors. An eigenvector of a square matrix is a special vector that, when multiplied by the matrix, produces a parallel vector. Such vectors provide a new way to look at matrix multiplication, and help to solve many intractable problems. Eigenvectors are practical tools in linear algebra with applications in differential equations, probability, statistics, and in related disciplines such as economics, physics, chemistry, and computer graphics.

## 3.1 Introduction to Determinants

### Determinants of 1 × 1, 2 × 2, and 3 × 3 Matrices

For a $1 \times 1$ matrix $\mathbf{A} = [a_{11}]$, the **determinant** $|\mathbf{A}|$ is defined to be $a_{11}$, its only entry. For example, the determinant of $\mathbf{A} = [-4]$ is simply $|\mathbf{A}| = -4$. We will represent a determinant by placing absolute value signs around the matrix, even though the determinant could be negative.[1]

For a $2 \times 2$ matrix $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, the **determinant** $|\mathbf{A}|$ is defined to be $a_{11}a_{22} - a_{12}a_{21}$. For example, the determinant of $\mathbf{A} = \begin{bmatrix} 4 & -3 \\ 2 & 5 \end{bmatrix}$ is $|\mathbf{A}| = \begin{vmatrix} 4 & -3 \\ 2 & 5 \end{vmatrix} = (4)(5) - (-3)(2) = 26$. Recall that in Section 2.4 we proved $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ has an inverse if and only if $|\mathbf{A}| = a_{11}a_{22} - a_{12}a_{21} \neq 0$.

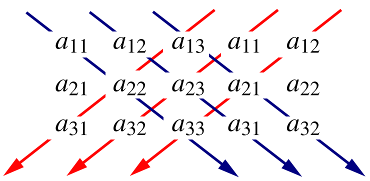
For the $3 \times 3$ matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

we define the **determinant** $|\mathbf{A}|$ to be the following expression, which has six terms:

$$|\mathbf{A}| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}.$$

This expression may look complicated, but its terms can be obtained by multiplying the following entries linked by arrows. Notice that the first two columns of the original $3 \times 3$ matrix have been repeated. Also, the arrows pointing right indicate terms with a positive sign, while those pointing left indicate terms with a negative sign.




This technique is sometimes referred to as the **basketweaving** method for calculating the determinant of a $3 \times 3$ matrix.

[1] The notation $\det(\mathbf{A})$ is sometimes used instead of $|\mathbf{A}|$ for the determinant of a matrix $\mathbf{A}$ in many textbooks as well as in some software packages.

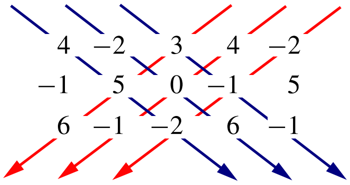
**Elementary Linear Algebra.** https://doi.org/10.1016/B978-0-12-822978-1.00013-4

**Example 1**

Find the determinant of

$$\mathbf{A} = \begin{bmatrix} 4 & -2 & 3 \\ -1 & 5 & 0 \\ 6 & -1 & -2 \end{bmatrix}.$$

Repeating the first two columns and forming terms using the basketweaving method, we have




which gives

$$(4)(5)(-2) + (-2)(0)(6) + (3)(-1)(-1) - (3)(5)(6) - (4)(0)(-1) - (-2)(-1)(-2).$$

This reduces to $-40 + 0 + 3 - 90 - 0 - (-4) = -123$. Thus,

$$|\mathbf{A}| = \begin{vmatrix} 4 & -2 & 3 \\ -1 & 5 & 0 \\ 6 & -1 & -2 \end{vmatrix} = -123.$$

## Application: Areas and Volumes

The next theorem illustrates why $2 \times 2$ and $3 \times 3$ determinants are sometimes interpreted as areas and volumes, respectively.

**Theorem 3.1**

**(1)** *Let* $\mathbf{x} = [x_1, x_2]$ *and* $\mathbf{y} = [y_1, y_2]$ *be two nonparallel vectors in* $\mathbb{R}^2$ *beginning at a common point (see Fig. 3.1(a)). Then the area of the parallelogram determined by* **x** *and* **y** *is the absolute value of the determinant*

$$\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}.$$

**(2)** *Let* $\mathbf{x} = [x_1, x_2, x_3]$, $\mathbf{y} = [y_1, y_2, y_3]$, *and* $\mathbf{z} = [z_1, z_2, z_3]$ *be three vectors not all in the same plane beginning at a common initial point (see Fig. 3.1(b)). Then the volume of the parallelepiped determined by* **x**, **y**, *and* **z** *is the absolute value of the determinant*

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}.$$

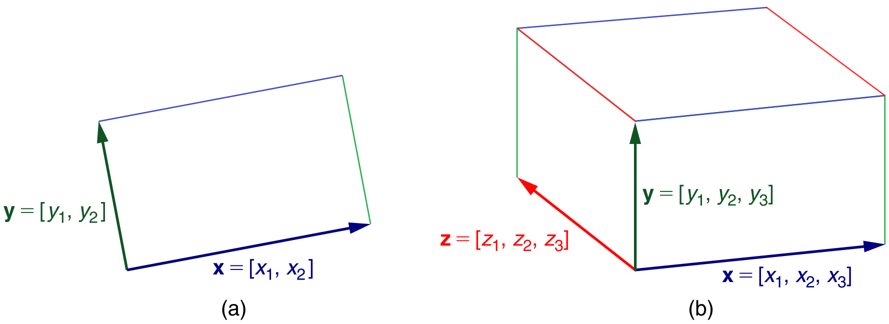



**FIGURE 3.1** (a) The parallelogram determined by **x** and **y** (Theorem 3.1); (b) the parallelepiped determined by **x**, **y**, and **z** (Theorem 3.1)

The proof of this theorem is straightforward (see Exercises 10 and 12).

**Example 2**

The volume of the parallelepiped whose sides are $\mathbf{x} = [-2, 1, 3]$, $\mathbf{y} = [3, 0, -2]$, and $\mathbf{z} = [-1, 3, 7]$ is given by the absolute value of the determinant

$$\begin{vmatrix} -2 & 1 & 3 \\ 3 & 0 & -2 \\ -1 & 3 & 7 \end{vmatrix}.$$

Calculating this determinant, we obtain $-4$, so the volume is $|-4| = 4$.

## Cofactors

Before defining determinants for square matrices larger than $3 \times 3$, we first introduce a few new terms.

**Definition** Let $\mathbf{A}$ be an $n \times n$ matrix, with $n \geq 2$. The $(i, j)$ **submatrix**, $\mathbf{A}_{ij}$, of $\mathbf{A}$, is the $(n-1) \times (n-1)$ matrix obtained by deleting all entries of the $i$th row and all entries of the $j$th column of $\mathbf{A}$. The $(i, j)$ **minor**, $|\mathbf{A}_{ij}|$, of $\mathbf{A}$, is the determinant of the submatrix $\mathbf{A}_{ij}$ of $\mathbf{A}$.

**Example 3**

Consider the following matrices:

$$\mathbf{A} = \begin{bmatrix} 5 & -2 & 1 \\ 0 & 4 & -3 \\ 2 & -7 & 6 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 9 & -1 & 4 & 7 \\ -3 & 2 & 6 & -2 \\ -8 & 0 & 1 & 3 \\ 4 & 7 & -5 & -1 \end{bmatrix}.$$

The $(1, 3)$ submatrix of $\mathbf{A}$ obtained by deleting all entries in the 1st row and all entries in the 3rd column is $\mathbf{A}_{13} = \begin{bmatrix} 0 & 4 \\ 2 & -7 \end{bmatrix}$, and the $(3, 4)$ submatrix of $\mathbf{B}$ obtained by deleting all entries in the 3rd row and all entries in the 4th column is

$$\mathbf{B}_{34} = \begin{bmatrix} 9 & -1 & 4 \\ -3 & 2 & 6 \\ 4 & 7 & -5 \end{bmatrix}.$$

The corresponding minors associated with these submatrices are

$$|\mathbf{A}_{13}| = \begin{vmatrix} 0 & 4 \\ 2 & -7 \end{vmatrix} = -8 \quad \text{and} \quad |\mathbf{B}_{34}| = \begin{vmatrix} 9 & -1 & 4 \\ -3 & 2 & 6 \\ 4 & 7 & -5 \end{vmatrix} = -593.$$

An $n \times n$ matrix has a total of $n^2$ minors—one for each entry of the matrix. In particular, a $3 \times 3$ matrix has nine minors. For the matrix $\mathbf{A}$ in Example 3, the minors are

$$|\mathbf{A}_{11}| = \begin{vmatrix} 4 & -3 \\ -7 & 6 \end{vmatrix} = 3 \qquad |\mathbf{A}_{12}| = \begin{vmatrix} 0 & -3 \\ 2 & 6 \end{vmatrix} = 6 \qquad |\mathbf{A}_{13}| = \begin{vmatrix} 0 & 4 \\ 2 & -7 \end{vmatrix} = -8$$

$$|\mathbf{A}_{21}| = \begin{vmatrix} -2 & 1 \\ -7 & 6 \end{vmatrix} = -5 \qquad |\mathbf{A}_{22}| = \begin{vmatrix} 5 & 1 \\ 2 & 6 \end{vmatrix} = 28 \qquad |\mathbf{A}_{23}| = \begin{vmatrix} 5 & -2 \\ 2 & -7 \end{vmatrix} = -31$$

$$|\mathbf{A}_{31}| = \begin{vmatrix} -2 & 1 \\ 4 & -3 \end{vmatrix} = 2 \qquad |\mathbf{A}_{32}| = \begin{vmatrix} 5 & 1 \\ 0 & -3 \end{vmatrix} = -15 \qquad |\mathbf{A}_{33}| = \begin{vmatrix} 5 & -2 \\ 0 & 4 \end{vmatrix} = 20.$$

We now define a "cofactor" for each entry based on its minor.

**Definition** Let $\mathbf{A}$ be an $n \times n$ matrix, with $n \geq 2$. The $(i, j)$ **cofactor** of $\mathbf{A}$, $\mathcal{A}_{ij}$, is $(-1)^{i+j}$ times the $(i, j)$ minor of $\mathbf{A}$—that is, $\mathcal{A}_{ij} = (-1)^{i+j} |\mathbf{A}_{ij}|$.

**Example 4**

For the matrices **A** and **B** in Example 3, the cofactor $\mathcal{A}_{13}$ of **A** is

$$\mathcal{A}_{13} = (-1)^{1+3}\,|\mathbf{A}_{13}| = (-1)^4(-8) = -8,$$

and the cofactor $\mathcal{B}_{34}$ of **B** is

$$\mathcal{B}_{34} = (-1)^{3+4}\,|\mathbf{B}_{34}| = (-1)^7(-593) = 593.$$

An $n \times n$ matrix has $n^2$ cofactors, one for each matrix entry. In particular, a $3 \times 3$ matrix has nine cofactors. For the matrix **A** from Example 3, these cofactors are

$$\begin{aligned}
\mathcal{A}_{11} &= (-1)^{1+1}\,|\mathbf{A}_{11}| = (-1)^2\,(3) &&= \phantom{-}3\\
\mathcal{A}_{12} &= (-1)^{1+2}\,|\mathbf{A}_{12}| = (-1)^3\,(6) &&= -6\\
\mathcal{A}_{13} &= (-1)^{1+3}\,|\mathbf{A}_{13}| = (-1)^4\,(-8) &&= -8\\
\mathcal{A}_{21} &= (-1)^{2+1}\,|\mathbf{A}_{21}| = (-1)^3\,(-5) &&= \phantom{-}5\\
\mathcal{A}_{22} &= (-1)^{2+2}\,|\mathbf{A}_{22}| = (-1)^4\,(28) &&= \phantom{-}28\\
\mathcal{A}_{23} &= (-1)^{2+3}\,|\mathbf{A}_{23}| = (-1)^5\,(-31) &&= \phantom{-}31\\
\mathcal{A}_{31} &= (-1)^{3+1}\,|\mathbf{A}_{31}| = (-1)^4\,(2) &&= \phantom{-}2\\
\mathcal{A}_{32} &= (-1)^{3+2}\,|\mathbf{A}_{32}| = (-1)^5\,(-15) &&= \phantom{-}15\\
\mathcal{A}_{33} &= (-1)^{3+3}\,|\mathbf{A}_{33}| = (-1)^6\,(20) &&= \phantom{-}20.
\end{aligned}$$

## Formal Definition of the Determinant

We are now ready to define the determinant of a general $n \times n$ matrix. We will see shortly that the following definition agrees with our earlier formulas for determinants of size $1 \times 1$, $2 \times 2$, and $3 \times 3$.

**Definition** Let **A** be an $n \times n$ (square) matrix. The **determinant** of **A**, denoted $|\mathbf{A}|$, is defined as follows:
If $n = 1$ (so that $\mathbf{A} = [a_{11}]$), then $|\mathbf{A}| = a_{11}$.
If $n > 1$, then $|\mathbf{A}| = a_{n1}\mathcal{A}_{n1} + a_{n2}\mathcal{A}_{n2} + \cdots + a_{nn}\mathcal{A}_{nn}$.

For $n > 1$, this defines the determinant as a sum of products. Each entry $a_{ni}$ of the last row of the matrix **A** is multiplied by its corresponding cofactor $\mathcal{A}_{ni}$, and we sum the results. This process is often referred to as **cofactor expansion** (or **Laplace expansion**) **along the last row** of the matrix. Since the cofactors of an $n \times n$ matrix are calculated by finding determinants of appropriate $(n-1) \times (n-1)$ submatrices, we see that this definition is actually recursive. That is, we can find the determinant of any matrix once we know how to find the determinant of any smaller size matrix!

**Example 5**

Consider again the matrix from Example 3:

$$\mathbf{A} = \begin{bmatrix} 5 & -2 & 1 \\ 0 & 4 & -3 \\ 2 & -7 & 6 \end{bmatrix}.$$

Multiplying every entry of the last row by its cofactor, and summing, we have

$$|\mathbf{A}| = a_{31}\mathcal{A}_{31} + a_{32}\mathcal{A}_{32} + a_{33}\mathcal{A}_{33} = 2(2) + (-7)(15) + 6(20) = 19.$$

You can verify that using "basketweaving" also produces $|\mathbf{A}| = 19$.

Note that this new definition for the determinant agrees with the previous definitions for $2 \times 2$ and $3 \times 3$ matrices. For, if **B** is a $2 \times 2$ matrix, then cofactor expansion on **B** yields

$$\begin{aligned}|\mathbf{B}| &= b_{21}\mathcal{B}_{21} + b_{22}\mathcal{B}_{22} \\ &= b_{21}(-1)^{2+1}|\mathbf{B}_{21}| + b_{22}(-1)^{2+2}|\mathbf{B}_{22}| \\ &= -b_{21}(b_{12}) + b_{22}(b_{11}) \\ &= b_{11}b_{22} - b_{12}b_{21},\end{aligned}$$

which is correct. Similarly, if **C** is a $3 \times 3$ matrix, then

$$\begin{aligned}|\mathbf{C}| &= c_{31}\mathcal{C}_{31} + c_{32}\mathcal{C}_{32} + c_{33}\mathcal{C}_{33} \\ &= c_{31}(-1)^{3+1}|\mathbf{C}_{31}| + c_{32}(-1)^{3+2}|\mathbf{C}_{32}| + c_{33}(-1)^{3+3}|\mathbf{C}_{33}| \\ &= c_{31}\begin{vmatrix} c_{12} & c_{13} \\ c_{22} & c_{23}\end{vmatrix} - c_{32}\begin{vmatrix} c_{11} & c_{13} \\ c_{21} & c_{23}\end{vmatrix} + c_{33}\begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22}\end{vmatrix} \\ &= c_{31}(c_{12}c_{23} - c_{13}c_{22}) - c_{32}(c_{11}c_{23} - c_{13}c_{21}) + c_{33}(c_{11}c_{22} - c_{12}c_{21}) \\ &= c_{11}c_{22}c_{33} + c_{12}c_{23}c_{31} + c_{13}c_{21}c_{32} - c_{13}c_{22}c_{31} - c_{11}c_{23}c_{32} - c_{12}c_{21}c_{33},\end{aligned}$$

which agrees with the basketweaving formula for a $3 \times 3$ determinant.

We now compute the determinant of a $4 \times 4$ matrix.

**Example 6**

Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 0 & 5 \\ 4 & 1 & 3 & -1 \\ 2 & -1 & 3 & 6 \\ 5 & 0 & 2 & -1 \end{bmatrix}.$$

Then, using cofactor expansion along the last row, we have

$$\begin{aligned}|\mathbf{A}| &= a_{41}\mathcal{A}_{41} + a_{42}\mathcal{A}_{42} + a_{43}\mathcal{A}_{43} + a_{44}\mathcal{A}_{44} \\ &= 5(-1)^{4+1}|\mathbf{A}_{41}| + 0(-1)^{4+2}|\mathbf{A}_{42}| + 2(-1)^{4+3}|\mathbf{A}_{43}| + (-1)(-1)^{4+4}|\mathbf{A}_{44}| \\ &= -5\begin{vmatrix} 2 & 0 & 5 \\ 1 & 3 & -1 \\ -1 & 3 & 6\end{vmatrix} + 0 - 2\begin{vmatrix} 3 & 2 & 5 \\ 4 & 1 & -1 \\ 2 & -1 & 6\end{vmatrix} - 1\begin{vmatrix} 3 & 2 & 0 \\ 4 & 1 & 3 \\ 2 & -1 & 3\end{vmatrix}.\end{aligned}$$

At this point, we could use basketweaving to finish the calculation. Instead, we evaluate each of the remaining determinants using cofactor expansion along the last row to illustrate the recursive nature of the method. Now,

$$\begin{aligned}\begin{vmatrix} 2 & 0 & 5 \\ 1 & 3 & -1 \\ -1 & 3 & 6\end{vmatrix} &= (-1)(-1)^{3+1}\begin{vmatrix} 0 & 5 \\ 3 & -1\end{vmatrix} + 3(-1)^{3+2}\begin{vmatrix} 2 & 5 \\ 1 & -1\end{vmatrix} + 6(-1)^{3+3}\begin{vmatrix} 2 & 0 \\ 1 & 3\end{vmatrix} \\ &= (-1)(0-15) + (-3)(-2-5) + (6)(6-0) \\ &= 15 + 21 + 36 = 72,\end{aligned}$$

$$\begin{aligned}\begin{vmatrix} 3 & 2 & 5 \\ 4 & 1 & -1 \\ 2 & -1 & 6\end{vmatrix} &= (2)(-1)^{3+1}\begin{vmatrix} 2 & 5 \\ 1 & -1\end{vmatrix} + (-1)(-1)^{3+2}\begin{vmatrix} 3 & 5 \\ 4 & -1\end{vmatrix} + 6(-1)^{3+3}\begin{vmatrix} 3 & 2 \\ 4 & 1\end{vmatrix} \\ &= (2)(-2-5) + (1)(-3-20) + (6)(3-8) \\ &= -14 - 23 - 30 = -67, \quad \text{and}\end{aligned}$$

$$\begin{aligned}\begin{vmatrix} 3 & 2 & 0 \\ 4 & 1 & 3 \\ 2 & -1 & 3\end{vmatrix} &= (2)(-1)^{3+1}\begin{vmatrix} 2 & 0 \\ 1 & 3\end{vmatrix} + (-1)(-1)^{3+2}\begin{vmatrix} 3 & 0 \\ 4 & 3\end{vmatrix} + 3(-1)^{3+3}\begin{vmatrix} 3 & 2 \\ 4 & 1\end{vmatrix} \\ &= (2)(6-0) + (1)(9-0) + (3)(3-8) \\ &= 12 + 9 - 15 = 6.\end{aligned}$$

Hence, $|\mathbf{A}| = (-5)(72) - 2(-67) - 1(6) = -360 + 134 - 6 = -232$.

The computation of the $4 \times 4$ determinant in Example 6 is quite cumbersome. Finding the determinant of a $5 \times 5$ matrix would involve the computation of five $4 \times 4$ determinants! As the size of the matrix increases, the calculation of the determinant becomes increasingly tedious. In Section 3.2, we present an alternative method for calculating determinants that is computationally more efficient for larger matrices. After that, we will generally use methods other than cofactor expansion, except in cases in which enough zeroes in the matrix allow us to avoid computing many of the corresponding cofactors. (For instance, in Example 6, we did not need to calculate $\mathcal{A}_{42}$ because $a_{42} = 0$ in that case.)

## New Vocabulary

basketweaving
cofactor
cofactor expansion (along the last row of a matrix)
determinant
minor
submatrix

## Highlights

- For an $n \times n$ matrix $\mathbf{A}$ with $n \geq 2$, the $(i, j)$ minor of $\mathbf{A}$ is $|\mathbf{A}_{ij}|$, the determinant of the $(i, j)$ submatrix $\mathbf{A}_{ij}$ of $\mathbf{A}$.
- For an $n \times n$ matrix $\mathbf{A}$ with $n \geq 2$, the $(i, j)$ cofactor $\mathcal{A}_{ij}$ of $\mathbf{A}$ is $(-1)^{i+j}|\mathbf{A}_{ij}|$.
- The determinant of a $1 \times 1$ matrix $[a_{11}]$ is $a_{11}$.
- For an $n \times n$ matrix $\mathbf{A}$ with $n \geq 2$, $|\mathbf{A}| = a_{n1}\mathcal{A}_{n1} + a_{n2}\mathcal{A}_{n2} + \cdots + a_{nn}\mathcal{A}_{nn}$.
- The determinant of a $2 \times 2$ matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad - bc$.
- The determinant of a $3 \times 3$ matrix is easily found using basketweaving.
- The area of the parallelogram determined by nonparallel vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^2$ is the absolute value of the determinant of the matrix whose rows are $\mathbf{x}$ and $\mathbf{y}$.
- The volume of the parallelepiped determined by noncoplanar vectors $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ in $\mathbb{R}^3$ is the absolute value of the determinant of the matrix whose rows are $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$.

## Exercises for Section 3.1

1. Calculate the determinant of each of the following matrices using the quick formulas given at the beginning of this section:

★ (a) $\begin{bmatrix} -2 & 5 \\ 3 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 5 & -3 \\ 2 & 7 \end{bmatrix}$

★ (c) $\begin{bmatrix} 6 & -12 \\ -4 & 8 \end{bmatrix}$

(d) $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

★ (e) $\begin{bmatrix} 2 & 0 & 5 \\ -4 & 1 & 7 \\ 0 & 3 & -3 \end{bmatrix}$

(f) $\begin{bmatrix} 3 & -5 & 7 \\ 1 & 4 & -2 \\ -1 & 6 & -4 \end{bmatrix}$

★ (g) $\begin{bmatrix} 5 & 0 & 0 \\ 3 & -2 & 0 \\ -1 & 8 & 4 \end{bmatrix}$

(h) $\begin{bmatrix} -3 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{bmatrix}$

★ (i) $\begin{bmatrix} 3 & 1 & -2 \\ -1 & 4 & 5 \\ 3 & 1 & -2 \end{bmatrix}$

★ (j) $[-3]$

(k) $\begin{bmatrix} 0 & -5 & 3 \\ 5 & 0 & 2 \\ -3 & -2 & 0 \end{bmatrix}$

2. Calculate the indicated minors for each given matrix.

★ (a) $|\mathbf{A}_{21}|$, for $\mathbf{A} = \begin{bmatrix} -2 & 4 & 3 \\ 3 & -1 & 6 \\ 5 & -2 & 4 \end{bmatrix}$

(b) $|\mathbf{B}_{24}|$, for $\mathbf{B} = \begin{bmatrix} 0 & 2 & -3 & 1 \\ 1 & 4 & 2 & -1 \\ 3 & -2 & 4 & 0 \\ 4 & -1 & 1 & 0 \end{bmatrix}$

★ **(c)** $|\mathbf{C}_{42}|$, for $\mathbf{C} = \begin{bmatrix} -3 & 3 & 0 & 5 \\ 2 & 1 & -1 & 4 \\ 6 & -3 & 4 & 0 \\ -1 & 5 & 1 & -2 \end{bmatrix}$

**3.** Calculate the indicated cofactors for each given matrix.

★ **(a)** $\mathcal{A}_{22}$, for $\mathbf{A} = \begin{bmatrix} 4 & 1 & -3 \\ 0 & 2 & -2 \\ 9 & 14 & -7 \end{bmatrix}$

**(b)** $\mathcal{B}_{23}$, for $\mathbf{B} = \begin{bmatrix} 9 & -3 & 2 \\ 5 & 1 & -1 \\ 4 & 7 & 6 \end{bmatrix}$

★ **(c)** $\mathcal{C}_{43}$, for $\mathbf{C} = \begin{bmatrix} -5 & 2 & 2 & 13 \\ -8 & 2 & -5 & 22 \\ -6 & -3 & 0 & -16 \\ 4 & -1 & 7 & -8 \end{bmatrix}$

★ **(d)** $\mathcal{D}_{12}$, for $\mathbf{D} = \begin{bmatrix} x+1 & x & x-7 \\ x-4 & x+5 & x-3 \\ x-1 & x & x+2 \end{bmatrix}$, where $x \in \mathbb{R}$

**(e)** $\mathcal{E}_{32}$, for $\mathbf{E} = \begin{bmatrix} x-8 & 7 & -2 \\ 3 & x+12 & -5 \\ 8 & -1 & x-9 \end{bmatrix}$

**4.** Calculate the determinant of each of the matrices in Exercise 1 using the formal definition of the determinant. (Detailed answers are provided in the Student Solutions Manual for the parts starred in Exercise 1.)

**5.** Calculate the determinant of each of the following matrices:

★ **(a)** $\begin{bmatrix} 5 & 2 & 1 & 0 \\ -1 & 3 & 5 & 2 \\ 4 & 1 & 0 & 2 \\ 0 & 2 & 3 & 0 \end{bmatrix}$

**(b)** $\begin{bmatrix} 3 & 7 & -2 & 1 \\ 0 & 4 & 1 & -3 \\ 6 & 0 & 9 & 2 \\ -4 & 2 & 0 & 5 \end{bmatrix}$

**(c)** $\begin{bmatrix} 4 & 3 & -1 & 0 & 6 \\ 0 & 2 & 5 & 9 & -4 \\ 0 & 0 & 5 & 11 & 7 \\ 0 & 0 & 0 & -2 & 18 \\ 0 & 0 & 0 & 0 & 7 \end{bmatrix}$

★ **(d)** $\begin{bmatrix} 0 & 4 & 1 & 3 & -2 \\ 2 & 2 & 3 & -1 & 0 \\ 3 & 1 & 2 & -5 & 1 \\ 1 & 0 & -4 & 0 & 0 \\ 0 & 3 & 0 & 0 & 2 \end{bmatrix}$

**6.** For a general $4 \times 4$ matrix $\mathbf{A}$, write out the formula for $|\mathbf{A}|$ using cofactor expansion along the last row, and simplify as far as possible. (Your final answer should have 24 terms, each being a product of four entries of $\mathbf{A}$.)

★ **7.** Give a counterexample to show that for square matrices $\mathbf{A}$ and $\mathbf{B}$ of the same size, it is not always true that $|\mathbf{A}+\mathbf{B}| = |\mathbf{A}| + |\mathbf{B}|$.

**8.** This exercise introduces the cross product of vectors in $\mathbb{R}^3$.

**(a)** Show that the **cross product** $\mathbf{a} \times \mathbf{b} = [a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1]$ of $\mathbf{a} = [a_1, a_2, a_3]$ and $\mathbf{b} = [b_1, b_2, b_3]$ can be expressed in "determinant notation" as

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

**(b)** Show that $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

**9.** Calculate the area of the parallelogram in $\mathbb{R}^2$ determined by the following:

★ **(a)** $\mathbf{x} = [3, 2]$, $\mathbf{y} = [4, 5]$

**(b)** $\mathbf{x} = [-10, 15]$, $\mathbf{y} = [14, -21]$

★ **(c)** $\mathbf{x} = [5, -1]$, $\mathbf{y} = [-3, 3]$

**(d)** $\mathbf{x} = [-5, 3]$, $\mathbf{y} = [-7, 9]$

▶ **10.** Prove part (1) of Theorem 3.1. (Hint: See Fig. 3.2. The area of the parallelogram is the length of the base $\mathbf{x}$ multiplied by the length of the perpendicular height $\mathbf{h}$. Note that if $\mathbf{p} = \mathbf{proj_x y}$, then $\mathbf{h} = \mathbf{y} - \mathbf{p}$.)

**11.** Calculate the volume of the parallelepiped in $\mathbb{R}^3$ determined by the following:

★ **(a)** $\mathbf{x} = [-2, 3, 1]$, $\mathbf{y} = [4, 2, 0]$, $\mathbf{z} = [-1, 3, 2]$

**(b)** $\mathbf{x} = [5, 2, -7]$, $\mathbf{y} = [9, 1, -1]$, $\mathbf{z} = [-3, 4, 6]$

★ **(c)** $\mathbf{x} = [-3, 4, 0]$, $\mathbf{y} = [6, -2, 1]$, $\mathbf{z} = [0, -3, 3]$

**(d)** $\mathbf{x} = [3, 6, 1]$, $\mathbf{y} = [7, -2, 4]$, $\mathbf{z} = [2, 9, -5]$

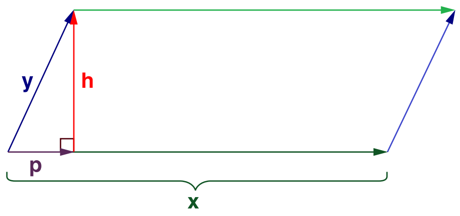



FIGURE 3.2 Parallelogram determined by **x** and **y**

▶ **12.** Prove part (2) of Theorem 3.1. (Hint: See Fig. 3.3. Let **h** be the perpendicular dropped from **z** to the plane of the parallelogram. From Exercise 8, **x** × **y** is perpendicular to both **x** and **y**, and so **h** is actually the projection of **z** onto **x** × **y**. Hence, the volume of the parallelepiped is the area of the parallelogram determined by **x** and **y** multiplied by the length of **h**. Use a calculation similar to that in Exercise 10 to show that the area of the parallelogram is $\sqrt{(x_2y_3 - x_3y_2)^2 + (x_1y_3 - x_3y_1)^2 + (x_1y_2 - x_2y_1)^2}$.)

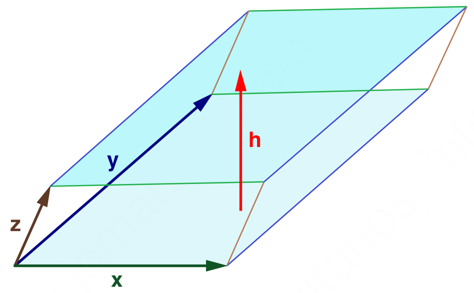



FIGURE 3.3 Parallelepiped determined by **x**, **y**, and **z**

**13.** This exercise asks you to prove Corollary 3.4, which appears in Section 3.2. However, your proof should not rely on results appearing beyond Section 3.1.

**(a)** If **A** is an $n \times n$ matrix, and $c$ is a scalar, prove that $|c\mathbf{A}| = c^n\,|\mathbf{A}|$. (Hint: Use a proof by induction on $n$.)

**(b)** Use part (a) together with part (2) of Theorem 3.1 to explain why, when each side of a parallelepiped is doubled, the volume is multiplied by 8.

**14.** Show that, for $x \in \mathbb{R}$, $x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ is the determinant of

$$\begin{bmatrix} x & -1 & 0 & 0 \\ 0 & x & -1 & 0 \\ 0 & 0 & x & -1 \\ a_0 & a_1 & a_2 & a_3 + x \end{bmatrix}.$$

**15.** Solve the following determinant equations for $x \in \mathbb{R}$:

★ **(a)** $\begin{vmatrix} x & 2 \\ 5 & x+3 \end{vmatrix} = 0$

**(b)** $\begin{vmatrix} -5 & x-4 \\ 2x+1 & 2 \end{vmatrix} = 0$

★ **(c)** $\begin{vmatrix} x-3 & 5 & -19 \\ 0 & x-1 & 6 \\ 0 & 0 & x-2 \end{vmatrix} = 0$

**(d)** $\begin{vmatrix} x+3 & -6 & 9 \\ 4 & x-9 & 14 \\ 0 & -3 & x+6 \end{vmatrix} = 0$

**16.** This exercise introduces the $3 \times 3$ Vandermonde matrix.

**(a)** Show that the determinant of the $3 \times 3$ **Vandermonde matrix**

$$\begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix}$$

is equal to $(a-b)(b-c)(c-a)$.

★ **(b)** Using part (a), calculate the determinant of

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & -2 \\ 4 & 9 & 4 \end{bmatrix}.$$

**17.** The purpose of this exercise is to show that it is impossible to have an equilateral triangle whose three vertices all lie on **lattice points** in the plane—that is, points whose coordinates are both integers. Suppose $T$ is such an equilateral triangle. Use the following steps to reach a contradiction:

**(a)** If $s$ is the length of a side of $T$, use elementary geometry to find a formula for the area of $T$ in terms of $s$.

**(b)** Use your answer for part (a) to show that the area of $T$ is an irrational number. (You may assume $\sqrt{3}$ is irrational.)

**(c)** Suppose the three vertices of a triangle in the plane are given. Use part (1) of Theorem 3.1 to express the area of the triangle using a determinant.

**(d)** Use your answer for part (c) to show that the area of $T$ is a rational number, thus contradicting part (b).

★ **18.** True or False:

**(a)** The basketweaving technique can be used to find determinants of $3 \times 3$ and larger square matrices.

**(b)** The area of the parallelogram determined by nonparallel vectors $[x_1, x_2]$ and $[y_1, y_2]$ is $|x_1y_2 - x_2y_1|$.

**(c)** An $n \times n$ matrix has $2n$ associated cofactors.

**(d)** The cofactor $\mathcal{B}_{23}$ for a square matrix $\mathbf{B}$ equals the minor $|\mathbf{B}_{23}|$.

**(e)** The determinant of a $4 \times 4$ matrix $\mathbf{A}$ is $a_{41}\mathcal{A}_{41} + a_{42}\mathcal{A}_{42} + a_{43}\mathcal{A}_{43} + a_{44}\mathcal{A}_{44}$.

## 3.2 Determinants and Row Reduction

In this section, we provide a method for calculating the determinant of a matrix using row reduction. For large matrices, this technique is computationally more efficient than cofactor expansion. We will also use the relationship between determinants and row reduction to establish a link between determinants and rank.

### Determinants of Upper Triangular Matrices

We begin by proving the following simple formula for the determinant of an upper triangular matrix. Our goal will be to reduce every other determinant computation to this special case using row reduction.

**Theorem 3.2** *Let* $\mathbf{A}$ *be an upper triangular* $n \times n$ *matrix. Then* $|\mathbf{A}| = a_{11}a_{22}\cdots a_{nn}$*, the product of the entries of* $\mathbf{A}$ *along the main diagonal.*

Because we have defined the determinant recursively, we prove Theorem 3.2 by induction.

*Proof.* We use induction on $n$.

**Base Step:** $n = 1$. In this case, $\mathbf{A} = [a_{11}]$, and $|\mathbf{A}| = a_{11}$, which verifies the formula in the theorem.

**Inductive Step:** Let $n > 1$. Assume that for any upper triangular $(n-1) \times (n-1)$ matrix $\mathbf{B}$, $|\mathbf{B}| = b_{11}b_{22}\cdots b_{(n-1)(n-1)}$. We must prove that the formula given in the theorem holds for any $n \times n$ matrix $\mathbf{A}$.

Now, $|\mathbf{A}| = a_{n1}\mathcal{A}_{n1} + a_{n2}\mathcal{A}_{n2} + \cdots + a_{nn}\mathcal{A}_{nn} = 0\mathcal{A}_{n1} + 0\mathcal{A}_{n2} + \cdots + 0\mathcal{A}_{(n-1)(n-1)} + a_{nn}\mathcal{A}_{nn}$, because $a_{ni} = 0$ for $i < n$, since $\mathbf{A}$ is upper triangular. Thus, $|\mathbf{A}| = a_{nn}\mathcal{A}_{nn} = a_{nn}(-1)^{n+n}|\mathbf{A}_{nn}| = a_{nn}|\mathbf{A}_{nn}|$ (since $n+n$ is even). However, the $(n-1) \times (n-1)$ submatrix $\mathbf{A}_{nn}$ is itself an upper triangular matrix, since $\mathbf{A}$ is upper triangular. Thus, by the inductive hypothesis, $|\mathbf{A}_{nn}| = a_{11}a_{22}\cdots a_{(n-1)(n-1)}$. Hence, $|\mathbf{A}| = a_{nn}(a_{11}a_{22}\cdots a_{(n-1)(n-1)}) = a_{11}a_{22}\cdots a_{nn}$, completing the proof. □

**Example 1**

By Theorem 3.2,

$$\begin{vmatrix} 4 & 2 & 0 & 1 \\ 0 & 3 & 9 & 6 \\ 0 & 0 & -1 & 5 \\ 0 & 0 & 0 & 7 \end{vmatrix} = (4)(3)(-1)(7) = -84.$$

As a special case of Theorem 3.2, notice that for all $n \geq 1$, we have $|\mathbf{I}_n| = 1$, since $\mathbf{I}_n$ is upper triangular with all its main diagonal entries equal to 1.

## Effect of Row Operations on the Determinant

The following theorem describes explicitly how each type of row operation affects the determinant:

**Theorem 3.3** *Let* $\mathbf{A}$ *be an* $n \times n$ *matrix, with determinant* $|\mathbf{A}|$, *and let* $c$ *be a scalar.*

**(1)** *If* $R_1$ *is the Type (I) row operation* $\langle i \rangle \leftarrow c\,\langle i \rangle$, *then* $|R_1(\mathbf{A})| = c|\mathbf{A}|$.

**(2)** *If* $R_2$ *is the Type (II) row operation* $\langle j \rangle \leftarrow c\,\langle i \rangle + \langle j \rangle$, *then* $|R_2(\mathbf{A})| = |\mathbf{A}|$.

**(3)** *If* $R_3$ *is the Type (III) row operation* $\langle i \rangle \leftrightarrow \langle j \rangle$, *then* $|R_3(\mathbf{A})| = -|\mathbf{A}|$.

All three parts of Theorem 3.3 are proved by induction. The proof of part (1) is easiest and is outlined in Exercise 8. Part (2) is easier to prove after part (3) is proven, and we outline the proof of part (2) in Exercises 9 and 10. The proof of part (3) is done by induction. Most of the proof of part (3) is given after the next example, except for one tedious case which has been placed in Appendix A.

### Example 2

Let

$$\mathbf{A} = \begin{bmatrix} 5 & -2 & 1 \\ 4 & 3 & -1 \\ 2 & 1 & 0 \end{bmatrix}.$$

You can quickly verify by the basketweaving method that $|\mathbf{A}| = 7$. Consider the following matrices:

$$\mathbf{B}_1 = \begin{bmatrix} 5 & -2 & 1 \\ 4 & 3 & -1 \\ -6 & -3 & 0 \end{bmatrix}, \quad \mathbf{B}_2 = \begin{bmatrix} 5 & -2 & 1 \\ 4 & 3 & -1 \\ 12 & -3 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{B}_3 = \begin{bmatrix} 4 & 3 & -1 \\ 5 & -2 & 1 \\ 2 & 1 & 0 \end{bmatrix}.$$

Now, $\mathbf{B}_1$ is obtained from $\mathbf{A}$ by the Type (I) row operation $\langle 3 \rangle \leftarrow -3\,\langle 3 \rangle$. Hence, part (1) of Theorem 3.3 asserts that $|\mathbf{B}_1| = -3\,|\mathbf{A}| = (-3)(7) = -21$.

Next, $\mathbf{B}_2$ is obtained from $\mathbf{A}$ by the Type (II) row operation $\langle 3 \rangle \leftarrow 2\,\langle 1 \rangle + \langle 3 \rangle$. By part (2) of Theorem 3.3, $|\mathbf{B}_2| = |\mathbf{A}| = 7$.

Finally, $\mathbf{B}_3$ is obtained from $\mathbf{A}$ by the Type (III) row operation $\langle 1 \rangle \leftrightarrow \langle 2 \rangle$. Then, by part (3) of Theorem 3.3, $|\mathbf{B}_3| = -|\mathbf{A}| = -7$.

You can use basketweaving on $\mathbf{B}_1$, $\mathbf{B}_2$, and $\mathbf{B}_3$ to verify that the values given for their determinants are indeed correct.

*Proof.* Proof of Part (3) of Theorem 3.3: We proceed by induction on $n$. Notice that for $n = 1$, we cannot have a Type (III) row operation, so $n = 2$ for the Base Step.

**Base Step:** $n = 2$. Then $R$ must be the row operation $\langle 1 \rangle \leftrightarrow \langle 2 \rangle$, and $|R(\mathbf{A})| = \left| R\left( \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right) \right| = \begin{vmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{vmatrix} = a_{21}a_{12} - a_{22}a_{11} = -(a_{11}a_{22} - a_{12}a_{21}) = -|\mathbf{A}|$.

**Inductive Step:** Assume $n \geq 3$, and that switching two rows of an $(n-1) \times (n-1)$ matrix results in a matrix whose determinant has the opposite sign. We consider three separate cases.

**Case 1:** Suppose $R$ is the row operation $\langle i \rangle \leftrightarrow \langle j \rangle$, where $i \neq n$ and $j \neq n$. Let $\mathbf{B} = R(\mathbf{A})$. Then, since the last row of $\mathbf{A}$ is not changed, $b_{nk} = a_{nk}$, for $1 \leq k \leq n$. Also, $\mathbf{B}_{nk}$, the $(n,k)$ submatrix of $\mathbf{B}$, equals $R(\mathbf{A}_{nk})$ (why?). Therefore, by the inductive hypothesis, $|\mathbf{B}_{nk}| = -|\mathbf{A}_{nk}|$, implying $\mathcal{B}_{nk} = (-1)^{n+k}|\mathbf{B}_{nk}| = (-1)^{n+k}(-1)|\mathbf{A}_{nk}| = -\mathcal{A}_{nk}$, for $1 \leq k \leq n$. Hence, $|\mathbf{B}| = b_{n1}\mathcal{B}_{n1} + \cdots + b_{nn}\mathcal{B}_{nn} = a_{n1}(-\mathcal{A}_{n1}) + \cdots + a_{nn}(-\mathcal{A}_{nn}) = -(a_{n1}\mathcal{A}_{n1} + \cdots + a_{nn}\mathcal{A}_{nn}) = -|\mathbf{A}|$.

**Case 2:** Suppose $R$ is the row operation $\langle n-1 \rangle \leftrightarrow \langle n \rangle$, switching the last two rows. This case is proved by brute-force calculation, the details of which appear in Appendix A.

**Case 3:** Suppose $R$ is the row operation $\langle i \rangle \leftrightarrow \langle n \rangle$, with $i \leq n-2$. In this case, we express $R$ as a sequence of row swaps from the two previous cases. Let $R_1$ be the row operation $\langle i \rangle \leftrightarrow \langle n-1 \rangle$ and $R_2$ be the row operation $\langle n-1 \rangle \leftrightarrow \langle n \rangle$. Then $\mathbf{B} = R(\mathbf{A}) = R_1(R_2(R_1(\mathbf{A})))$ (why?). Using the previous two cases, we have $|\mathbf{B}| = |R(\mathbf{A})| = |R_1(R_2(R_1(\mathbf{A})))| = -|R_2(R_1(\mathbf{A}))| = (-1)^2|R_1(\mathbf{A})| = (-1)^3|\mathbf{A}| = -|\mathbf{A}|$.

This completes the proof. □

Theorem 3.3 can be used to prove that if a matrix $\mathbf{A}$ has a row with all entries zero, or has two identical rows, then $|\mathbf{A}| = 0$ (see Exercises 11 and 12).

Part (1) of Theorem 3.3 can be used to multiply each of the $n$ rows of a matrix $\mathbf{A}$ by $c$ in turn, thus proving the following corollary[2]:

**Corollary 3.4** *If* $\mathbf{A}$ *is an* $n \times n$ *matrix, and* $c$ *is any scalar, then* $|c\mathbf{A}| = c^n|\mathbf{A}|$.

**Example 3**

A quick calculation shows that

$$\begin{vmatrix} 0 & 2 & 1 \\ 3 & -3 & -2 \\ 16 & 7 & 1 \end{vmatrix} = -1.$$

Therefore,

$$\begin{vmatrix} 0 & -4 & -2 \\ -6 & 6 & 4 \\ -32 & -14 & -2 \end{vmatrix} = \left| -2 \begin{bmatrix} 0 & 2 & 1 \\ 3 & -3 & -2 \\ 16 & 7 & 1 \end{bmatrix} \right| = (-2)^3 \begin{vmatrix} 0 & 2 & 1 \\ 3 & -3 & -2 \\ 16 & 7 & 1 \end{vmatrix} = (-8)(-1) = 8.$$

## Calculating the Determinant by Row Reduction

We will now illustrate how to use row operations to calculate the determinant of a given matrix $\mathbf{A}$ by finding an upper triangular matrix $\mathbf{B}$ that is row equivalent to $\mathbf{A}$.

**Example 4**

Let

$$\mathbf{A} = \begin{bmatrix} 0 & -14 & -8 \\ 1 & 3 & 2 \\ -2 & 0 & 6 \end{bmatrix}.$$

We row reduce $\mathbf{A}$ to upper triangular form, as follows, keeping track of the effect on the determinant at each step:

$$\mathbf{A} = \begin{bmatrix} 0 & -14 & -8 \\ 1 & 3 & 2 \\ -2 & 0 & 6 \end{bmatrix}$$

$$\text{(III)}: \langle 1 \rangle \leftrightarrow \langle 2 \rangle \quad \Longrightarrow \quad \mathbf{B}_1 = \begin{bmatrix} 1 & 3 & 2 \\ 0 & -14 & -8 \\ -2 & 0 & 6 \end{bmatrix} \quad (|\mathbf{B}_1| = -|\mathbf{A}|)$$

$$\text{(II)}: \langle 3 \rangle \leftarrow 2\langle 1 \rangle + \langle 3 \rangle \quad \Longrightarrow \quad \mathbf{B}_2 = \begin{bmatrix} 1 & 3 & 2 \\ 0 & -14 & -8 \\ 0 & 6 & 10 \end{bmatrix} \quad (|\mathbf{B}_2| = |\mathbf{B}_1| = -|\mathbf{A}|)$$

$$\text{(I)}: \langle 2 \rangle \leftarrow -\tfrac{1}{14}\langle 2 \rangle \quad \Longrightarrow \quad \mathbf{B}_3 = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & \frac{4}{7} \\ 0 & 6 & 10 \end{bmatrix} \quad \left(|\mathbf{B}_3| = -\tfrac{1}{14}|\mathbf{B}_2| = +\tfrac{1}{14}|\mathbf{A}|\right)$$

$$\text{(II)}: \langle 3 \rangle \leftarrow -6\langle 2 \rangle + \langle 3 \rangle \quad \Longrightarrow \quad \mathbf{B} = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & \frac{4}{7} \\ 0 & 0 & \frac{46}{7} \end{bmatrix}. \quad \left(|\mathbf{B}| = |\mathbf{B}_3| = +\tfrac{1}{14}|\mathbf{A}|\right)$$

[2] You were also asked to prove this result in Exercise 13 of Section 3.1 directly from the definition of the determinant using induction.

Because the last matrix **B** is in upper triangular form, we stop. (Notice that we do not target the entries above the main diagonal, as in reduced row echelon form.) From Theorem 3.2, $|\mathbf{B}| = (1)(1)\left(\frac{46}{7}\right) = \frac{46}{7}$. Since $|\mathbf{B}| = +\frac{1}{14}|\mathbf{A}|$, we see that $|\mathbf{A}| = 14|\mathbf{B}| = 14(\frac{46}{7}) = 92$.

A more convenient method of calculating $|\mathbf{A}|$ is to create a variable $P$ (for "product") with initial value 1, and update $P$ appropriately as each row operation is performed. That is, we replace the current value of $P$ by

$$\begin{cases} P \times c & \text{for Type (I) row operations} \\ P \times (-1) & \text{for Type (III) row operations} \end{cases}.$$

Of course, Type (II) row operations do not affect the determinant. Then, using the final value of $P$, we can solve for $|\mathbf{A}|$ using $|\mathbf{A}| = (1/P)\,|\mathbf{B}|$, where **B** is the upper triangular result of the row reduction process. This method is illustrated in the next example.

**Example 5**

Let us redo the calculation for $|\mathbf{A}|$ in Example 4. We create a variable $P$ and initialize $P$ to 1. Listed below are the row operations used in that example to convert **A** into upper triangular form **B**, with $|\mathbf{B}| = \frac{46}{7}$. After each operation, we update the value of $P$ accordingly.

| Row Operation | Effect | $P$ |
|---|---|---|
| (III): $\langle 1\rangle \leftrightarrow \langle 2\rangle$ | Multiply $P$ by $-1$ | $-1$ |
| (II): $\langle 3\rangle \leftarrow 2\,\langle 1\rangle + \langle 3\rangle$ | No change | $-1$ |
| (I): $\langle 2\rangle \leftarrow -\frac{1}{14}\,\langle 2\rangle$ | Multiply $P$ by $-\frac{1}{14}$ | $\frac{1}{14}$ |
| (II): $\langle 3\rangle \leftarrow -6\,\langle 2\rangle + \langle 3\rangle$ | No change | $\frac{1}{14}$ |

Then $|\mathbf{A}|$ equals the reciprocal of the final value of $P$ times $|\mathbf{B}|$; that is, $|\mathbf{A}| = (1/P)|\mathbf{B}| = 14 \times \frac{46}{7} = 92$.

## Determinant Criterion for Matrix Singularity

The next theorem gives an alternative way of determining whether the inverse of a given square matrix exists.

**Theorem 3.5** *An $n \times n$ matrix **A** is nonsingular if and only if $|\mathbf{A}| \neq 0$.*

*Proof.* Let **D** be the unique matrix in reduced row echelon form for **A**. Now, using Theorem 3.3, we see that a single row operation of any type cannot convert a matrix having a nonzero determinant to a matrix having a zero determinant (why?). Because **A** is converted to **D** using a finite number of such row operations, Theorem 3.3 assures us that $|\mathbf{A}|$ and $|\mathbf{D}|$ are either both zero or both nonzero.

Now, if **A** is nonsingular (which implies $\mathbf{D} = \mathbf{I}_n$), we know that $|\mathbf{D}| = 1 \neq 0$ and therefore $|\mathbf{A}| \neq 0$, and we have completed half of the proof.

For the other half, assume that $|\mathbf{A}| \neq 0$. Then $|\mathbf{D}| \neq 0$. Because **D** is a square matrix with a staircase pattern of pivots, it is upper triangular. Because $|\mathbf{D}| \neq 0$, Theorem 3.2 asserts that all main diagonal entries of **D** are nonzero. Hence, they are all pivots, and $\mathbf{D} = \mathbf{I}_n$. Therefore, row reduction transforms **A** to $\mathbf{I}_n$, so **A** is nonsingular. □

Notice that Theorem 3.5 agrees with Theorem 2.14 in asserting that an inverse for $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ exists if and only if $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0$.

Theorem 2.15 and Theorem 3.5 together imply the following:

**Corollary 3.6** *Let **A** be an $n \times n$ matrix. Then $\operatorname{rank}(\mathbf{A}) = n$ if and only if $|\mathbf{A}| \neq 0$.*

**Example 6**

Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 6 \\ -3 & 5 \end{bmatrix}$. Now, $|\mathbf{A}| = 23 \neq 0$. Hence, $\text{rank}(\mathbf{A}) = 2$ by Corollary 3.6. Also, because $\mathbf{A}$ is the coefficient matrix of the system

$$\begin{cases} x + 6y = 20 \\ -3x + 5y = 9 \end{cases}$$

and $|\mathbf{A}| \neq 0$, this system has a unique solution by Theorem 2.16. In fact, the solution is $(2, 3)$.

On the other hand, for the matrix

$$\mathbf{B} = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & -7 \\ -1 & 2 & 6 \end{bmatrix},$$

we have $|\mathbf{B}| = 0$. Thus, $\text{rank}(\mathbf{B}) < 3$. Also, because $\mathbf{B}$ is the coefficient matrix for the homogeneous system

$$\begin{cases} x_1 + 5x_2 + x_3 = 0 \\ 2x_1 + x_2 - 7x_3 = 0 \\ -x_1 + 2x_2 + 6x_3 = 0 \end{cases},$$

this system has nontrivial solutions by Theorem 2.7. You can verify that its complete solution set is $\{c(4, -1, 1) \mid c \in \mathbb{R}\}$.

For reference, we summarize many of the results obtained in Chapters 2 and 3 in Table 3.1. You should be able to justify each equivalence in Table 3.1 by citing a relevant definition or result.

**TABLE 3.1 Equivalent conditions for singular and nonsingular matrices**

| For an $n \times n$ matrix $\mathbf{A}$, the following are all equivalent: | For an $n \times n$ matrix $\mathbf{A}$, the following are all equivalent: |
|---|---|
| $\mathbf{A}$ is singular ($\mathbf{A}^{-1}$ does not exist). | $\mathbf{A}$ is nonsingular ($\mathbf{A}^{-1}$ exists). |
| $\text{Rank}(\mathbf{A}) \neq n$. | $\text{Rank}(\mathbf{A}) = n$. |
| $\|\mathbf{A}\| = 0$. | $\|\mathbf{A}\| \neq 0$. |
| $\mathbf{A}$ is not row equivalent to $\mathbf{I}_n$. | $\mathbf{A}$ is row equivalent to $\mathbf{I}_n$. |
| $\mathbf{AX} = \mathbf{0}$ has a nontrivial solution for $\mathbf{X}$. | $\mathbf{AX} = \mathbf{0}$ has only the trivial solution for $\mathbf{X}$. |
| $\mathbf{AX} = \mathbf{B}$ does not have a unique solution (no solutions or infinitely many solutions). | $\mathbf{AX} = \mathbf{B}$ has a unique solution for $\mathbf{X}$ (namely, $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$). |

## Highlights

- If $\mathbf{A}$ is an upper triangular $n \times n$ matrix, then $|\mathbf{A}| = a_{11}a_{22}\cdots a_{nn}$, the product of the entries along the main diagonal of $\mathbf{A}$.
- If $R_1$ is the Type (I) row operation $\langle i \rangle \leftarrow c\,\langle i \rangle$, then $|R_1(\mathbf{A})| = c|\mathbf{A}|$.
- If $R_2$ is the Type (II) row operation $\langle j \rangle \leftarrow c\,\langle i \rangle + \langle j \rangle$, then $|R_2(\mathbf{A})| = |\mathbf{A}|$.
- If $R_3$ is the Type (III) row operation $\langle i \rangle \leftrightarrow \langle j \rangle$, then $|R_3(\mathbf{A})| = -\,|\mathbf{A}|$.
- If $\mathbf{A}$ is an $n \times n$ matrix and $\mathbf{B} = c\mathbf{A}$ for some scalar $c$, then $|\mathbf{B}| = c^n|\mathbf{A}|$.
- The determinant of a matrix $\mathbf{A}$ can be found by row reducing $\mathbf{A}$ to an upper triangular form $\mathbf{B}$ using a running product $\mathbf{P}$, where $\mathbf{P}$ is adjusted appropriately for each row operation of Type (I) or (III). Then $|\mathbf{A}| = \frac{1}{P}|\mathbf{B}|$.
- An $n \times n$ matrix $\mathbf{A}$ is nonsingular iff $|\mathbf{A}| \neq \mathbf{0}$ iff $\text{rank}(\mathbf{A}) = n$.

# Exercises for Section 3.2

1. Each of the following matrices is obtained from $\mathbf{I}_3$ by performing a single row operation. Identify the operation, and use Theorem 3.3 to give the determinant of each matrix.

★ (a) $\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

★ (c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix}$

★ (f) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

**2.** Calculate the determinant of each of the following matrices by using row reduction to produce an upper triangular form:

★ (a) $\begin{bmatrix} 10 & 4 & 21 \\ 0 & -4 & 3 \\ -5 & -1 & -12 \end{bmatrix}$

(b) $\begin{bmatrix} 5 & -2 & 3 \\ 3 & 5 & 1 \\ -2 & 7 & 3 \end{bmatrix}$

★ (c) $\begin{bmatrix} 1 & -1 & 5 & 1 \\ -2 & 1 & -7 & 1 \\ -3 & 2 & -12 & -2 \\ 2 & -1 & 9 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 1 & -3 & 4 \\ 4 & 7 & 1 & 7 \\ 0 & 5 & 11 & 2 \\ 6 & -7 & -15 & 20 \end{bmatrix}$

★ (e) $\begin{bmatrix} 5 & 3 & -8 & 4 \\ \frac{15}{2} & \frac{1}{2} & -1 & -7 \\ -\frac{5}{2} & \frac{3}{2} & -4 & 1 \\ 10 & -3 & 8 & -8 \end{bmatrix}$

(f) $\begin{bmatrix} 4 & 3 & -1 & 8 & 6 \\ 0 & 5 & 3 & 1 & -4 \\ -8 & 4 & 7 & -12 & -18 \\ 0 & -5 & -6 & 7 & 17 \\ -4 & 2 & 0 & 7 & 25 \end{bmatrix}$

**3.** By calculating the determinant of each matrix, decide whether it is nonsingular.

★ (a) $\begin{bmatrix} 5 & 6 \\ -3 & -4 \end{bmatrix}$

(b) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

★ (c) $\begin{bmatrix} -12 & 7 & -27 \\ 4 & -1 & 2 \\ 3 & 2 & -8 \end{bmatrix}$

(d) $\begin{bmatrix} 5 & 2 & -7 \\ 8 & 6 & 1 \\ -3 & 9 & 5 \end{bmatrix}$

(e) $\begin{bmatrix} 3 & -1 & 16 \\ -4 & 1 & -19 \\ 2 & 0 & 6 \end{bmatrix}$

**4.** By calculating the determinant of the coefficient matrix, decide whether each of the following homogeneous systems has a nontrivial solution. (You do not need to find the actual solutions.)

★ (a) $\begin{cases} -6x + 3y - 22z = 0 \\ -7x + 4y - 31z = 0 \\ 11x - 6y + 46z = 0 \end{cases}$

(b) $\begin{cases} 3x_1 + 4x_2 - 2x_3 = 0 \\ 4x_1 - x_2 + 8x_3 = 0 \\ 5x_1 \qquad + 7x_3 = 0 \end{cases}$

(c) $\begin{cases} x_1 - x_2 + x_3 + 7x_4 = 0 \\ 2x_1 - x_2 + 4x_3 + 24x_4 = 0 \\ 2x_1 - 4x_2 \qquad + 2x_4 = 0 \\ 3x_1 - x_2 + 4x_3 + 29x_4 = 0 \end{cases}$

**5.** Let $\mathbf{A}$ be an upper triangular matrix. Prove that $|\mathbf{A}| \neq 0$ if and only if all the main diagonal elements of $\mathbf{A}$ are nonzero.

★ **6.** Find the determinant of the following matrix:

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & a_{16} \\ 0 & 0 & 0 & 0 & a_{25} & a_{26} \\ 0 & 0 & 0 & a_{34} & a_{35} & a_{36} \\ 0 & 0 & a_{43} & a_{44} & a_{45} & a_{46} \\ 0 & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}.$$

(Hint: Use part (3) of Theorem 3.3 and then Theorem 3.2.)

**7.** Suppose that $\mathbf{AB} = \mathbf{AC}$ and $|\mathbf{A}| \neq 0$. Show that $\mathbf{B} = \mathbf{C}$.

**8.** The purpose of this exercise is to outline a proof by induction of part (1) of Theorem 3.3. Because Theorem 3.3 is used extensively to prove further results in the textbook, in the following proof you may not use any results that appear after Theorem 3.3 in order to avoid circular reasoning. Let $\mathbf{A}$ be an $n \times n$ matrix, let $R$ be the row operation $\langle i \rangle \leftarrow c\,\langle i \rangle$, and let $\mathbf{B} = R(\mathbf{A})$.
   - **(a)** Prove $|\mathbf{B}| = c|\mathbf{A}|$ when $n = 1$. (This is the Base Step.)
   - **(b)** State the inductive hypothesis for the Inductive Step.
   - **(c)** Complete the Inductive Step for the case in which $R$ is not performed on the last row of $\mathbf{A}$.
   - **(d)** Complete the Inductive Step for the case in which $R$ is performed on the last row of $\mathbf{A}$.

**9.** The purpose of this exercise and the next is to outline a proof by induction of part (2) of Theorem 3.3. This exercise completes the Base Step.
   - **(a)** Explain why $n \neq 1$ in this problem.
   - **(b)** Prove that applying the Type (II) row operation $\langle 1 \rangle \leftarrow c\,\langle 2 \rangle + \langle 1 \rangle$ to a $2 \times 2$ matrix does not change the determinant.
   - **(c)** Repeat part (b) for the Type (II) row operation $\langle 2 \rangle \leftarrow c\,\langle 1 \rangle + \langle 2 \rangle$.

**10.** The purpose of this exercise is to prove the Inductive Step for part (2) of Theorem 3.3. You may assume that part (3) of Theorem 3.3 has already been proved. Let $\mathbf{A}$ be an $n \times n$ matrix, for $n \geq 3$, and let $R$ be the Type (II) row operation $\langle i \rangle \leftarrow c\,\langle j \rangle + \langle i \rangle$.
   - **(a)** State the inductive hypothesis and the statement to be proved for the Inductive Step. (Assume for size $n-1$, and prove for size $n$.)
   - **(b)** Prove the Inductive Step in the case where $i \neq n$ and $j \neq n$. (Your proof should be similar to that for Case 1 in the proof of part (3) of Theorem 3.3.)
   - **(c)** Consider the case $i = n$. Suppose $k \neq j$ and $k \neq n$. Let $R_1$ be the Type (III) row operation $\langle k \rangle \leftrightarrow \langle n \rangle$ and $R_2$ be the Type (II) row operation $\langle k \rangle \leftarrow c\,\langle j \rangle + \langle k \rangle$. Prove that $R(\mathbf{A}) = R_1(R_2(R_1(\mathbf{A})))$.
   - **(d)** Finish the proof of the Inductive Step for the case $i = n$. (Your proof should be similar to that for Case 3 in the proof of part (3) of Theorem 3.3.)
   - **(e)** Finally, consider the case $j = n$. Suppose $k \neq i$ and $k \neq n$. Let $R_1$ be the Type (III) row operation $\langle k \rangle \leftrightarrow \langle n \rangle$ and $R_3$ be the Type (II) row operation $\langle i \rangle \leftarrow c\,\langle k \rangle + \langle i \rangle$. Prove that $R(\mathbf{A}) = R_1(R_3(R_1(\mathbf{A})))$.
   - **(f)** Finish the proof of the Inductive Step for the case $j = n$.

**11.** Let $\mathbf{A}$ be an $n \times n$ matrix having an entire row of zeroes.
   - **(a)** Use part (1) of Theorem 3.3 to prove that $|\mathbf{A}| = 0$.
   - **(b)** Use Corollary 3.6 to provide an alternate proof that $|\mathbf{A}| = 0$.

**12.** Let $\mathbf{A}$ be an $n \times n$ matrix having two identical rows.
   - **(a)** Use part (3) of Theorem 3.3 to prove that $|\mathbf{A}| = 0$.
   - **(b)** Use Corollary 3.6 to provide an alternate proof that $|\mathbf{A}| = 0$.

**13.** Let $\mathbf{A}$ be an $n \times n$ matrix.
   - **(a)** Show that if the entries of some row of $\mathbf{A}$ are proportional to those in another row, then $|\mathbf{A}| = 0$.
   - **(b)** Show that if the entries in every row of $\mathbf{A}$ add up to zero, then $|\mathbf{A}| = 0$. (Hint: Consider the system $\mathbf{AX} = \mathbf{0}$, and note that the $n \times 1$ vector $\mathbf{X}$ having every entry equal to 1 is a nontrivial solution.)

**14.** This exercise explores the determinant of a special type of matrix. (The result in part (a) is stated as Lemma 5.29 in Section 5.6.)
   - **(a)** Use row reduction to show that the determinant of the $n \times n$ matrix symbolically represented by $\begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{O} & \mathbf{D} \end{bmatrix}$ is $|\mathbf{B}|\,|\mathbf{D}|$, where

**B** is an $m \times m$ submatrix,
**C** is an $m \times (n-m)$ submatrix,
**D** is an $(n-m) \times (n-m)$ submatrix, and
**O** is an $(n-m) \times m$ zero submatrix.

**(b)** Use part (a) to compute

$$\begin{vmatrix} -2 & 5 & 8 & -1 \\ 3 & -4 & 6 & -2 \\ 0 & 0 & 7 & -3 \\ 0 & 0 & -1 & 2 \end{vmatrix}.$$

**15.** Suppose that $f\colon \mathcal{M}_{nn} \to \mathbb{R}$ such that $f(\mathbf{I}_n) = 1$, and that whenever a single row operation is performed on $\mathbf{A} \in \mathcal{M}_{nn}$ to create **B**,

$$f(\mathbf{B}) = \begin{cases} cf(\mathbf{A}) & \text{for a Type (I) row operation with } c \neq 0 \\ f(\mathbf{A}) & \text{for a Type (II) row operation} \\ -f(\mathbf{A}) & \text{for a Type (III) row operation} \end{cases}.$$

Prove that $f(\mathbf{A}) = |\mathbf{A}|$, for all $\mathbf{A} \in \mathcal{M}_{nn}$. (Hint: If **A** is row equivalent to $\mathbf{I}_n$, then the given properties of $f$ guarantee that $f(\mathbf{A}) = |\mathbf{A}|$ (why?). Otherwise, **A** is row equivalent to a matrix with a row of zeroes, and $|\mathbf{A}| = 0$. In this case, apply a Type (I) row operation with $c = 2$ to obtain $f(\mathbf{A}) = 0$.)

★ **16.** Let **A** be an $n \times n$ matrix with $|\mathbf{A}| = 0$. Show that there is an $n \times n$ matrix **B** such that $\mathbf{B} \neq \mathbf{O}_n$ and $\mathbf{AB} = \mathbf{O}_n$. (Hint: Notice that $\mathbf{AX} = \mathbf{0}$ has nontrivial solutions, and consider a matrix **B** having such nontrivial solutions as its columns.)

★ **17.** True or False:

**(a)** The determinant of a square matrix is the product of its main diagonal entries.
**(b)** Two Type (III) row operations performed in succession have no overall effect on the determinant.
**(c)** If every row of a $4 \times 4$ matrix is multiplied by 3, the determinant is also multiplied by 3.
**(d)** If two rows of a square matrix **A** are identical, then $|\mathbf{A}| = 1$.
**(e)** A square matrix **A** is nonsingular if and only if $|\mathbf{A}| = 0$.
**(f)** An $n \times n$ matrix **A** has determinant zero if and only if $\operatorname{rank}(\mathbf{A}) < n$.

## 3.3 Further Properties of the Determinant

In this section, we investigate the determinant of a product and the determinant of a transpose. We generalize the cofactor expansion formula for the determinant so that rows other than the last may be used, and we show that cofactor expansion also works along columns of the matrix. Finally, we present Cramer's Rule, an alternative technique for solving certain linear systems using determinants.

### Determinant of a Matrix Product

We begin by proving that the determinant of a product of two matrices **A** and **B** is equal to the product of their determinants $|\mathbf{A}|$ and $|\mathbf{B}|$.

**Theorem 3.7** *If* **A** *and* **B** *are both $n \times n$ matrices, then* $|\mathbf{AB}| = |\mathbf{A}|\,|\mathbf{B}|$.

*Proof.* First, suppose **A** is singular. Then $|\mathbf{A}| = 0$ by Theorem 3.5. If $|\mathbf{AB}| = 0$, then $|\mathbf{AB}| = |\mathbf{A}|\,|\mathbf{B}|$ and we will be done. We assume $|\mathbf{AB}| \neq 0$ and get a contradiction. Since $|\mathbf{AB}| \neq 0$, $(\mathbf{AB})^{-1}$ exists, and $\mathbf{I}_n = \mathbf{AB}(\mathbf{AB})^{-1}$. Hence, $\mathbf{B}(\mathbf{AB})^{-1}$ is a right inverse for **A**. But then by Theorem 2.10, $\mathbf{A}^{-1}$ exists, contradicting the fact that **A** is singular.

Now suppose **A** is nonsingular. In the special case where $\mathbf{A} = \mathbf{I}_n$, we have $|\mathbf{A}| = 1$ (why?), and so $|\mathbf{AB}| = |\mathbf{I}_n\mathbf{B}| = |\mathbf{B}| = 1|\mathbf{B}| = |\mathbf{A}|\,|\mathbf{B}|$. Finally, if **A** is any other nonsingular matrix, then **A** is row equivalent to $\mathbf{I}_n$, so there is a sequence $R_1, R_2, \ldots, R_k$ of row operations such that $R_k(\cdots(R_2(R_1(\mathbf{I}_n)))\cdots) = \mathbf{A}$. (These are the inverses of the row operations that row reduce **A** to $\mathbf{I}_n$.) Now, each row operation $R_i$ has an associated real number $r_i$, so that applying $R_i$ to a matrix multiplies its determinant by $r_i$ (as in Theorem 3.3). Hence,

$$\begin{aligned}
|\mathbf{AB}| &= |R_k(\cdots(R_2(R_1(\mathbf{I}_n)))\cdots)\mathbf{B}| && \\
&= |R_k(\cdots(R_2(R_1(\mathbf{I}_n\mathbf{B})))\cdots)| && \text{by Theorem 2.1, part (2)}\\
&= r_k\cdots r_2r_1|\mathbf{I}_n\mathbf{B}| && \text{by Theorem 3.3}\\
&= r_k\cdots r_2r_1|\mathbf{I}_n||\mathbf{B}| && \text{by the } \mathbf{I}_n \text{ special case}\\
&= |R_k(\cdots(R_2(R_1(\mathbf{I}_n)))\cdots)||\mathbf{B}| && \text{by Theorem 3.3}\\
&= |\mathbf{A}|\,|\mathbf{B}|\,.
\end{aligned}$$

□

**Example 1**

Let

$$\mathbf{A}=\begin{bmatrix}3&2&1\\5&0&-2\\-3&1&4\end{bmatrix} \quad\text{and}\quad \mathbf{B}=\begin{bmatrix}1&-1&0\\4&2&-1\\-2&0&3\end{bmatrix}.$$

Quick calculations show that $|\mathbf{A}|=-17$ and $|\mathbf{B}|=16$. Therefore, the determinant of

$$\mathbf{AB}=\begin{bmatrix}9&1&1\\9&-5&-6\\-7&5&11\end{bmatrix}$$

is $|\mathbf{AB}|=|\mathbf{A}|\,|\mathbf{B}|=(-17)\,(16)=-272$.

One consequence of Theorem 3.7 is that $|\mathbf{AB}|=0$ if and only if $|\mathbf{A}|=0$ or $|\mathbf{B}|=0$. (See Exercise 6(a).) Therefore, it follows that $\mathbf{AB}$ is singular if and only if $\mathbf{A}$ or $\mathbf{B}$ is singular. Another important result is

**Corollary 3.8** *If* $\mathbf{A}$ *is nonsingular, then* $\left|\mathbf{A}^{-1}\right|=\frac{1}{|\mathbf{A}|}$.

*Proof.* If $\mathbf{A}$ is nonsingular, then $\mathbf{AA}^{-1}=\mathbf{I}_n$. By Theorem 3.7, $|\mathbf{A}||\mathbf{A}^{-1}|=|\mathbf{I}_n|=1$, so $\left|\mathbf{A}^{-1}\right|=1/\,|\mathbf{A}|$. □

## Determinant of the Transpose

Our next goal is to prove Theorem 3.10—that every matrix has the same determinant as its transpose. To accomplish this, we first establish a related lemma. The first two parts of this lemma show, in special cases, how the relationship between the determinant of a matrix and the determinant of its transpose is affected by performing a single row operation. In the third part, using a strategy that should be familiar, the result is generalized to a finite sequence of row operations using a proof by induction.

**Lemma 3.9**

**(1)** *If $R$ is any row operation, then* $|R(\mathbf{I}_n)|=\left|(R(\mathbf{I}_n))^T\right|$.

**(2)** *If* $\mathbf{B}$ *is a matrix such that* $|\mathbf{B}|=\left|\mathbf{B}^T\right|$, *and $R$ is any row operation, then*

$$|R(\mathbf{B})|=\left|(R(\mathbf{B}))^T\right|.$$

**(3)** *If $R_1,\ldots,R_k$ are any row operations, then*

$$|R_k(\cdots R_2(R_1(\mathbf{I}_n))\cdots)|=\left|(R_k(\cdots R_2(R_1(\mathbf{I}_n))\cdots))^T\right|.$$

*Proof.* (1) If $R$ is a Type (I) or Type (III) row operation, then $R(\mathbf{I}_n)$ is symmetric (see parts (a) and (b) of Exercise 4), so part (1) is true in both cases. If $R$ is the Type (II) row operation $\langle i\rangle\leftarrow c\,\langle j\rangle+\langle i\rangle$, then let $S$ be the Type (II) row operation $\langle j\rangle\leftarrow c\,\langle i\rangle+\langle j\rangle$. We can show that $(R(\mathbf{I}_n))^T=S(\mathbf{I}_n)$ (see part (c) of Exercise 4). Then, by part (2) of Theorem 3.3, both $|R(\mathbf{I}_n)|$ and $|S(\mathbf{I}_n)|$ equal $|\mathbf{I}_n|$, so part (1) also holds in this case.

(2) Suppose $|\mathbf{B}| = \left|\mathbf{B}^T\right|$ and $R$ is any row operation. Then $\left|(R(\mathbf{B}))^T\right| = \left|(R(\mathbf{I}_n\mathbf{B}))^T\right| = \left|(R(\mathbf{I}_n)\mathbf{B})^T\right|$ (by part (1) of Theorem 2.1) $= \left|\mathbf{B}^T (R(\mathbf{I}_n))^T\right|$ (by Theorem 1.18) $= \left|\mathbf{B}^T\right| \left|(R(\mathbf{I}_n))^T\right|$ (by Theorem 3.7) $= \left|(R(\mathbf{I}_n))^T\right| \left|\mathbf{B}^T\right| = |R(\mathbf{I}_n)|\,|\mathbf{B}|$ (by part (1) and the assumption for part (2)) $= |R(\mathbf{I}_n)\mathbf{B}|$ (by Theorem 3.7) $= |R(\mathbf{I}_n\mathbf{B})|$ (by Theorem 2.1) $= |R(\mathbf{B})|$.

(3) We use a proof by induction on $k$.

Base Step: Here, $k = 1$, and we must show $|R_1(\mathbf{I}_n)| = \left|(R_1(\mathbf{I}_n))^T\right|$. But this follows immediately from part (1).

Inductive Step: Let $R_1, \ldots, R_k, R_{k+1}$ be any row operations, let $\mathbf{C}_k = R_k(\cdots R_2(R_1(\mathbf{I}_n))\cdots)$, and let $\mathbf{C}_{k+1} = R_{k+1}(R_k(\cdots R_2(R_1(\mathbf{I}_n))\cdots))$. We assume $|\mathbf{C}_k| = \left|(\mathbf{C}_k)^T\right|$ and prove that $|\mathbf{C}_{k+1}| = \left|(\mathbf{C}_{k+1})^T\right|$. But $|\mathbf{C}_{k+1}| = |R_{k+1}(\mathbf{C}_k)| = \left|(R_{k+1}(\mathbf{C}_k))^T\right|$ (by part (2) and the induction assumption) $= \left|(\mathbf{C}_{k+1})^T\right|$. □

**Theorem 3.10** *If* $\mathbf{A}$ *is an* $n \times n$ *matrix, then* $|\mathbf{A}| = \left|\mathbf{A}^T\right|$.

*Proof.* Suppose that $\mathbf{A}$ is singular. Then $|\mathbf{A}| = 0$ by Theorem 3.5. We must show that $\left|\mathbf{A}^T\right| = 0$. Suppose, instead, that $\left|\mathbf{A}^T\right| \neq 0$. Then $\mathbf{A}^T$ is nonsingular, by Theorem 3.5. Hence, $\mathbf{A} = \left(\mathbf{A}^T\right)^T$ is nonsingular, by part (4) of Theorem 2.12, giving a contradiction. Hence, $\left|\mathbf{A}^T\right| = 0 = |\mathbf{A}|$.

Now, suppose that $\mathbf{A}$ is nonsingular. Then $\mathbf{A}$ is row equivalent to $\mathbf{I}_n$ (by Theorem 2.15). Hence, there are row operations $R_1, \ldots, R_k$ such that $\mathbf{A} = R_k(\cdots R_2(R_1(\mathbf{I}_n))\cdots)$. Therefore, by part (3) of Lemma 3.9,

$$|\mathbf{A}| = |R_k(\cdots R_2(R_1(\mathbf{I}_n))\cdots)| = \left|(R_k(\cdots R_2(R_1(\mathbf{I}_n))\cdots))^T\right| = \left|\mathbf{A}^T\right|.$$

□

**Example 2**

A quick calculation shows that if

$$\mathbf{A} = \begin{bmatrix} -1 & 4 & 1 \\ 2 & 0 & 3 \\ -1 & -1 & 2 \end{bmatrix},$$

then $|\mathbf{A}| = -33$. Hence, by Theorem 3.10,

$$\left|\mathbf{A}^T\right| = \begin{vmatrix} -1 & 2 & -1 \\ 4 & 0 & -1 \\ 1 & 3 & 2 \end{vmatrix} = -33.$$

Theorem 3.10 can be used to prove "column versions" of several earlier results involving determinants. For example, the determinant of a *lower* triangular matrix equals the product of its main diagonal entries, just as for an upper triangular matrix. Also, if a square matrix has an entire *column* of zeroes, or if it has two identical *columns*, then its determinant is zero, just as with rows.

Also, column operations analogous to the familiar row operations can be defined. For example, a Type (I) column operation multiplies all entries of a given column of a matrix by a nonzero scalar. Theorem 3.10 can be combined with Theorem 3.3 to show that each type of column operation has the same effect on the determinant of a matrix as its corresponding row operation.

**Example 3**

Let

$$\mathbf{A} = \begin{bmatrix} 2 & 5 & 1 \\ 1 & 2 & 3 \\ -3 & 1 & -1 \end{bmatrix}.$$

After the Type (II) *column* operation $\langle\text{col. }2\rangle \leftarrow -3\langle\text{col. }1\rangle + \langle\text{col. }2\rangle$, we have

$$\mathbf{B} = \begin{bmatrix} 2 & -1 & 1 \\ 1 & -1 & 3 \\ -3 & 10 & -1 \end{bmatrix}.$$

A quick calculation checks that $|\mathbf{A}| = -43 = |\mathbf{B}|$. Thus, this Type (II) column operation has no effect on the determinant, as we would expect.

## A More General Cofactor Expansion

Our definition of the determinant specifies that we multiply the elements $a_{ni}$ of the last row of an $n \times n$ matrix $\mathbf{A}$ by their corresponding cofactors $\mathcal{A}_{ni}$, and sum the results. The next theorem shows the same result is obtained when a cofactor expansion is performed across *any* row or *any* column of the matrix!

**Theorem 3.11** *Let* $\mathbf{A}$ *be an* $n \times n$ *matrix, with* $n \geq 2$. *Then,*

**(1)** $a_{i1}\mathcal{A}_{i1} + a_{i2}\mathcal{A}_{i2} + \cdots + a_{in}\mathcal{A}_{in} = |\mathbf{A}|$, *for each* $i$, $1 \leq i \leq n$

**(2)** $a_{1j}\mathcal{A}_{1j} + a_{2j}\mathcal{A}_{2j} + \cdots + a_{nj}\mathcal{A}_{nj} = |\mathbf{A}|$, *for each* $j$, $1 \leq j \leq n$.

The formulas for $|\mathbf{A}|$ given in Theorem 3.11 are called the **cofactor expansion** (or, **Laplace expansion**) **along the** $\boldsymbol{i}$**th row** (part (1)) **and** $\boldsymbol{j}$**th column** (part (2)). An outline of the proof of this theorem is provided in Exercises 14 and 15. The proof that any row can be used, not simply the last row, is established by considering the effect of certain row swaps on the matrix. Then the $|\mathbf{A}| = \left|\mathbf{A}^T\right|$ formula is used to explain why any column expansion is allowable.

**Example 4**

Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 5 & 0 & 1 & -2 \\ 2 & 2 & 3 & 1 \\ -1 & 3 & 2 & 5 \\ 6 & 0 & 1 & 1 \end{bmatrix}.$$

After some calculation, we find that the 16 cofactors of $\mathbf{A}$ are

$$\begin{array}{llll} \mathcal{A}_{11} = -12 & \mathcal{A}_{12} = -74 & \mathcal{A}_{13} = 50 & \mathcal{A}_{14} = 22 \\ \mathcal{A}_{21} = 9 & \mathcal{A}_{22} = 42 & \mathcal{A}_{23} = -51 & \mathcal{A}_{24} = -3 \\ \mathcal{A}_{31} = -6 & \mathcal{A}_{32} = -46 & \mathcal{A}_{33} = 34 & \mathcal{A}_{34} = 2 \\ \mathcal{A}_{41} = -3 & \mathcal{A}_{42} = 40 & \mathcal{A}_{43} = -19 & \mathcal{A}_{44} = -17 \end{array}$$

We will use these values to compute $|\mathbf{A}|$ by a cofactor expansion across several different rows and columns of $\mathbf{A}$. Along the 2nd row, we have

$$\begin{aligned} |\mathbf{A}| &= a_{21}\mathcal{A}_{21} + a_{22}\mathcal{A}_{22} + a_{23}\mathcal{A}_{23} + a_{24}\mathcal{A}_{24} \\ &= 2(9) + 2(42) + 3(-51) + 1(-3) = -54. \end{aligned}$$

Along the 2nd column, we have

$$\begin{aligned} |\mathbf{A}| &= a_{12}\mathcal{A}_{12} + a_{22}\mathcal{A}_{22} + a_{32}\mathcal{A}_{32} + a_{42}\mathcal{A}_{42} \\ &= 0(-74) + 2(42) + 3(-46) + 0(40) = -54. \end{aligned}$$

Along the 4th column, we have

$$\begin{aligned} |\mathbf{A}| &= a_{14}\mathcal{A}_{14} + a_{24}\mathcal{A}_{24} + a_{34}\mathcal{A}_{34} + a_{44}\mathcal{A}_{44} \\ &= -2(22) + 1(-3) + 5(2) + 1(-17) = -54. \end{aligned}$$

Note in Example 4 that cofactor expansion is easiest along the second column because that column has two zeroes (entries $a_{12}$ and $a_{42}$). In this case, only two cofactors, $\mathcal{A}_{22}$ and $\mathcal{A}_{32}$ were really needed to compute $|\mathbf{A}|$. We generally choose the row or column containing the largest number of zero entries for cofactor expansion.

## Cramer's Rule

We conclude this section by stating an explicit formula, known as **Cramer's Rule**, for the solution to a system of $n$ equations and $n$ variables when it is unique:

**Theorem 3.12 (Cramer's Rule)** *Let* $\mathbf{AX} = \mathbf{B}$ *be a system of* $n$ *equations in* $n$ *variables with* $|\mathbf{A}| \neq 0$. *For* $1 \leq i \leq n$, *let* $\mathbf{A}_i$ *be the* $n \times n$ *matrix obtained by replacing the* $i$*th column of* $\mathbf{A}$ *with* $\mathbf{B}$. *Then the entries of the unique solution* $\mathbf{X}$ *are*

$$x_1 = \frac{|\mathbf{A}_1|}{|\mathbf{A}|}, \quad x_2 = \frac{|\mathbf{A}_2|}{|\mathbf{A}|}, \quad \ldots, \quad x_n = \frac{|\mathbf{A}_n|}{|\mathbf{A}|}.$$

The proof of this theorem is outlined in Exercise 18. Cramer's Rule cannot be used for a system $\mathbf{AX} = \mathbf{B}$ in which $|\mathbf{A}| = 0$ (why?). It is frequently used on $3 \times 3$ systems having a unique solution, because the determinants involved can be calculated quickly by hand.

### Example 5

We will solve

$$\begin{cases} 5x_1 - 3x_2 - 10x_3 = -9 \\ 2x_1 + 2x_2 - 3x_3 = 4 \\ -3x_1 - x_2 + 5x_3 = -1 \end{cases}$$

using Cramer's Rule. This system is equivalent to $\mathbf{AX} = \mathbf{B}$ where

$$\mathbf{A} = \begin{bmatrix} 5 & -3 & -10 \\ 2 & 2 & -3 \\ -3 & -1 & 5 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} -9 \\ 4 \\ -1 \end{bmatrix}.$$

A quick calculation shows that $|\mathbf{A}| = -2$. Let

$$\mathbf{A}_1 = \begin{bmatrix} -9 & -3 & -10 \\ 4 & 2 & -3 \\ -1 & -1 & 5 \end{bmatrix}, \ \mathbf{A}_2 = \begin{bmatrix} 5 & -9 & -10 \\ 2 & 4 & -3 \\ -3 & -1 & 5 \end{bmatrix} \text{ and } \mathbf{A}_3 = \begin{bmatrix} 5 & -3 & -9 \\ 2 & 2 & 4 \\ -3 & -1 & -1 \end{bmatrix}.$$

The matrix $\mathbf{A}_1$ is identical to $\mathbf{A}$, except in the 1st column, where its entries are taken from $\mathbf{B}$. $\mathbf{A}_2$ and $\mathbf{A}_3$ are created in an analogous manner. A quick computation shows that $|\mathbf{A}_1| = 8$, $|\mathbf{A}_2| = -6$, and $|\mathbf{A}_3| = 4$. Therefore,

$$x_1 = \frac{|\mathbf{A}_1|}{|\mathbf{A}|} = \frac{8}{-2} = -4, \quad x_2 = \frac{|\mathbf{A}_2|}{|\mathbf{A}|} = \frac{-6}{-2} = 3, \quad \text{and} \quad x_3 = \frac{|\mathbf{A}_3|}{|\mathbf{A}|} = \frac{4}{-2} = -2.$$

Hence, the unique solution to the given system is $(x_1, x_2, x_3) = (-4, 3, -2)$.

Notice that solving the system in Example 5 essentially amounts to calculating four determinants: $|\mathbf{A}|$, $|\mathbf{A}_1|$, $|\mathbf{A}_2|$, and $|\mathbf{A}_3|$.

## New Vocabulary

cofactor expansion (along any row or column)

Cramer's Rule

## Highlights

- For $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$, $|\mathbf{AB}| = |\mathbf{A}|\,|\mathbf{B}|$.
- If $\mathbf{A}$ is a nonsingular matrix, then $\left|\mathbf{A}^{-1}\right| = 1/\,|\mathbf{A}|$.
- If $\mathbf{A}$ is a square matrix, then $\left|\mathbf{A}^T\right| = |\mathbf{A}|$.
- The determinant of a square matrix can be found using cofactor expansion along any row or column.
- A system $\mathbf{AX} = \mathbf{B}$ where $|\mathbf{A}| \neq 0$ can be solved using Cramer's Rule: For each $i$, $x_i = |\mathbf{A}_i|/|\mathbf{A}|$, where $\mathbf{A}_i = \mathbf{A}$ except that the $i$th column of $\mathbf{A}_i$ equals $\mathbf{B}$.

## Exercises for Section 3.3

1. For a general $4 \times 4$ matrix $\mathbf{A}$, write out the formula for $|\mathbf{A}|$ using a cofactor expansion along the indicated row or column.
   ★ (a) Third row
   (b) Second row
   ★ (c) Fourth column
   (d) Second column

2. Find the determinant of each of the following matrices by performing a cofactor expansion along the indicated row or column:
   ★ (a) Second row of $\begin{bmatrix} 2 & -1 & 4 \\ 0 & 3 & -2 \\ 5 & -2 & -3 \end{bmatrix}$
   (b) First row of $\begin{bmatrix} 2 & 5 & -10 \\ 3 & -4 & 12 \\ -1 & 7 & 7 \end{bmatrix}$
   ★ (c) First column of $\begin{bmatrix} 4 & -2 & 3 \\ 5 & -1 & -2 \\ 3 & 3 & 2 \end{bmatrix}$
   (d) Third column of $\begin{bmatrix} 0 & 2 & -1 & 3 \\ 1 & 5 & 0 & 8 \\ -4 & 0 & 7 & 2 \\ 2 & 8 & 0 & -6 \end{bmatrix}$

3. Use Cramer's Rule to solve each of the following systems:
   ★ (a) $\begin{cases} 3x_1 - x_2 - x_3 = -8 \\ 2x_1 - x_2 - 2x_3 = 3 \\ -9x_1 + x_2 = 39 \end{cases}$
   (b) $\begin{cases} 4x_1 + 3x_2 + x_3 = 56 \\ 8x_1 - 5x_2 + 2x_3 = 68 \\ -x_1 + 6x_2 + 7x_3 = 71 \end{cases}$
   (c) $\begin{cases} 7x_1 + 5x_2 - 8x_3 = 15 \\ 2x_1 + 4x_2 + 3x_3 = 95 \\ 2x_1 + 2x_2 - x_3 = 26 \end{cases}$
   ★ (d) $\begin{cases} -5x_1 + 2x_2 - 2x_3 + x_4 = -10 \\ 2x_1 - x_2 + 2x_3 - 2x_4 = -9 \\ 5x_1 - 2x_2 + 3x_3 - x_4 = 7 \\ -6x_1 + 2x_2 - 2x_3 + x_4 = -14 \end{cases}$

▶ 4. Provide the indicated details for some of the steps in the proof of Lemma 3.9.
   (a) If $R$ is a Type (I) operation, prove that $R(\mathbf{I}_n)$ is symmetric.
   (b) If $R$ is a Type (III) operation, prove that $R(\mathbf{I}_n)$ is symmetric.
   (c) If $R$ is the Type (II) operation $\langle i\rangle \leftarrow c\langle j\rangle + \langle i\rangle$ and $S$ is the Type (II) row operation $\langle j\rangle \leftarrow c\langle i\rangle + \langle j\rangle$, prove that $(R(\mathbf{I}_n))^T = S(\mathbf{I}_n)$.

5. Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices.
   (a) Show that $\mathbf{A}$ is nonsingular if and only if $\mathbf{A}^T$ is nonsingular.
   (b) Show that $|\mathbf{AB}| = |\mathbf{BA}|$. (Remember that, in general, $\mathbf{AB} \neq \mathbf{BA}$.)

6. Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices.
   (a) Show that $|\mathbf{AB}| = 0$ if and only if $|\mathbf{A}| = 0$ or $|\mathbf{B}| = 0$.
   (b) Show that if $\mathbf{AB} = -\mathbf{BA}$ and $n$ is odd, then $\mathbf{A}$ or $\mathbf{B}$ is singular.

7. Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices.
   (a) Show that $\left|\mathbf{AA}^T\right| \geq 0$.
   (b) Show that $\left|\mathbf{AB}^T\right| = \left|\mathbf{A}^T\right| |\mathbf{B}|$.

8. Let $\mathbf{A}$ be an $n \times n$ skew-symmetric matrix.
   (a) If $n$ is odd, show that $|\mathbf{A}| = 0$.
   ★ (b) If $n$ is even, give an example where $|\mathbf{A}| \neq 0$.

9. An **orthogonal** matrix is a (square) matrix $\mathbf{A}$ with $\mathbf{A}^T = \mathbf{A}^{-1}$.
   (a) Why is $\mathbf{I}_n$ orthogonal?
   ★ (b) Find a $3 \times 3$ orthogonal matrix other than $\mathbf{I}_3$.
   (c) Show that $|\mathbf{A}| = \pm 1$ if $\mathbf{A}$ is orthogonal.

10. Show that there is no matrix $\mathbf{A}$ such that

$$\mathbf{A}^2 = \begin{bmatrix} 5 & 3 & -4 \\ 1 & -2 & 8 \\ 7 & 3 & 11 \end{bmatrix}.$$

11. Give a proof by induction in each case.
   (a) **General form of Theorem 3.7:** Assuming Theorem 3.7, prove $|\mathbf{A}_1\mathbf{A}_2\cdots\mathbf{A}_k| = |\mathbf{A}_1||\mathbf{A}_2|\cdots|\mathbf{A}_k|$ for any $n \times n$ matrices $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_k$.

(b) Prove $\left|\mathbf{A}^k\right| = |\mathbf{A}|^k$ for any $n \times n$ matrix $\mathbf{A}$ and any integer $k \geq 1$.
(c) Let $\mathbf{A}$ be an $n \times n$ matrix. Show that if $\mathbf{A}^k = \mathbf{O}_n$, for some integer $k \geq 1$, then $|\mathbf{A}| = 0$.

12. Suppose that $|\mathbf{A}|$ is an integer.
(a) Prove that $|\mathbf{A}^n|$ is not prime, for $n \geq 2$. (Recall that a **prime** number is an integer $> 1$ with no positive integer divisors except itself and 1.)
(b) Prove that if $\mathbf{A}^n = \mathbf{I}$, for some $n \geq 1$, $n$ odd, then $|\mathbf{A}| = 1$.

13. We say that a matrix $\mathbf{B}$ is **similar** to a matrix $\mathbf{A}$ if there exists some (nonsingular) matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{B}$.
(a) Show that if $\mathbf{B}$ is similar to $\mathbf{A}$, then they are both square matrices of the same size.
★ (b) Find two different matrices $\mathbf{B}$ similar to $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.
(c) Show that every square matrix $\mathbf{A}$ is similar to itself.
(d) Show that if $\mathbf{B}$ is similar to $\mathbf{A}$, then $\mathbf{A}$ is similar to $\mathbf{B}$.
(e) Prove that if $\mathbf{A}$ is similar to $\mathbf{B}$ and $\mathbf{B}$ is similar to $\mathbf{C}$, then $\mathbf{A}$ is similar to $\mathbf{C}$.
(f) Prove that if $\mathbf{A}$ is similar to $\mathbf{I}_n$, then $\mathbf{A} = \mathbf{I}_n$.
(g) Show that if $\mathbf{A}$ and $\mathbf{B}$ are similar, then $|\mathbf{A}| = |\mathbf{B}|$.

▶ 14. This exercise will prove part (1) of Theorem 3.11.
(a) Show that if part (1) of Theorem 3.11 is true for some $i = k$ with $2 \leq k \leq n$, then it is also true for $i = k - 1$. (Hint: Let $\mathbf{B} = R(\mathbf{A})$, where $R$ is the row operation (III): $\langle k \rangle \leftrightarrow \langle k-1 \rangle$. Show that $|\mathbf{B}_{kj}| = |\mathbf{A}_{(k-1)j}|$ for each $j$. Then apply part (1) of Theorem 3.11 along the $k$th row of $\mathbf{B}$.)
(b) Use part (a) to complete the proof of part (1) of Theorem 3.11.

▶ 15. This exercise will prove part (2) of Theorem 3.11.
(a) Let $\mathbf{A}$ be an $n \times n$ matrix. Show that $(\mathbf{A}_{jm})^T = (\mathbf{A}^T)_{mj}$, for $1 \leq j, m \leq n$, where $(\mathbf{A}^T)_{mj}$ refers to the $(m, j)$ submatrix of $\mathbf{A}^T$. (Hint: Show the $(i,k)$ entry of $(\mathbf{A}_{jm})^T = (i,k)$ entry of $(\mathbf{A}^T)_{mj}$ in each of the following cases: (1) $1 \leq k < j$ and $1 \leq i < m$; (2) $j \leq k < n$ and $1 \leq i < m$; (3) $1 \leq k < j$ and $m \leq i < n$; (4) $j \leq k < n$ and $m \leq i < n$.)
(b) Use part (a), part (1) of Theorem 3.11, and Theorem 3.10 to prove part (2) of Theorem 3.11.

16. Let $\mathbf{A}$ be an $n \times n$ matrix. Prove that $a_{i1}\mathcal{A}_{j1} + a_{i2}\mathcal{A}_{j2} + \cdots + a_{in}\mathcal{A}_{jn} = 0$, for $i \neq j$, $1 \leq i, j \leq n$. (Hint: Form a new matrix $\mathbf{B}$, which has all entries equal to $\mathbf{A}$, except that both the $i$th and $j$th rows of $\mathbf{B}$ equal the $i$th row of $\mathbf{A}$. Show that the cofactor expansion along the $j$th row of $\mathbf{B}$ equals $a_{i1}\mathcal{A}_{j1} + a_{i2}\mathcal{A}_{j2} + \cdots + a_{in}\mathcal{A}_{jn}$. Then apply part (1) of Theorem 3.11 and Exercise 12 in Section 3.2.)

17. Let $\mathbf{A}$ be an $n \times n$ matrix, and let $\mathbf{B}$ be the $n \times n$ matrix whose $(i, j)$ entry is $\mathcal{A}_{ij}$. Use part (1) of Theorem 3.11 and Exercise 16 to prove that $\mathbf{A}\mathbf{B}^T = (|\mathbf{A}|)\,\mathbf{I}_n$. (Note: The matrix $\mathbf{B}^T$ in this exercise is called the **classical adjoint** of the matrix $\mathbf{A}$.)

▶ 18. This exercise outlines a proof of Cramer's Rule. Consider the linear system having augmented matrix $[\mathbf{A}|\ \mathbf{B}]$, with $\mathbf{A}$ nonsingular. Let $\mathbf{A}_i$ be the matrix defined in Theorem 3.12. Let $R$ be a row operation (of any type).
(a) Show that the $i$th matrix (as defined in Theorem 3.12) for the linear system whose augmented matrix is $R([\mathbf{A}|\ \mathbf{B}])$ is equal to $R(\mathbf{A}_i)$.
(b) Prove that $\dfrac{|R(\mathbf{A}_i)|}{|R(\mathbf{A})|} = \dfrac{|\mathbf{A}_i|}{|\mathbf{A}|}$.
(c) Show that the solution for $\mathbf{AX} = \mathbf{B}$ as given in Theorem 3.12 is correct in the case when $\mathbf{A} = \mathbf{I}_n$.
(d) Use parts (a), (b) and (c) to prove that the solution for $\mathbf{AX} = \mathbf{B}$ as given in Theorem 3.12 is correct for any nonsingular matrix $\mathbf{A}$.

19. Suppose $\mathbf{A}$ is an $n \times n$ matrix with $|\mathbf{A}| = 0$.
(a) Prove that there is an $n \times n$ matrix $\mathbf{B}$ with $\mathbf{B} \neq \mathbf{O}_n$ such that $\mathbf{BA} = \mathbf{O}_n$. (Hint: See Exercise 16 in Section 3.2 and apply the transpose.)
(b) If $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 17 & 18 \\ 6 & 26 & 24 \end{bmatrix}$, find a $3 \times 3$ matrix $\mathbf{B}$ with $\mathbf{B} \neq \mathbf{O}_n$ such that $\mathbf{BA} = \mathbf{O}_n$.

★ 20. True or False:
(a) If $\mathbf{A}$ is a nonsingular matrix, then $|\mathbf{A}^{-1}| = \frac{1}{|\mathbf{A}^T|}$.
(b) If $\mathbf{A}$ is a $5 \times 5$ matrix, a cofactor expansion along the 2nd row gives the same result as a cofactor expansion along the 3rd column.
(c) If $\mathbf{B}$ is obtained from a Type (III) column operation on a square matrix $\mathbf{A}$, then $|\mathbf{B}| = |\mathbf{A}|$.

**(d)** If a square matrix has two identical columns, then its determinant equals zero.
**(e)** The determinant of a lower triangular matrix equals the product of its main diagonal entries.
**(f)** For the system $\begin{cases} 4x_1 - \ \ 2x_2 - \ \ x_3 = -6 \\ \quad\ -3x_2 + 4x_3 = \ \ 5 \\ \qquad\qquad\quad\ x_3 = \ \ 3 \end{cases}$, $x_2 = -\frac{1}{12}\begin{vmatrix} 4 & -6 & -1 \\ 0 & 5 & 4 \\ 0 & 3 & 1 \end{vmatrix}$.

## 3.4 Eigenvalues and Diagonalization

In this section, we define eigenvalues and eigenvectors in the context of matrices, in order to find, when possible, a diagonal form for a square matrix. Some of the theoretical details involved cannot be discussed fully until we have introduced vector spaces and linear transformations, which are covered in Chapters 4 and 5. Thus, we will take a more comprehensive look at eigenvalues and eigenvectors at the end of Chapter 5, as well as in Chapters 6 and 7.

### Eigenvalues and Eigenvectors

**Definition** Let $\mathbf{A}$ be an $n \times n$ matrix. A real number $\lambda$ is an **eigenvalue** of $\mathbf{A}$ if and only if there is a nonzero $n$-vector $\mathbf{X}$ such that $\mathbf{AX} = \lambda\mathbf{X}$. Also, any nonzero vector $\mathbf{X}$ for which $\mathbf{AX} = \lambda\mathbf{X}$ is an **eigenvector** for $\mathbf{A}$ corresponding to the eigenvalue $\lambda$.

In some textbooks, eigenvalues are called **characteristic values** and eigenvectors are called **characteristic vectors**.

Notice that an eigenvalue can be zero. However, by definition, an eigenvector is never the zero vector.

If $\mathbf{X}$ is an eigenvector associated with an eigenvalue $\lambda$ for an $n \times n$ matrix $\mathbf{A}$, then the matrix product $\mathbf{AX}$ is equivalent to performing the scalar product $\lambda\mathbf{X}$. Thus, $\mathbf{AX}$ is parallel to the vector $\mathbf{X}$, **dilating** (or lengthening) $\mathbf{X}$ if $|\lambda| > 1$ and **contracting** (or shortening) $\mathbf{X}$ if $|\lambda| < 1$. Of course, if $\lambda = 0$, then $\mathbf{AX} = \mathbf{0}$.

**Example 1**

Consider the $3 \times 3$ matrix

$$\mathbf{A} = \begin{bmatrix} -4 & 8 & -12 \\ 6 & -6 & 12 \\ 6 & -8 & 14 \end{bmatrix}.$$

Now, $\lambda = 2$ is an eigenvalue for $\mathbf{A}$ because a nonzero vector $\mathbf{X}$ exists such that $\mathbf{AX} = 2\mathbf{X}$. In particular,

$$\mathbf{A}\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 & 8 & -12 \\ 6 & -6 & 12 \\ 6 & -8 & 14 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}.$$

Hence, $\mathbf{X} = [4, 3, 0]$ is an eigenvector corresponding to the eigenvalue 2. In fact, any nonzero scalar multiple $c$ of $[4, 3, 0]$ is also an eigenvector corresponding to 2, because $\mathbf{A}(c\mathbf{X}) = c(\mathbf{AX}) = c(2\mathbf{X}) = 2(c\mathbf{X})$. Therefore, there are infinitely many eigenvectors corresponding to the eigenvalue $\lambda = 2$.

**Definition** Let $\mathbf{A}$ be an $n \times n$ matrix and $\lambda$ be an eigenvalue for $\mathbf{A}$. Then the set $E_\lambda = \{\mathbf{X} \mid \mathbf{AX} = \lambda\mathbf{X}\}$ is called the **eigenspace** of $\lambda$.

The eigenspace $E_\lambda$ for a particular eigenvalue $\lambda$ of $\mathbf{A}$ consists of the set of all eigenvectors for $\mathbf{A}$ associated with $\lambda$, together with the zero vector $\mathbf{0}$, since $\mathbf{A0} = \mathbf{0} = \lambda\mathbf{0}$, for any $\lambda$. Thus, for the matrix $\mathbf{A}$ in Example 1, the eigenspace $E_2$ contains (at least) all of the scalar multiples of $[4, 3, 0]$.

### The Characteristic Polynomial of a Matrix

Our next goal is to find a method for determining all the eigenvalues and eigenvectors of an $n \times n$ matrix $\mathbf{A}$. Now, if $\mathbf{X}$ is an eigenvector for $\mathbf{A}$ corresponding to the eigenvalue $\lambda$, then we have

$$\mathbf{AX} = \lambda\mathbf{X} = \lambda\mathbf{I}_n\mathbf{X}, \quad \text{or,} \quad (\lambda\mathbf{I}_n - \mathbf{A})\mathbf{X} = \mathbf{0}.$$

Therefore, $\mathbf{X}$ is a nontrivial solution to the homogeneous system whose coefficient matrix is $\lambda\mathbf{I}_n - \mathbf{A}$. Theorem 2.7 and Corollary 3.6 then show that $|\lambda\mathbf{I}_n - \mathbf{A}| = 0$. Since all of the steps in this argument are reversible, we have proved

**Theorem 3.13** *Let $\mathbf{A}$ be an $n \times n$ matrix and let $\lambda$ be a real number. Then $\lambda$ is an eigenvalue of $\mathbf{A}$ if and only if $|\lambda\mathbf{I}_n - \mathbf{A}| = 0$. The eigenvectors corresponding to $\lambda$ are the nontrivial solutions of the homogeneous system $(\lambda\mathbf{I}_n - \mathbf{A})\mathbf{X} = \mathbf{0}$. The eigenspace $E_\lambda$ is the complete solution set for this homogeneous system.*

Because the determinant $|\lambda\mathbf{I}_n - \mathbf{A}|$ is useful for finding eigenvalues, we make the following definition:

**Definition** If $\mathbf{A}$ is an $n \times n$ matrix, then the **characteristic polynomial** of $\mathbf{A}$ is the polynomial $p_{\mathbf{A}}(x) = |x\mathbf{I}_n - \mathbf{A}|$.

It can be shown that if $\mathbf{A}$ is an $n \times n$ matrix, then $p_{\mathbf{A}}(x)$ is a polynomial of degree $n$ (see Exercise 23(b)). From calculus, we know that $p_{\mathbf{A}}(x)$ has at most $n$ real roots. Now, using this terminology, we can rephrase the first assertion of Theorem 3.13 as

The eigenvalues of an $n \times n$ matrix $\mathbf{A}$ are precisely the real roots of the characteristic polynomial $p_{\mathbf{A}}(x)$.

## Example 2

The characteristic polynomial of $\mathbf{A} = \begin{bmatrix} 12 & -51 \\ 2 & -11 \end{bmatrix}$ is

$$\begin{aligned} p_{\mathbf{A}}(x) &= |x\mathbf{I}_2 - \mathbf{A}| \\ &= \left| \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} - \begin{bmatrix} 12 & -51 \\ 2 & -11 \end{bmatrix} \right| = \begin{vmatrix} x-12 & 51 \\ -2 & x+11 \end{vmatrix} \\ &= (x-12)(x+11) + 102 \\ &= x^2 - x - 30 = (x-6)(x+5). \end{aligned}$$

Therefore, the eigenvalues of $\mathbf{A}$ are the solutions to $p_{\mathbf{A}}(x) = 0$; that is, $\lambda_1 = 6$ and $\lambda_2 = -5$. We now find the eigenspace for each of the eigenvalues of $\mathbf{A}$.

**Eigenvalue $\lambda_1 = 6$**: For this eigenvalue, we need to solve the homogeneous system $(\lambda_1\mathbf{I}_2 - \mathbf{A})\mathbf{X} = \mathbf{0}$; that is, $(6\mathbf{I}_2 - \mathbf{A})\mathbf{X} = \mathbf{0}$. Since

$$6\mathbf{I}_2 - \mathbf{A} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} - \begin{bmatrix} 12 & -51 \\ 2 & -11 \end{bmatrix} = \begin{bmatrix} -6 & 51 \\ -2 & 17 \end{bmatrix},$$

the augmented matrix for this system is

$$[6\mathbf{I}_2 - \mathbf{A} \,|\, \mathbf{0}] = \left[\begin{array}{cc|c} -6 & 51 & 0 \\ -2 & 17 & 0 \end{array}\right], \text{ which row reduces to } \left[\begin{array}{cc|c} 1 & -\frac{17}{2} & 0 \\ 0 & 0 & 0 \end{array}\right].$$

The complete solution set for this system is $\left\{ b\left[\frac{17}{2}, 1\right] \,\middle|\, b \in \mathbb{R} \right\}$. This is the eigenspace $E_6$ for the eigenvalue $\lambda_1 = 6$.

To simplify the description of eigenspaces, we often eliminate fractions by replacing the eigenvectors describing the solution set with appropriate scalar multiples. (This does not add or subtract any vectors from the eigenspace. Why?) In this particular case we multiply $\left[\frac{17}{2}, 1\right]$ by 2, and express this eigenspace as

$$E_6 = \{ b[17, 2] \mid b \in \mathbb{R} \}.$$

Thus, the eigenvectors for $\lambda_1 = 6$ are precisely the nonzero scalar multiples of $\mathbf{X}_1 = [17, 2]$. We can check that $[17, 2]$ is an eigenvector corresponding to $\lambda_1 = 6$ by noting that

$$\mathbf{A}\mathbf{X}_1 = \begin{bmatrix} 12 & -51 \\ 2 & -11 \end{bmatrix} \begin{bmatrix} 17 \\ 2 \end{bmatrix} = \begin{bmatrix} 102 \\ 12 \end{bmatrix} = 6 \begin{bmatrix} 17 \\ 2 \end{bmatrix} = 6\mathbf{X}_1.$$

**Eigenvalue $\lambda_2 = -5$**: For this eigenvalue, we need to solve the homogeneous system $(\lambda_2 \mathbf{I}_2 - \mathbf{A})\mathbf{X} = \mathbf{0}$; that is, $(-5\mathbf{I}_2 - \mathbf{A})\mathbf{X} = \mathbf{0}$. Since

$$-5\mathbf{I}_2 - \mathbf{A} = \begin{bmatrix} -5 & 0 \\ 0 & -5 \end{bmatrix} - \begin{bmatrix} 12 & -51 \\ 2 & -11 \end{bmatrix} = \begin{bmatrix} -17 & 51 \\ -2 & 6 \end{bmatrix},$$

the augmented matrix for this system is

$$[-5\mathbf{I}_2 - \mathbf{A} \,|\, \mathbf{0}] = \left[\begin{array}{cc|c} -17 & 51 & 0 \\ -2 & 6 & 0 \end{array}\right], \text{ which row reduces to } \left[\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

The complete solution set for this system is the eigenspace

$$E_{-5} = \{ b[3,1] \mid b \in \mathbb{R} \}.$$

Thus, the eigenvectors for $\lambda_2 = -5$ are precisely the nonzero scalar multiples of $\mathbf{X}_2 = [3, 1]$. You should check that for this vector $\mathbf{X}_2$, we have $\mathbf{A}\mathbf{X}_2 = -5\mathbf{X}_2$.

## Example 3

The characteristic polynomial of

$$\mathbf{B} = \begin{bmatrix} 7 & 1 & -1 \\ -11 & -3 & 2 \\ 18 & 2 & -4 \end{bmatrix} \quad \text{is}$$

$$p_{\mathbf{B}}(x) = \left| \begin{bmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{bmatrix} - \begin{bmatrix} 7 & 1 & -1 \\ -11 & -3 & 2 \\ 18 & 2 & -4 \end{bmatrix} \right| = \begin{vmatrix} x-7 & -1 & 1 \\ 11 & x+3 & -2 \\ -18 & -2 & x+4 \end{vmatrix},$$

which simplifies to $p_{\mathbf{B}}(x) = x^3 - 12x - 16 = (x+2)^2(x-4)$. Hence, $\lambda_1 = -2$ and $\lambda_2 = 4$ are the eigenvalues for $\mathbf{B}$.

**Eigenvalue $\lambda_1 = -2$**: We need to solve the homogeneous system $(-2\mathbf{I}_3 - \mathbf{B})\mathbf{X} = \mathbf{0}$. Since

$$-2\mathbf{I}_3 - \mathbf{B} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} - \begin{bmatrix} 7 & 1 & -1 \\ -11 & -3 & 2 \\ 18 & 2 & -4 \end{bmatrix} = \begin{bmatrix} -9 & -1 & 1 \\ 11 & 1 & -2 \\ -18 & -2 & 2 \end{bmatrix},$$

the augmented matrix for this system is

$$[2\mathbf{I}_3 - \mathbf{B} \,|\, \mathbf{0}] = \left[\begin{array}{ccc|c} -9 & -1 & 1 & 0 \\ 11 & 1 & -2 & 0 \\ -18 & -2 & 2 & 0 \end{array}\right], \text{ which row reduces to } \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{7}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus, the complete solution set for this system is the eigenspace $E_{-2} = \left\{ c\left[\frac{1}{2}, -\frac{7}{2}, 1\right] \,\middle|\, c \in \mathbb{R} \right\}$. After multiplying to remove fractions, this is equivalent to

$$E_{-2} = \{ c[1, -7, 2] \mid c \in \mathbb{R} \}.$$

Hence, the eigenvectors for $\lambda_1 = -2$ are precisely the nonzero multiples of $\mathbf{X}_1 = [1, -7, 2]$. You can verify that $\mathbf{B}\mathbf{X}_1 = -2\mathbf{X}_1$.

**Eigenvalue $\lambda_2 = 4$**: We need to solve the homogeneous system $(4\mathbf{I}_3 - \mathbf{B})\mathbf{X} = \mathbf{0}$. Since

$$4\mathbf{I}_3 - \mathbf{B} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 7 & 1 & -1 \\ -11 & -3 & 2 \\ 18 & 2 & -4 \end{bmatrix} = \begin{bmatrix} -3 & -1 & 1 \\ 11 & 7 & -2 \\ -18 & -2 & 8 \end{bmatrix},$$

the augmented matrix for this system is

$$[2\mathbf{I}_3 - \mathbf{B} \,|\, \mathbf{0}] = \left[\begin{array}{ccc|c} -3 & -1 & 1 & 0 \\ 11 & 7 & -2 & 0 \\ -18 & -2 & 8 & 0 \end{array}\right], \text{ which row reduces to } \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus, the complete solution set for this system is the eigenspace $E_4 = \left\{ c\left[\frac{1}{2}, -\frac{1}{2}, 1\right] \;\middle|\; c \in \mathbb{R} \right\}$. After multiplying to remove fractions, this is equivalent to

$$E_4 = \{ c\,[1, -1, 2] \mid c \in \mathbb{R} \}.$$

Thus, the eigenvectors for $\lambda_2 = 4$ are precisely the nonzero multiples of $\mathbf{X}_2 = [1, -1, 2]$. You can verify that $\mathbf{BX}_2 = 4\mathbf{X}_2$.

## Example 4

Recall the matrix $\mathbf{A} = \begin{bmatrix} -4 & 8 & -12 \\ 6 & -6 & 12 \\ 6 & -8 & 14 \end{bmatrix}$ from Example 1. We will find all of the eigenvalues and eigenspaces for $\mathbf{A}$. The characteristic polynomial for $\mathbf{A}$ is $|x\mathbf{I}_3 - \mathbf{A}|$, which is

$$\left| \begin{bmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{bmatrix} - \begin{bmatrix} -4 & 8 & -12 \\ 6 & -6 & 12 \\ 6 & -8 & 14 \end{bmatrix} \right| = \begin{vmatrix} x+4 & -8 & 12 \\ -6 & x+6 & -12 \\ -6 & 8 & x-14 \end{vmatrix}.$$

Setting this equal to 0, we obtain after some simplification, $x^3 - 4x^2 + 4x = x(x-2)^2 = 0$, which yields two solutions: $\lambda_1 = 2$, and $\lambda_2 = 0$. (We already noted in Example 1 that 2 is an eigenvalue for $\mathbf{A}$.)

**Eigenvalue $\boldsymbol{\lambda_1 = 2}$:** We need to solve the homogeneous system $(2\mathbf{I}_3 - \mathbf{A})\mathbf{X} = \mathbf{0}$. Since

$$2\mathbf{I}_3 - \mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} -4 & 8 & -12 \\ 6 & -6 & 12 \\ 6 & -8 & 14 \end{bmatrix} = \begin{bmatrix} 6 & -8 & 12 \\ -6 & 8 & -12 \\ -6 & 8 & -12 \end{bmatrix},$$

the augmented matrix for this system is

$$[2\mathbf{I}_3 - \mathbf{A} \mid \mathbf{0}] = \left[\begin{array}{ccc|c} 6 & -8 & 12 & 0 \\ -6 & 8 & -12 & 0 \\ -6 & 8 & -12 & 0 \end{array}\right], \text{ which row reduces to } \left[\begin{array}{ccc|c} 1 & -\frac{4}{3} & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus, after multiplying to remove fractions, the complete solution set for this system is the eigenspace

$$E_2 = \{ a[4, 3, 0] + b[-2, 0, 1] \mid a, b \in \mathbb{R} \}.$$

Setting $a = 1$, $b = 0$ produces the eigenvector $[4, 3, 0]$ from Example 1. If $\mathbf{X}_1 = [4, 3, 0]$, notice that $\mathbf{AX}_1 = 2\mathbf{X}_1$. However, with $a = 0$, $b = 1$, we also discover the eigenvector $\mathbf{X}_2 = [-2, 0, 1]$. You can verify that $\mathbf{AX}_2 = 2\mathbf{X}_2$. Also, any nontrivial linear combination of $\mathbf{X}_1$ and $\mathbf{X}_2$ (that is, a linear combination having at least one nonzero coefficient) is also an eigenvector for $\mathbf{A}$ corresponding to $\lambda_1$ (why?). In fact, the eigenspace $E_2$ consists precisely of all the linear combinations of $\mathbf{X}_1$ and $\mathbf{X}_2$.

**Eigenvalue $\boldsymbol{\lambda_2 = 0}$:** Similarly, we can find eigenvectors for $\lambda_2$ by row reducing

$$[0\mathbf{I}_3 - \mathbf{A} \mid \mathbf{0}] = \left[\begin{array}{ccc|c} 4 & -8 & 12 & 0 \\ -6 & 6 & -12 & 0 \\ -6 & 8 & -14 & 0 \end{array}\right] \text{ to obtain } \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

which has the solution set

$$E_0 = \{ c[-1, 1, 1] \mid c \in \mathbb{R} \}.$$

Therefore, the eigenvectors for $\mathbf{A}$ corresponding to $\lambda_2 = 0$ are the nonzero scalar multiples of $\mathbf{X}_3 = [-1, 1, 1]$. You should check that $\mathbf{AX}_3 = 0\mathbf{X}_3$.

Calculating the characteristic polynomial of a $4 \times 4$ or larger matrix can be tedious. Computing the roots of the characteristic polynomial may also be difficult. Thus, in practice, you should use a calculator or computer with appropriate software to compute the eigenvalues of a matrix. Numerical techniques for finding eigenvalues without the characteristic polynomial are discussed in Section 9.3.

## Diagonalization

One of the most important uses of eigenvalues and eigenvectors is in the diagonalization of matrices. Because diagonal matrices have such a simple structure, it is relatively easy to compute a matrix product when one of the matrices is diagonal. As we will see later, other important matrix computations are also easier when using diagonal matrices. Hence, if a given square matrix can be replaced by a corresponding diagonal matrix, it could greatly simplify computations involving the original matrix. Therefore, our next goal is to present a formal method for using eigenvalues and eigenvectors to find a diagonal form for a given square matrix, if possible. Before stating the method, we motivate it with an example.

### Example 5

Consider again the $3 \times 3$ matrix

$$\mathbf{A} = \begin{bmatrix} -4 & 8 & -12 \\ 6 & -6 & 12 \\ 6 & -8 & 14 \end{bmatrix}.$$

In Example 4, we found the eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 0$ of $\mathbf{A}$. We also found eigenvectors $\mathbf{X} = [4, 3, 0]$ and $\mathbf{Y} = [-2, 0, 1]$ for $\lambda_1 = 2$ and an eigenvector $\mathbf{Z} = [-1, 1, 1]$ for $\lambda_2 = 0$. We will use these three vectors as columns for a $3 \times 3$ matrix

$$\mathbf{P} = \begin{bmatrix} 4 & -2 & -1 \\ 3 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Now, $|\mathbf{P}| = -1$ (verify!), and so $\mathbf{P}$ is nonsingular. A quick calculation yields

$$\mathbf{P}^{-1} = \begin{bmatrix} 1 & -1 & 2 \\ 3 & -4 & 7 \\ -3 & 4 & -6 \end{bmatrix}.$$

We can now use $\mathbf{A}$, $\mathbf{P}$, and $\mathbf{P}^{-1}$ to compute a diagonal matrix $\mathbf{D}$:

$$\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} 1 & -1 & 2 \\ 3 & -4 & 7 \\ -3 & 4 & -6 \end{bmatrix}\begin{bmatrix} -4 & 8 & -12 \\ 6 & -6 & 12 \\ 6 & -8 & 14 \end{bmatrix}\begin{bmatrix} 4 & -2 & -1 \\ 3 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Notice that each main diagonal entry $d_{ii}$ of $\mathbf{D}$ is an eigenvalue having an associated eigenvector in the corresponding column of $\mathbf{P}$.

Example 5 motivates the following definition[3]:

**Definition** A matrix $\mathbf{B}$ is **similar** to a matrix $\mathbf{A}$ if there exists some nonsingular matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{B}$.

Because $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$ in Example 5, the diagonal matrix $\mathbf{D}$ in that example is similar to the original matrix $\mathbf{A}$. Also, the computation

$$\left(\mathbf{P}^{-1}\right)^{-1}\mathbf{D}\left(\mathbf{P}^{-1}\right) = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} = \mathbf{P}\mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{P}^{-1} = \left(\mathbf{P}\mathbf{P}^{-1}\right)\mathbf{A}\left(\mathbf{P}\mathbf{P}^{-1}\right) = \mathbf{A}$$

shows that $\mathbf{A}$ is also similar to $\mathbf{D}$. Adapting this argument (or see Exercise 13 of Section 3.3), we see that, in general, for any matrices $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{A}$ is similar to $\mathbf{B}$ if and only if $\mathbf{B}$ is similar to $\mathbf{A}$. Thus, we will frequently just say that $\mathbf{A}$ and $\mathbf{B}$ are similar (to each other).

Other properties of the similarity relation between matrices were stated in Exercise 13 of Section 3.3. For example, similar matrices must be square, have the same size, and have equal determinants. Exercise 6 in this section shows that similar matrices have identical characteristic polynomials.

The next theorem shows that the diagonalization process presented in Example 5 works for many matrices.

[3] This definition of similar matrices was also given in Exercise 13 of Section 3.3.

**Theorem 3.14** *Let* $\mathbf{A}$ *and* $\mathbf{P}$ *be* $n \times n$ *matrices such that each column of* $\mathbf{P}$ *is an eigenvector for* $\mathbf{A}$. *If* $\mathbf{P}$ *is nonsingular, then* $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$ *is a diagonal matrix similar to* $\mathbf{A}$. *The* $i$*th main diagonal entry* $d_{ii}$ *of* $\mathbf{D}$ *is the eigenvalue for the eigenvector forming the* $i$*th column of* $\mathbf{P}$.

The proof of Theorem 3.14 is not difficult, and we leave it, with hints, as Exercise 20. Thus, the following technique can be used to diagonalize a matrix:

**Method for Diagonalizing an $n \times n$ Matrix A (if possible) (Diagonalization Method)**

**Step 1:** Calculate $p_{\mathbf{A}}(x) = |x\mathbf{I}_n - \mathbf{A}|$.

**Step 2:** Find all real roots of $p_{\mathbf{A}}(x)$ (that is, all real solutions to $p_{\mathbf{A}}(x) = 0$). These are the eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k$ for $\mathbf{A}$.

**Step 3:** For each eigenvalue $\lambda_m$ in turn:

Row reduce the augmented matrix $[\lambda_m\mathbf{I}_n - \mathbf{A} \,|\, \mathbf{0}]$. Use the result to obtain the fundamental solutions of the homogeneous system $(\lambda_m\mathbf{I}_n - \mathbf{A})\mathbf{X} = \mathbf{0}$. (These are found by setting each independent variable in turn equal to 1 while setting all other independent variables equal to 0. You can eliminate fractions from these solutions by replacing them with nonzero scalar multiples.)

We will often refer to the particular eigenvectors that are obtained in this manner as the **fundamental eigenvectors** for $\lambda_m$.

**Step 4:** If after repeating Step 3 for each eigenvalue, you have less than $n$ fundamental eigenvectors overall for $\mathbf{A}$, then $\mathbf{A}$ cannot be diagonalized. Stop.

**Step 5:** Otherwise, form a matrix $\mathbf{P}$ whose columns are these $n$ fundamental eigenvectors. (This matrix $\mathbf{P}$ is nonsingular.)

**Step 6:** To check your work, verify that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$ is a diagonal matrix whose $d_{ii}$ entry is the eigenvalue for the fundamental eigenvector forming the $i$th column of $\mathbf{P}$. Also note that $\mathbf{A} = \mathbf{PDP}^{-1}$.

The assertions in Step 4 that $\mathbf{A}$ cannot be diagonalized, and in Step 5 that $\mathbf{P}$ is nonsingular, will not be proved here, but will follow from results in Section 5.6.

### Example 6

Consider the $4 \times 4$ matrix

$$\mathbf{A} = \begin{bmatrix} -4 & 7 & 1 & 4 \\ 6 & -16 & -3 & -9 \\ 12 & -27 & -4 & -15 \\ -18 & 43 & 7 & 24 \end{bmatrix}.$$

**Step 1:** A lengthy calculation gives $p_{\mathbf{A}}(x) = x^4 - 3x^2 - 2x = x(x-2)(x+1)^2$.

**Step 2:** The eigenvalues of $\mathbf{A}$ are the roots of $p_{\mathbf{A}}(x)$, namely, $\lambda_1 = -1$, $\lambda_2 = 2$, and $\lambda_3 = 0$.

**Step 3:** We first compute eigenvectors for $\lambda_1 = -1$. Row reducing $[(-1)\mathbf{I}_4 - \mathbf{A} \,|\, \mathbf{0}]$ yields

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Setting the first independent variable (corresponding to column 3) equal to 1 and the second independent variable (column 4) equal to 0 gives a fundamental eigenvector $\mathbf{X}_1 = [-2, -1, 1, 0]$. Setting the second independent variable equal to 1 and the first independent variable equal to 0 gives a fundamental eigenvector $\mathbf{X}_2 = [-1, -1, 0, 1]$.

Similarly, we row reduce $[2\mathbf{I}_4 - \mathbf{A} \,|\, \mathbf{0}]$ to obtain the eigenvector $\left[\frac{1}{6}, -\frac{1}{3}, -\frac{2}{3}, 1\right]$. We multiply this by 6 to avoid fractions, yielding a fundamental eigenvector $\mathbf{X}_3 = [1, -2, -4, 6]$. Finally, from $[0\mathbf{I}_4 - \mathbf{A} \,|\, \mathbf{0}]$, we obtain a fundamental eigenvector $\mathbf{X}_4 = [1, -3, -3, 7]$.

**Step 4:** We have produced 4 fundamental eigenvectors for this $4 \times 4$ matrix, so we proceed to Step 5.

**Step 5:** Let

$$\mathbf{P} = \begin{bmatrix} -2 & -1 & 1 & 1 \\ -1 & -1 & -2 & -3 \\ 1 & 0 & -4 & -3 \\ 0 & 1 & 6 & 7 \end{bmatrix},$$

the matrix whose columns are our fundamental eigenvectors $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4$.

**Step 6:** Calculating $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$, we verify that $\mathbf{D}$ is the diagonal matrix whose corresponding entries on the main diagonal are the eigenvalues $-1$, $-1$, 2, and 0, respectively.

In Chapter 4, we will learn more about fundamental eigenvectors. Be careful! Remember that for an eigenvalue $\lambda$, any fundamental eigenvectors are only particular vectors in the eigenspace $E_\lambda$. In fact, $E_\lambda$ contains an infinite number of eigenvectors, not just the fundamental eigenvectors.

Theorem 3.14 requires a nonsingular matrix $\mathbf{P}$ whose columns are eigenvectors for $\mathbf{A}$, as in Examples 5 and 6. However, such a matrix $\mathbf{P}$ does not always exist in general. Thus, we have the following definition:

**Definition** An $n \times n$ matrix $\mathbf{A}$ is **diagonalizable** if and only if there exists a nonsingular $n \times n$ matrix $\mathbf{P}$ such that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$ is diagonal.

It can be shown that if a matrix $\mathbf{P}$ exists as described in this definition, then the columns of $\mathbf{P}$ must be eigenvectors of $\mathbf{A}$ (see Exercise 21).

## Nondiagonalizable Matrices

In the next two examples, we illustrate some square matrices that are not diagonalizable.

### Example 7

Consider the matrix

$$\mathbf{B} = \begin{bmatrix} 7 & 1 & -1 \\ -11 & -3 & 2 \\ 18 & 2 & -4 \end{bmatrix}$$

from Example 3, where we found $p_{\mathbf{B}}(x) = (x+2)^2(x-4)$, thus giving us the eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 4$. Using Step 3 of the Diagonalization Method produces the fundamental eigenvector $[1, -7, 2]$ for $\lambda_1 = -2$, and the fundamental eigenvector $[1, -1, 2]$ for $\lambda_2 = 4$. Since the method yields only two fundamental eigenvectors for this $3 \times 3$ matrix, $\mathbf{B}$ cannot be diagonalized.

### Example 8

Consider the $2 \times 2$ matrix

$$\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

for some angle $\theta$ (in radians). In Chapter 5, we will see that if a 2-vector $\mathbf{X}$ has its initial point at the origin, then $\mathbf{AX}$ is the vector obtained by rotating $\mathbf{X}$ counterclockwise about the origin through an angle of $\theta$ radians. Now,

$$p_{\mathbf{A}}(x) = \begin{vmatrix} (x-\cos\theta) & \sin\theta \\ -\sin\theta & (x-\cos\theta) \end{vmatrix} = x^2 - (2\cos\theta)x + 1.$$

Using the Quadratic Formula to solve for eigenvalues yields

$$\lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} = \cos\theta \pm \sqrt{-\sin^2\theta}.$$

Thus, there are no eigenvalues unless $\theta$ is an integral multiple of $\pi$. When there are no eigenvalues, there cannot be any eigenvectors, and so in most cases $\mathbf{A}$ cannot be diagonalized.

The lack of eigenvectors for $\mathbf{A}$ makes perfect sense geometrically. If we rotate a vector $\mathbf{X}$ beginning at the origin through an angle which is not a multiple of $\pi$ radians, then the new vector $\mathbf{AX}$ points in a direction that is not parallel to $\mathbf{X}$. Thus, $\mathbf{AX}$ cannot be a scalar multiple of $\mathbf{X}$, and hence there are no eigenvalues. If $\theta$ is an even multiple of $\pi$, then $\mathbf{A} = \mathbf{I}_2$, and $\mathbf{X}$ is rotated into itself. Therefore, 1 is an eigenvalue. (Here, $\mathbf{AX} = +1\mathbf{X}$.) If $\theta$ is an odd multiple of $\pi$, then $\mathbf{AX}$ is in the opposite direction as $\mathbf{X}$, so $-1$ is an eigenvalue. (Here, $\mathbf{AX} = -1\mathbf{X}$.)

## Algebraic Multiplicity of an Eigenvalue

**Definition** Let $\mathbf{A}$ be an $n \times n$ matrix, and let $\lambda$ be an eigenvalue for $\mathbf{A}$. Suppose that $(x-\lambda)^k$ is the highest power of $(x-\lambda)$ that divides $p_{\mathbf{A}}(x)$. Then $k$ is called the **algebraic multiplicity of** $\lambda$.

### Example 9

Recall the matrix $\mathbf{A}$ in Example 6 whose characteristic polynomial is $p_{\mathbf{A}}(x) = x(x-2)(x+1)^2$. The algebraic multiplicity of $\lambda_1 = -1$ is 2 (because the factor $(x+1)$ appears to the second power in $p_{\mathbf{A}}(x)$), while the algebraic multiplicities of $\lambda_2 = 2$ and $\lambda_3 = 0$ are both 1.

In Chapter 5, we will establish that, for any eigenvalue, the number of fundamental eigenvectors produced by the Diagonalization Method is always *less than or equal to* its algebraic multiplicity. (In Example 9, the algebraic multiplicity of each eigenvalue is actually equal to the number of fundamental eigenvectors for that eigenvalue, as shown by Step 3 of the Diagonalization Method in Example 6.)

### Example 10

Recall the nondiagonalizable matrix $\mathbf{B}$ from Example 3 with $p_{\mathbf{B}}(x) = (x+2)^2(x-4)$. The eigenvalue $\lambda_1 = -2$ for $\mathbf{B}$ has algebraic multiplicity 2 because the factor $(x+2)$ appears to the second power in $p_{\mathbf{B}}(x)$. By the remark just before this example, we know that Step 3 of the Diagonalization Method must produce two or fewer fundamental eigenvectors for $\lambda_1 = -2$. In fact, in Example 7, we obtained only one fundamental eigenvector for $\lambda_1 = -2$.

### Example 11

Consider the $3 \times 3$ matrix

$$\mathbf{A} = \begin{bmatrix} -3 & -1 & -2 \\ -2 & 16 & -18 \\ 2 & 9 & -7 \end{bmatrix},$$

for which $p_{\mathbf{A}}(x) = |x\mathbf{I}_3 - \mathbf{A}| = x^3 - 6x^2 + 25x = x(x^2 - 6x + 25)$ (verify!). Since $x^2 - 6x + 25$ has no real solutions (try the Quadratic Formula), $\mathbf{A}$ has only one eigenvalue, $\lambda = 0$, which has algebraic multiplicity 1. Thus, the Diagonalization Method can produce only one fundamental eigenvector for $\lambda$ overall. Therefore, according to Step 4, $\mathbf{A}$ cannot be diagonalized**.**

Example 11 illustrates that if the sum of the algebraic multiplicities of all the eigenvalues for an $n \times n$ matrix $\mathbf{A}$ is less than $n$, then there is no need to proceed beyond Step 2 of the Diagonalization Method. This is because we are assured that Step 3 can not produce a sufficient number of fundamental eigenvectors, and so $\mathbf{A}$ cannot be diagonalized.

## Application: Large Powers of a Matrix

If $\mathbf{D}$ is a diagonal matrix, any positive integer power of $\mathbf{D}$ can be obtained by merely raising each of the diagonal entries of $\mathbf{D}$ to that power (why?). For example,

$$\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}^{12} = \begin{bmatrix} 3^{12} & 0 \\ 0 & (-2)^{12} \end{bmatrix} = \begin{bmatrix} 531441 & 0 \\ 0 & 4096 \end{bmatrix}.$$

Now, suppose that $\mathbf{A}$ and $\mathbf{P}$ are $n \times n$ matrices such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$, a diagonal matrix. We know $\mathbf{A} = \mathbf{PDP}^{-1}$. But then,

$$\mathbf{A}^2 = \mathbf{AA} = \left(\mathbf{PDP}^{-1}\right)\left(\mathbf{PDP}^{-1}\right) = \mathbf{PD}\left(\mathbf{P}^{-1}\mathbf{P}\right)\mathbf{DP}^{-1} = \mathbf{PDI}_n\mathbf{DP}^{-1} = \mathbf{PD}^2\mathbf{P}^{-1}.$$

More generally, a straightforward proof by induction shows that for all positive integers $k$, $\mathbf{A}^k = \mathbf{PD}^k\mathbf{P}^{-1}$ (see Exercise 15). Hence, calculating positive powers of $\mathbf{A}$ is relatively easy if the corresponding matrices $\mathbf{P}$ and $\mathbf{D}$ are known.

### Example 12

We will use eigenvalues and eigenvectors to compute $\mathbf{A}^{11}$ for the matrix

$$\mathbf{A} = \begin{bmatrix} -4 & 7 & 1 & 4 \\ 6 & -16 & -3 & -9 \\ 12 & -27 & -4 & -15 \\ -18 & 43 & 7 & 24 \end{bmatrix}$$

in Example 6. Recall that in that example, we found

$$\mathbf{P} = \begin{bmatrix} -2 & -1 & 1 & 1 \\ -1 & -1 & -2 & -3 \\ 1 & 0 & -4 & -3 \\ 0 & 1 & 6 & 7 \end{bmatrix} \quad \text{and} \quad \mathbf{D} = \mathbf{P}^{-1}\mathbf{AP} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then, $\mathbf{A} = \mathbf{PDP}^{-1}$, and so

$$\begin{aligned} \mathbf{A}^{11} &= \mathbf{PD}^{11}\mathbf{P}^{-1} \\ &= \begin{bmatrix} -2 & -1 & 1 & 1 \\ -1 & -1 & -2 & -3 \\ 1 & 0 & -4 & -3 \\ 0 & 1 & 6 & 7 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2048 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -4 & 11 & 4 & 7 \\ 6 & -19 & -7 & -12 \\ -1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -2050 & 4099 & 1 & 2050 \\ 4098 & -8200 & -3 & -4101 \\ 8196 & -16395 & -4 & -8199 \\ -12294 & 24595 & 7 & 12300 \end{bmatrix}. \end{aligned}$$

The technique illustrated in Example 12 can also be adapted to calculate square roots and cube roots of matrices (see Exercises 7 and 8).

## Roundoff Error Involving Eigenvalues

If the numerical value that is obtained for an eigenvalue $\lambda$ is slightly in error, perhaps due to rounding, then the corresponding matrix $(\lambda\mathbf{I}_n - \mathbf{A})$ will have inaccurate entries. A calculator or computer software might then obtain only the trivial solution for $(\lambda\mathbf{I}_n - \mathbf{A})\,\mathbf{X} = \mathbf{0}$, erroneously yielding no eigenvectors.

### Example 13

Let $\mathbf{A} = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$. Then $p_{\mathbf{A}}(x) = x^2 - 2$, and so the eigenvalues for $\mathbf{A}$ are $\lambda_1 = \sqrt{2}$ and $\lambda_2 = -\sqrt{2}$. Suppose we try to find fundamental eigenvectors for $\lambda_1$ using 1.414 as an approximation for $\sqrt{2}$. Row reducing

$$[\,(1.414\mathbf{I}_2 - \mathbf{A})\,|\,\mathbf{0}] = \left[\begin{array}{cc|c} 1.414 & -2 & 0 \\ -1 & 1.414 & 0 \end{array}\right], \quad \text{we obtain} \quad \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right].$$

Thus, our approximation of the eigenvalue has resulted in a homogeneous system having only the trivial solution, despite the fact that $\left(\sqrt{2}\mathbf{I}_2 - \mathbf{A}\right)\mathbf{X} = \mathbf{0}$ actually has nontrivial solutions. In fact, $\mathbf{X} = [\sqrt{2}, 1]$ is an eigenvector for $\mathbf{A}$ corresponding to $\lambda_1 = \sqrt{2}$.

There are several efficient numerical techniques that can be used other than the Diagonalization Method that produce an eigenvector when we are working with an approximate eigenvalue. While we do not consider them in this section, appropriate techniques to resolve this problem can be found in Sections 8.10 and 9.3. Other more advanced techniques can be found in the literature. You should not encounter a problem with roundoff doing the exercises in this section.

- **Supplemental Material:** You have now covered the prerequisites for Section 7.2, "Complex Eigenvalues and Complex Eigenvectors" and for Section 9.3, "The Power Method for Finding Eigenvalues."
- **Application:** You have now covered the prerequisites for Section 8.6, "Linear Recurrence Relations and the Fibonacci Sequence."

## New Vocabulary

algebraic multiplicity (of an eigenvalue)
characteristic polynomial (of a matrix)
diagonalizable matrix
eigenspace
eigenvalue (characteristic value)
eigenvector (characteristic vector)
nondiagonalizable matrix
nontrivial linear combination
similar matrices

## Highlights

- For a square matrix $\mathbf{A}$, $\lambda$ is an eigenvalue if and only if there is some nonzero vector $\mathbf{X}$ for which $\mathbf{AX} = \lambda\mathbf{X}$. ($\mathbf{X}$ is then an eigenvector for $\lambda$.)
- For a square matrix $\mathbf{A}$, the eigenvalues are the roots of the characteristic polynomial $p_{\mathbf{A}}(x) = |x\mathbf{I}_n - \mathbf{A}|$.
- For a square matrix $\mathbf{A}$, the eigenvectors for an eigenvalue $\lambda$ are the nontrivial solutions of $(\lambda\mathbf{I}_n - \mathbf{A})\mathbf{X} = \mathbf{0}$.
- For a square matrix $\mathbf{A}$, the eigenspace $E_\lambda$ for an eigenvalue $\lambda$ is the set of all eigenvectors for $\lambda$ together with the zero vector. That is, $E_\lambda$ is the complete solution set for the homogeneous system $(\lambda\mathbf{I}_n - \mathbf{A})\mathbf{X} = \mathbf{0}$.
- Two matrices $\mathbf{A}$ and $\mathbf{B}$ are similar if and only if $\mathbf{B} = \mathbf{P}^{-1}\mathbf{AP}$ for some nonsingular matrix $\mathbf{P}$.
- If $\mathbf{A}$ and $\mathbf{B}$ are similar, then $\mathbf{A}$ and $\mathbf{B}$ are both square, both have the same size, $|\mathbf{A}| = |\mathbf{B}|$, and $p_{\mathbf{A}}(x) = p_{\mathbf{B}}(x)$.
- For a square matrix $\mathbf{A}$, fundamental eigenvectors for an eigenvalue $\lambda$ are found from the solution set of $(\lambda\mathbf{I}_n - \mathbf{A})\mathbf{X} = \mathbf{0}$ by setting each independent variable equal to 1 and all other independent variables equal to 0. (Fractions are often eliminated for simplicity by using an appropriate scalar multiple of each vector obtained.)
- If $\mathbf{A}$ is an $n \times n$ matrix, and the Diagonalization Method produces $n$ fundamental eigenvectors for $\mathbf{A}$, then $\mathbf{A}$ is diagonalizable. If $\mathbf{P}$ is a matrix whose columns are these $n$ fundamental eigenvectors, then $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$, a diagonal matrix whose main diagonal entries are the eigenvalues of $\mathbf{A}$.
- If $\mathbf{A}$ is an $n \times n$ matrix, and the Diagonalization Method produces fewer than $n$ fundamental eigenvectors, then $\mathbf{A}$ is nondiagonalizable.
- For a square matrix $\mathbf{A}$, the algebraic multiplicity of an eigenvalue $\lambda$ is the number of factors of $x - \lambda$ in $p_{\mathbf{A}}(x)$.
- If the algebraic multiplicity of an eigenvalue $\lambda$ is $k$, then the number of fundamental eigenvectors for $\lambda$ (from the Diagonalization Method) is $\leq k$.
- If $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$ is diagonal, then positive powers of $\mathbf{A}$ are readily computed using $\mathbf{A}^k = \mathbf{PD}^k\mathbf{P}^{-1}$.

## Exercises for Section 3.4

1. Find the characteristic polynomial of each given matrix. (Hints: For part (e), do a cofactor expansion along the third row. For part (f), consider Exercise 14 in Section 3.2.)

★ (a) $\begin{bmatrix} 3 & 1 \\ -2 & 4 \end{bmatrix}$

(b) $\begin{bmatrix} 4 & 0 & 0 \\ -1 & 5 & 0 \\ 2 & 3 & -2 \end{bmatrix}$

★ (c) $\begin{bmatrix} 2 & 1 & -1 \\ -6 & 6 & 0 \\ 3 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 4 & -4 & -12 \\ 5 & -8 & -9 \\ -3 & 6 & 3 \end{bmatrix}$

★ (e) $\begin{bmatrix} 0 & -1 & 0 & 1 \\ -5 & 2 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ 4 & -1 & 3 & 0 \end{bmatrix}$

**(f)** $\begin{bmatrix} 5 & -2 & 3 & 7 \\ 4 & -1 & 6 & -5 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 4 & 2 \end{bmatrix}$

**2.** Solve for the eigenspace $E_\lambda$ corresponding to the given eigenvalue $\lambda$ for each of the following matrices. Express $E_\lambda$ as a set of linear combinations of fundamental eigenvectors.

★ **(a)** $\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}, \lambda = 2$

**(b)** $\begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 2 \\ -3 & -3 & -2 \end{bmatrix}, \lambda = 1$

★ **(c)** $\begin{bmatrix} -5 & 2 & 0 \\ -8 & 3 & 0 \\ 4 & -2 & -1 \end{bmatrix}, \lambda = -1$

**(d)** $\begin{bmatrix} 1 & 4 & 2 & 6 \\ 0 & -1 & -2 & -4 \\ 2 & -4 & 1 & -6 \\ -1 & 4 & 2 & 8 \end{bmatrix}, \lambda = 3$

**3.** Find all eigenvalues corresponding to each given matrix and their corresponding algebraic multiplicities. Also, express each eigenspace as a set of linear combinations of fundamental eigenvectors.

★ **(a)** $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

**(b)** $\begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix}$

★ **(c)** $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -3 \\ 0 & 0 & -5 \end{bmatrix}$

**(d)** $\begin{bmatrix} 16 & 42 \\ -7 & -19 \end{bmatrix}$

★ **(e)** $\begin{bmatrix} 4 & 0 & -2 \\ 6 & 2 & -6 \\ 4 & 0 & -2 \end{bmatrix}$

**(f)** $\begin{bmatrix} 8 & 18 & -27 \\ 0 & -7 & 9 \\ 2 & 0 & -1 \end{bmatrix}$

**(g)** $\begin{bmatrix} -3 & -1 & 1 & -1 \\ 5 & 2 & 0 & 0 \\ -2 & -1 & -1 & 1 \\ 1 & 0 & -2 & 2 \end{bmatrix}$

★ **(h)** $\begin{bmatrix} 3 & -1 & 4 & -1 \\ 0 & 3 & -3 & 3 \\ -6 & 2 & -8 & 2 \\ -6 & -4 & -2 & -4 \end{bmatrix}$

**4.** Use the Diagonalization Method to determine whether each of the following matrices is diagonalizable. If so, specify the matrices $\mathbf{D}$ and $\mathbf{P}$ and check your work by verifying that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$.

★ **(a)** $\mathbf{A} = \begin{bmatrix} 19 & -48 \\ 8 & -21 \end{bmatrix}$

**(b)** $\mathbf{A} = \begin{bmatrix} -7 & 4 \\ -15 & 9 \end{bmatrix}$

★ **(c)** $\mathbf{A} = \begin{bmatrix} 13 & -34 \\ 5 & -13 \end{bmatrix}$

★ **(d)** $\mathbf{A} = \begin{bmatrix} -13 & -3 & 18 \\ -20 & -4 & 26 \\ -14 & -3 & 19 \end{bmatrix}$

**(e)** $\mathbf{A} = \begin{bmatrix} 7 & 0 & 8 \\ 12 & -1 & 12 \\ -6 & 0 & -7 \end{bmatrix}$

★ **(f)** $\mathbf{A} = \begin{bmatrix} 5 & -8 & -12 \\ -2 & 3 & 4 \\ 4 & -6 & -9 \end{bmatrix}$

★ **(g)** $\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 4 & 1 \\ 3 & -2 & 1 \end{bmatrix}$

**(h)** $\mathbf{A} = \begin{bmatrix} -2 & 7 & 8 \\ 0 & 9 & 9 \\ -2 & -2 & -1 \end{bmatrix}$

★ **(i)** $\mathbf{A} = \begin{bmatrix} 3 & 1 & -6 & -2 \\ 4 & 0 & -6 & -4 \\ 2 & 0 & -3 & -2 \\ 0 & 1 & -2 & 1 \end{bmatrix}$

(j) $\mathbf{A} = \begin{bmatrix} -3 & -4 & -12 & -14 \\ 5 & 9 & 12 & 10 \\ -5 & -10 & -7 & -2 \\ 3 & 6 & 4 & 1 \end{bmatrix}$

5. Use diagonalization to calculate the indicated powers of **A** in each case.
   ★ (a) $\mathbf{A}^{15}$, where $\mathbf{A} = \begin{bmatrix} 4 & -6 \\ 3 & -5 \end{bmatrix}$
   (b) $\mathbf{A}^{28}$, where $\mathbf{A} = \begin{bmatrix} -2 & 6 & -6 \\ 2 & -7 & 6 \\ 3 & -10 & 9 \end{bmatrix}$
   ★ (c) $\mathbf{A}^{49}$, $\mathbf{A} = \begin{bmatrix} 17 & 28 & -36 & 52 \\ 0 & -3 & 0 & 4 \\ 8 & 8 & -17 & 32 \\ 0 & -2 & 0 & 3 \end{bmatrix}$
   (d) $\mathbf{A}^{12}$, where $\mathbf{A} = \begin{bmatrix} 21 & 20 & 68 \\ 11 & 12 & 37 \\ -10 & -10 & -33 \end{bmatrix}$
   ★ (e) $\mathbf{A}^{10}$, where $\mathbf{A} = \begin{bmatrix} 7 & 9 & -12 \\ 10 & 16 & -22 \\ 8 & 12 & -16 \end{bmatrix}$

6. Let **A** and **B** be $n \times n$ matrices. Prove that if **A** is similar to **B**, then $p_{\mathbf{A}}(x) = p_{\mathbf{B}}(x)$.
7. Let **A** be a diagonalizable $n \times n$ matrix.
   (a) Show that **A** has a cube root—that is, that there is a matrix **B** such that $\mathbf{B}^3 = \mathbf{A}$.
   ★ (b) Give a sufficient condition for **A** to have a square root. Prove that your condition is valid.

★ 8. Find a matrix **A** such that $\mathbf{A}^3 = \begin{bmatrix} 15 & -14 & -14 \\ -13 & 16 & 17 \\ 20 & -22 & -23 \end{bmatrix}$. (Hint: See Exercise 7.)

9. Prove that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has two distinct eigenvalues if $(a-d)^2 + 4bc > 0$, one distinct eigenvalue if $(a-d)^2 + 4bc = 0$, and no eigenvalues if $(a-d)^2 + 4bc < 0$.
10. Let **A** be an $n \times n$ matrix, and let $k$ be a positive integer.
    (a) Prove that if $\lambda$ is an eigenvalue of **A**, then $\lambda^k$ is an eigenvalue of $\mathbf{A}^k$.
    ★ (b) Give a $2 \times 2$ matrix **A** and an integer $k$ that provide a counterexample to the converse of part (a).
11. Suppose that **A** is a nonsingular $n \times n$ matrix. Prove that

$$p_{\mathbf{A}^{-1}}(x) = (-x)^n \left|\mathbf{A}^{-1}\right| p_{\mathbf{A}}\left(\frac{1}{x}\right).$$

(Hint: First express $p_{\mathbf{A}}\left(\frac{1}{x}\right)$ as $\left|\left(\frac{1}{x}\right)\mathbf{I}_n - \mathbf{A}\right|$. Then collect the right-hand side into one determinant.)

12. Let **A** be an upper triangular $n \times n$ matrix. (Note: The following assertions are also true if **A** is a lower triangular matrix.)
    (a) Prove that $\lambda$ is an eigenvalue for **A** if and only if $\lambda$ appears on the main diagonal of **A**.
    (b) Show that the algebraic multiplicity of an eigenvalue $\lambda$ of **A** equals the number of times $\lambda$ appears on the main diagonal.
13. Let **A** be an $n \times n$ matrix. Prove that **A** and $\mathbf{A}^T$ have the same characteristic polynomial and hence the same eigenvalues.
14. (Note: You must have covered the material in Section 8.4 in order to do this exercise.) Suppose that **A** is a stochastic $n \times n$ matrix. Prove that $\lambda = 1$ is an eigenvalue for **A**. (Hint: Let $\mathbf{X} = [1, 1, \ldots, 1]$, and consider $\mathbf{A}^T\mathbf{X}$. Then use Exercise 13.) (This exercise implies that every stochastic matrix has a fixed point. However, not all initial conditions reach this fixed point, as demonstrated in Example 3 in Section 8.4.)
15. Let **A**, **P**, and **D** be $n \times n$ matrices with **P** nonsingular and $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$. Use a proof by induction to show that $\mathbf{A}^k = \mathbf{PD}^k\mathbf{P}^{-1}$, for every integer $k > 0$.
16. Let **A** be an $n \times n$ upper triangular matrix with all main diagonal entries distinct. Prove that **A** is diagonalizable.
17. Prove that a square matrix **A** is singular if and only if $\lambda = 0$ is an eigenvalue for **A**.
18. Let **A** be a diagonalizable matrix. Prove that $\mathbf{A}^T$ is diagonalizable. (Hint: Use the definition of diagonalizable rather than the Diagonalization Method.)

**19.** Let $\mathbf{A}$ be a nonsingular diagonalizable matrix. Prove that $\mathbf{A}^{-1}$ is diagonalizable. (Hint: Use the definition of diagonalizable rather than the Diagonalization Method. Also consider part (3) of Theorem 2.12 to prove that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$ is nonsingular.)

**20.** This exercise outlines a proof of Theorem 3.14. Let $\mathbf{A}$ and $\mathbf{P}$ be given as stated in the theorem.

- **(a)** Suppose $\lambda_i$ is the eigenvalue corresponding to $\mathbf{P}_i = i$th column of $\mathbf{P}$. Prove that the $i$th column of $\mathbf{AP}$ equals $\lambda_i \mathbf{P}_i$.
- **(b)** Use the fact that $\mathbf{P}^{-1}\mathbf{P} = \mathbf{I}_n$ to prove that $\mathbf{P}^{-1}\lambda_i\mathbf{P}_i = \lambda_i \mathbf{e}_i$.
- **(c)** Use parts (a) and (b) to finish the proof of Theorem 3.14.

**21.** Prove that if $\mathbf{A}$ and $\mathbf{P}$ are $n \times n$ matrices such that $\mathbf{P}$ is nonsingular and $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$ is diagonal, then for each $i$, $\mathbf{P}_i$, the $i$th column of $\mathbf{P}$, is an eigenvector for $\mathbf{A}$ corresponding to the eigenvalue $d_{ii}$. (Hint: Note that $\mathbf{PD} = \mathbf{AP}$, and calculate the $i$th column of both sides to show that $d_{ii}\mathbf{P}_i = \mathbf{AP}_i$.)

**22.** Prove the following: Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ be $n \times n$ matrices such that $\mathbf{C} = x\mathbf{A} + \mathbf{B}$. If at most $k$ rows of $\mathbf{A}$ have nonzero entries, then $|\mathbf{C}|$ is a polynomial in $x$ of degree $\leq k$. (Hint: Use induction on $n$.)

**23.** This exercise concerns properties of the characteristic polynomial of a matrix.

- **(a)** Show that the characteristic polynomial of a $2 \times 2$ matrix $\mathbf{A}$ is given by $x^2 - (\text{trace}(\mathbf{A}))x + |\mathbf{A}|$.
- **(b)** Prove that the characteristic polynomial of an $n \times n$ matrix always has degree $n$, with the coefficient of $x^n$ equal to 1. (Hint: Use induction and Exercise 22.)
- **(c)** If $\mathbf{A}$ is an $n \times n$ matrix, show that the constant term of $p_{\mathbf{A}}(x)$ is $(-1)^n\,|\mathbf{A}|$. (Hint: The constant term of $p_{\mathbf{A}}(x)$ equals $p_{\mathbf{A}}(0)$.)
- **(d)** If $\mathbf{A}$ is an $n \times n$ matrix, show that the coefficient of $x^{n-1}$ in $p_{\mathbf{A}}(x)$ is $-\text{trace}(\mathbf{A})$. (Hint: Use induction and Exercise 22.)

▶ **24.** The purpose of this exercise is to outline an algorithm for computing the characteristic polynomial of a square matrix $\mathbf{A}$ that can be easily programmed into a graphing calculator. We assume that the calculator has built-in functions for the determinant and for finding the reduced row echelon form for any square matrix. Assume that $p_{\mathbf{A}}(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$.

- **(a)** Use Exercise 23 to explain why, after a quick computation of the determinant of $\mathbf{A}$ and the trace of $\mathbf{A}$, the coefficients $a_n$, $a_{n-1}$, and $a_0$ can be easily determined. Hence, computing the characteristic polynomial of an $n \times n$ matrix $\mathbf{A}$ can be reduced to calculating the remaining $(n-2)$ coefficients.
- **(b)** Now, $p_{\mathbf{A}}(x) = |x\mathbf{I}_n - \mathbf{A}|$. Also, $p_{\mathbf{A}}(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$. Plug $x = 1$ into both of these expressions for $p_{\mathbf{A}}(x)$, and substitute in the values for the 3 known coefficients from part (a) to obtain a linear equation involving the $(n-2)$ unknown coefficients.
- **(c)** Plug in $x = 2$ through $(n-2)$ in a manner similar to that in part (b) to create $(n-3)$ additional equations involving the $(n-2)$ unknown coefficients.
- **(d)** Combine the equations in parts (b) and (c) to create a system containing $(n-2)$ equations in $(n-2)$ unknowns that can be solved to determine the remaining $(n-2)$ coefficients of $p_{\mathbf{A}}(x)$.

The answer in the Student Solutions Manual includes a program written in TI–BASIC 83: the programming language for the TI-83 and TI-84 graphing calculators. This program implements the algorithm outlined in this exercise. If you have such a calculator, you can use this program to compute characteristic polynomials.

**25.** Let $\mathbf{A}$ be an $n \times n$ matrix, and let $\mathbf{x}$ and $\mathbf{y}$ be column vectors in $\mathbb{R}^n$.

- **(a)** Prove that $(\mathbf{Ax}) \cdot \mathbf{y} = \mathbf{x} \cdot (\mathbf{A}^T\mathbf{y})$. (Hint: If $\mathbf{a}$ and $\mathbf{b}$ represent any column vectors in $\mathbb{R}^n$, then the product $\mathbf{a}^T\mathbf{b}$ represents a $1 \times 1$ matrix whose only entry is $\mathbf{a} \cdot \mathbf{b}$.)
- **(b)** If $\mathbf{A}$ is symmetric and $\mathbf{x}$ and $\mathbf{y}$ are eigenvectors for $\mathbf{A}$ corresponding to distinct eigenvalues, prove that $\mathbf{x} \cdot \mathbf{y} = 0$.

★ **26.** True or False:

- **(a)** If $\mathbf{A}$ is a square matrix, then 5 is an eigenvalue of $\mathbf{A}$ if $\mathbf{AX} = 5\mathbf{X}$ for some nonzero vector $\mathbf{X}$.
- **(b)** The eigenvalues of an $n \times n$ matrix $\mathbf{A}$ are the solutions of $x\mathbf{I}_n - \mathbf{A} = \mathbf{O}_n$.
- **(c)** If $\lambda$ is an eigenvalue for an $n \times n$ matrix $\mathbf{A}$, then any nontrivial solution of $(\lambda\mathbf{I}_n - \mathbf{A})\mathbf{X} = \mathbf{0}$ is an eigenvector for $\mathbf{A}$ corresponding to $\lambda$.
- **(d)** If $\mathbf{D}$ is the diagonal matrix created from an $n \times n$ matrix $\mathbf{A}$ by the Diagonalization Method, then the main diagonal entries of $\mathbf{D}$ are eigenvalues of $\mathbf{A}$.
- **(e)** If $\mathbf{A}$ and $\mathbf{P}$ are $n \times n$ matrices such that each column of $\mathbf{P}$ is an eigenvector for $\mathbf{A}$, then $\mathbf{P}$ is nonsingular and $\mathbf{P}^{-1}\mathbf{AP}$ is a diagonal matrix.
- **(f)** If $\mathbf{A}$ is a square matrix and $p_{\mathbf{A}}(x) = (x-3)^2(x+1)$, then the Diagonalization Method cannot produce more than one fundamental eigenvector for the eigenvalue $\lambda = -1$.

(g) If a $3 \times 3$ matrix $\mathbf{A}$ has 3 distinct eigenvalues, then $\mathbf{A}$ is diagonalizable.
(h) If $\mathbf{A} = \mathbf{PDP}^{-1}$, where $\mathbf{D}$ is a diagonal matrix, then $\mathbf{A}^n = \mathbf{P}^n\mathbf{D}^n(\mathbf{P}^{-1})^n$.

## Review Exercises for Chapter 3

1. Consider $\mathbf{A} = \begin{bmatrix} 4 & -5 & 2 & -3 \\ -6 & 1 & -2 & -4 \\ 3 & -8 & 5 & 2 \\ -7 & 0 & -1 & 9 \end{bmatrix}$.
   (a) Find the $(2,4)$ minor of $\mathbf{A}$.
   ★ (b) Find the $(3,4)$ cofactor of $\mathbf{A}$.
   (c) Find $|\mathbf{A}|$ using cofactor expansion along the last row of $\mathbf{A}$.
   ★ (d) Find $|\mathbf{A}|$ using cofactor expansion along the second column of $\mathbf{A}$.
2. Find the determinant of $\mathbf{A} = \begin{bmatrix} 6 & -5 & 1 \\ 4 & 3 & 7 \\ -8 & 0 & 2 \end{bmatrix}$ by basketweaving.
★ 3. Find the determinant of $\mathbf{A} = \begin{bmatrix} 2 & 0 & -3 & 3 \\ 4 & -2 & -1 & 3 \\ 1 & -1 & 0 & -2 \\ 2 & 1 & -2 & 1 \end{bmatrix}$ by row reducing $\mathbf{A}$ to upper triangular form.
4. Find the volume of the parallelepiped determined by vectors $\mathbf{x} = [7, -2, 1]$, $\mathbf{y} = [-4, 3, 1]$, $\mathbf{z} = [6, 5, -8]$.
★ 5. If $\mathbf{A}$ is a $4 \times 4$ matrix and $|\mathbf{A}| = -15$, what is $|\mathbf{B}|$, if $\mathbf{B}$ is obtained from $\mathbf{A}$ after the indicated row operation?
   (a) (I): $\langle 3\rangle \leftarrow -4\,\langle 3\rangle$
   (b) (II): $\langle 2\rangle \leftarrow 5\,\langle 1\rangle + \langle 2\rangle$
   (c) (III): $\langle 3\rangle \leftrightarrow \langle 4\rangle$
6. Suppose $\mathbf{A}$ is a $4 \times 4$ matrix and $|\mathbf{A}| = -2$.
   (a) Is $\mathbf{A}$ nonsingular?
   (b) What is rank($\mathbf{A}$)?
   (c) Is $\mathbf{A}$ row equivalent to $\mathbf{I}_4$?
   (d) Does $\mathbf{AX} = \mathbf{0}$ have a unique solution?
★ 7. If $\mathbf{A}$ and $\mathbf{B}$ are $3 \times 3$ matrices, with $|\mathbf{A}| = -7$ and $|\mathbf{B}| = \frac{1}{2}$, what is $|-3\mathbf{A}^T\mathbf{B}^{-1}|$?
★ 8. Suppose that $\mathbf{A}$ is a $2 \times 2$ matrix such that $\mathbf{A}\begin{bmatrix} 4 \\ -7 \end{bmatrix} = \mathbf{A}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.
   (a) What is $|\mathbf{A}|$?
   (b) Does $\mathbf{AX} = \mathbf{0}$ have a unique solution? If so, why? If not, give a nontrivial solution for $\mathbf{AX} = \mathbf{0}$.
9. Let $\mathbf{A}$ be a nonsingular $n \times n$ matrix, and let $\mathbf{B}$ be the $n \times n$ matrix whose $(i, j)$ entry is $\mathcal{A}_{ij}$. Prove that $\mathbf{A}$ and $\mathbf{B}^T$ commute. (Hint: Recall from Exercise 17 in Section 3.3 that $\mathbf{AB}^T = (|\mathbf{A}|)\,\mathbf{I}_n$. Use this to find a right (and hence, left) inverse for $\mathbf{A}$.)
★ 10. Solve the following system using Cramer's Rule: $\begin{cases} 2x_1 - 3x_2 + 2x_3 = 11 \\ 3x_1 + 4x_2 + 3x_3 = -9 \\ -x_1 + 2x_2 + x_3 = 3 \end{cases}$.
★ 11. This exercise involves potential powers of a square matrix $\mathbf{A}$.
   (a) Show that there is no matrix $\mathbf{A}$ such that $\mathbf{A}^4 = \begin{bmatrix} 5 & -4 & -2 \\ -8 & -3 & 3 \\ -2 & 4 & 7 \end{bmatrix}$.
   (b) Show that there is no matrix $\mathbf{A}$ such that $\mathbf{A}^{-1} = \begin{bmatrix} 3 & -2 & 5 \\ -1 & 1 & 4 \\ 1 & 0 & 13 \end{bmatrix}$.
12. If $\mathbf{B}$ is similar to $\mathbf{A}$, prove the following:
   (a) $\mathbf{B}^k$ is similar to $\mathbf{A}^k$ (for any integer $k > 0$).
   ★ (b) $|\mathbf{B}^T| = |\mathbf{A}^T|$.
   (c) $\mathbf{B}$ is nonsingular if and only if $\mathbf{A}$ is nonsingular.

(d) If $\mathbf{A}$ and $\mathbf{B}$ are both nonsingular, then $\mathbf{A}^{-1}$ is similar to $\mathbf{B}^{-1}$.

★ (e) $\mathbf{B} + \mathbf{I}_n$ is similar to $\mathbf{A} + \mathbf{I}_n$.

(f) Trace($\mathbf{B}$) = trace($\mathbf{A}$). (Hint: Use Exercise 28(c) in Section 1.5.)

(g) $\mathbf{B}$ is diagonalizable if and only if $\mathbf{A}$ is diagonalizable.

**13.** Let $\mathbf{A}$ be an $n \times n$ matrix with eigenvalue $\lambda$, and let $\mathbf{X}$ and $\mathbf{Y}$ be two eigenvectors for $\mathbf{A}$ corresponding to $\lambda$. Prove that any nonzero vector that is a linear combination of $\mathbf{X}$ and $\mathbf{Y}$ is also an eigenvector for $\mathbf{A}$ corresponding to $\lambda$.

**14.** For the given matrix $\mathbf{A}$, find the characteristic polynomial, all the eigenvalues, the eigenspace for each eigenvalue, a matrix $\mathbf{P}$ whose columns are fundamental eigenvectors for $\mathbf{A}$, and a diagonal matrix $\mathbf{D}$ similar to $\mathbf{A}$. Check your work by verifying that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$.

(a) $\mathbf{A} = \begin{bmatrix} -2 & 0 & 4 & 0 \\ -1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 1 & 1 & -1 & 1 \end{bmatrix}$

★ (b) $\mathbf{A} = \begin{bmatrix} 5 & 16 & -16 \\ -32 & -67 & 64 \\ -32 & -64 & 61 \end{bmatrix}$

**15.** Show that each of the following matrices is not diagonalizable according to the Diagonalization Method.

(a) $\mathbf{A} = \begin{bmatrix} -5 & -18 & -8 \\ -3 & -8 & -4 \\ 11 & 28 & 14 \end{bmatrix}$

★ (b) $\mathbf{A} = \begin{bmatrix} -468 & -234 & -754 & 299 \\ 324 & 162 & 525 & -204 \\ 144 & 72 & 231 & -93 \\ -108 & -54 & -174 & 69 \end{bmatrix}$

(Hint: $p_{\mathbf{A}}(x) = x^4 + 6x^3 + 9x^2$.)

★ **16.** For the matrix $\mathbf{A} = \begin{bmatrix} -21 & 22 & 16 \\ -28 & 29 & 20 \\ 8 & -8 & -5 \end{bmatrix}$, use diagonalization (as in Example 12 of Section 3.4) to find $\mathbf{A}^{13}$.

★ **17.** Let $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$, where $\mathbf{D} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ and $\mathbf{P} = \begin{bmatrix} 1 & 1 & 3 & 2 \\ -2 & 0 & 7 & 8 \\ 1 & 0 & -3 & -4 \\ 1 & 1 & 2 & 3 \end{bmatrix}$.

(a) What are the eigenvalues of $\mathbf{A}$?

(b) Without using row reduction, give the eigenspaces for each eigenvalue in part (a).

(c) What is $|\mathbf{A}|$?

★ **18.** True or False:

(a) If $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices, $n > 1$, with $|\mathbf{A}| = |\mathbf{B}|$, then $\mathbf{A} = \mathbf{B}$.

(b) If a square matrix $\mathbf{A}$ has all zero entries on its main diagonal, then $|\mathbf{A}| = 0$.

(c) The submatrix $\mathbf{A}_{ij}$ of any matrix $\mathbf{A}$ equals $(-1)^{i+j}\mathcal{A}_{ij}$.

(d) If the vectors $\mathbf{x}$ and $\mathbf{y}$ determine a parallelogram in $\mathbb{R}^2$, then the determinant of the matrix whose rows are $\mathbf{x}$ and $\mathbf{y}$ in either order gives the correct area for the parallelogram.

(e) The volume of the parallelepiped determined by three vectors in $\mathbb{R}^3$ equals the absolute value of the determinant of the matrix whose columns are the three vectors.

(f) A lower triangular matrix having a zero on the main diagonal must be singular.

(g) If an $n \times n$ matrix $\mathbf{B}$ is created by changing the order of the columns of a matrix $\mathbf{A}$, then either $|\mathbf{B}| = |\mathbf{A}|$ or $|\mathbf{B}| = -|\mathbf{A}|$.

(h) If $\mathbf{A}$ is an $n \times n$ matrix such that $\mathbf{Ae}_1 = \mathbf{0}$, then $|\mathbf{A}| = 0$.

(i) In general, for large square matrices, cofactor expansion along the last row is the most efficient method for calculating the determinant.

(j) Any two $n \times n$ matrices having the same nonzero determinant are row equivalent.

(k) If $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices, $n > 1$, with $|\mathbf{A}| = |\mathbf{B}|$, then $\mathbf{A}$ can be obtained from $\mathbf{B}$ by performing a Type (II) row operation.

(l) A homogeneous system of linear equations having the same number of equations as variables has a nontrivial solution if and only if its coefficient matrix has a nonzero determinant.

(m) If $|\mathbf{AB}| = 0$, then $|\mathbf{A}| = 0$ or $|\mathbf{B}| = 0$.

(n) Any two row equivalent $n \times n$ matrices have the same determinant.

(o) If $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices, $n > 1$, with $\mathbf{A}$ singular, then $|\mathbf{A} + \mathbf{B}| = |\mathbf{B}|$.

**(p)** If $\mathbf{A}$ is a square matrix such that $\mathbf{A} = \mathbf{A}^{-1}$, then $|\mathbf{A}| = 1$.
**(q)** Since an eigenspace $E_\lambda$ contains the zero vector as well as all fundamental eigenvectors corresponding to an eigenvalue $\lambda$, the total number of vectors in $E_\lambda$ is one more than the number of fundamental eigenvectors found in the Diagonalization Method for $\lambda$.
**(r)** The sum of the algebraic multiplicities of the eigenvalues for an $n \times n$ matrix can not exceed $n$.
**(s)** If $\mathbf{A}$ is an $n \times n$ matrix, then the coefficient of the $x^n$ term in $p_{\mathbf{A}}(x)$ is 1.
**(t)** If $\mathbf{A} = \mathbf{P}^{-1}\mathbf{BP}$, then $\mathbf{B} = \mathbf{P}^{-1}\mathbf{AP}$.
**(u)** Every nonsingular $n \times n$ matrix is similar to $\mathbf{I}_n$.
**(v)** For every root $\lambda$ of $p_{\mathbf{A}}(x)$, there is at least one nonzero vector $\mathbf{X}$ such that $\mathbf{AX} = \lambda\mathbf{X}$.
**(w)** If $\mathbf{A}$ is nonsingular and $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$ is a diagonal matrix, then $\mathbf{A}^{-1} = \mathbf{PRP}^{-1}$, where $\mathbf{R}$ is the diagonal matrix whose diagonal entries are the reciprocals of the corresponding diagonal entries of $\mathbf{D}$.
**(x)** If $\lambda$ is not an eigenvalue for an $n \times n$ matrix $\mathbf{A}$, then the homogeneous system $(\lambda\mathbf{I}_n - \mathbf{A})\mathbf{X} = \mathbf{0}$ has only the trivial solution.
**(y)** The sum of diagonalizable matrices is diagonalizable.
**(z)** The product of diagonalizable matrices is diagonalizable.

## Summary of Techniques

We summarize here many of the computational techniques developed in Chapters 2 and 3. These computations should be done using calculators or computer software packages if they cannot be done easily by hand.

### *Techniques for Solving a System $\mathbf{AX} = \mathbf{B}$ of $m$ Linear Equations in $n$ Unknowns*

- **Gaussian Elimination:** Use row operations to find a matrix in row echelon form that is row equivalent to $[\mathbf{A}|\mathbf{B}]$. Assign values to the independent variables and use back substitution to determine the values of the dependent variables. Advantages: finds the complete solution set for any linear system; fewer computational roundoff errors than Gauss-Jordan row reduction (Section 2.1).
- **Gauss-Jordan row reduction:** Use row operations to find the matrix in reduced row echelon form for $[\mathbf{A}|\mathbf{B}]$. Assign values to the independent variables and solve for the dependent variables. Advantages: easily computerized; finds the complete solution set for any linear system (Section 2.2).
- **Multiplication by inverse matrix:** Use when $m = n$ and $|\mathbf{A}| \neq 0$. The solution is $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$. Disadvantage: $\mathbf{A}^{-1}$ must be known or calculated first, and therefore the method is only useful when there are several systems to be solved with the same coefficient matrix $\mathbf{A}$ (Section 2.4).
- **Cramer's Rule**: Use when $m = n$ and $|\mathbf{A}| \neq 0$. The solution is $x_1 = |\mathbf{A}_1| / |\mathbf{A}|$, $x_2 = |\mathbf{A}_2| / |\mathbf{A}|, \ldots, x_n = |\mathbf{A}_n| / |\mathbf{A}|$, where $\mathbf{A}_i$ (for $1 \le i \le n$) and $\mathbf{A}$ are identical except that the $i$th column of $\mathbf{A}_i$ equals $\mathbf{B}$. Disadvantage: efficient only for small systems because it involves calculating $n + 1$ determinants of size $n$ (Section 3.3).

Other techniques for solving systems are discussed in Chapter 9. Among these are **LDU** Decomposition and iterative methods, such as the Gauss-Seidel and Jacobi techniques.

Also remember that if $m < n$ and $\mathbf{B} = \mathbf{0}$ (homogeneous case), then the number of solutions to $\mathbf{AX} = \mathbf{B}$ is infinite.

### *Techniques for Finding the Inverse (If It Exists) of an $n \times n$ Matrix $\mathbf{A}$*

- **$2 \times 2$ case:** The inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ exists if and only if $ad - bc \neq 0$. In that case, the inverse is given by $\left(\frac{1}{ad-bc}\right)\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ (Section 2.4).
- **Row reduction:** Row reduce $[\mathbf{A}|\mathbf{I}_n]$ to $[\mathbf{I}_n|\mathbf{A}^{-1}]$ (where $\mathbf{A}^{-1}$ does not exist if the process stops prematurely). Advantages: easily computerized; relatively efficient (Section 2.4).

### *Techniques for Finding the Determinant of an $n \times n$ Matrix $\mathbf{A}$*

- **$2 \times 2$ case:** $|\mathbf{A}| = a_{11}a_{22} - a_{12}a_{21}$ (Sections 2.4 and 3.1).
- **$3 \times 3$ case:** Basketweaving (Section 3.1).
- **Row reduction:** Row reduce $\mathbf{A}$ to an upper triangular form matrix $\mathbf{B}$, keeping track of the effect of each row operation on the determinant using a variable $P$. Then $|\mathbf{A}| = (\frac{1}{P})|\mathbf{B}|$, using the final value of $P$. Advantages: easily computerized; relatively efficient (Section 3.2).

- **Cofactor expansion:** Multiply each element along any row or column of $\mathbf{A}$ by its cofactor and sum the results. Advantage: useful for matrices with many zero entries. Disadvantage: not as fast as row reduction (Sections 3.1 and 3.3).

Also remember that $|\mathbf{A}| = 0$ if $\mathbf{A}$ is row equivalent to a matrix with a row or column of zeroes, or with two identical rows, or with two identical columns.

### Techniques for Finding the Eigenvalues of an $n \times n$ Matrix $\mathbf{A}$

- **Characteristic polynomial:** Find the roots of $p_{\mathbf{A}}(x) = |x\mathbf{I}_n - \mathbf{A}|$. (We only consider the real roots of $p_{\mathbf{A}}(x)$ in Chapters 1 through 6.) Disadvantages: tedious to calculate $p_{\mathbf{A}}(x)$; polynomial becomes more difficult to factor as degree of $p_{\mathbf{A}}(x)$ increases (Section 3.4).

A more computationally efficient technique for finding eigenvalues is the Power Method in Chapter 9. If the Power Method is used to compute an eigenvalue, it will also produce a corresponding eigenvector.

### Technique for Finding the Eigenvectors of an $n \times n$ Matrix $\mathbf{A}$

- **Row reduction:** For each eigenvalue $\lambda$ of $\mathbf{A}$, solve $(\lambda\mathbf{I}_n - \mathbf{A})\mathbf{X} = \mathbf{0}$ by row reducing the augmented matrix $[\,(\lambda\mathbf{I}_n - \mathbf{A})\,|\,\mathbf{0}]$ and taking the nontrivial solutions (Section 3.4).

Chapter 4

# Finite Dimensional Vector Spaces

**Driven to Abstraction**

In Chapter 1, we saw that the operations of addition and scalar multiplication on the set $\mathcal{M}_{mn}$ possess many of the same algebraic properties as addition and scalar multiplication on the set $\mathbb{R}^n$. In fact, there are many other sets of mathematical objects, such as functions, matrices, and infinite series, that possess properties in common with $\mathbb{R}^n$. It is profitable to study all of these together in a more abstract fashion by generalizing our discussion of vectors to related sets of objects. Ultimately, generalization is necessary in linear algebra because studying $\mathbb{R}^n$ alone can only take us so far. The advantage of working in this more abstract setting is that we can generate theorems that apply to all analogous cases, and in so doing, we reveal a theory with a wider range of real-world applications.

In this chapter, we define vector spaces to be algebraic structures with operations having properties similar to those of addition and scalar multiplication on $\mathbb{R}^n$. We then establish many important results related to vector spaces. In so doing, we extend many of the concepts covered in the first three chapters, but at a higher level of abstraction than before.

## 4.1 Introduction to Vector Spaces

### Definition of a Vector Space

We now introduce a general class of sets called **vector spaces**,[1] with operations of addition and scalar multiplication having the same eight properties from Theorems 1.3 and 1.12, as well as two **closure properties**.

**Definition** A **vector space** is a set $\mathcal{V}$ together with an operation called **vector addition** (a rule for adding two elements of $\mathcal{V}$ to obtain a third element of $\mathcal{V}$) and another operation called **scalar multiplication** (a rule for multiplying a real number times an element of $\mathcal{V}$ to obtain a second element of $\mathcal{V}$) on which the following 10 properties hold:

For every $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathcal{V}$, and for every $a$ and $b$ in $\mathbb{R}$:

| | | |
|---|---|---|
| (A) | $\mathbf{u}+\mathbf{v}\in\mathcal{V}$ | Closure Property of Addition |
| (B) | $a\mathbf{u}\in\mathcal{V}$ | Closure Property of Scalar Multiplication |
| (1) | $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ | Commutative Law of Addition |
| (2) | $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$ | Associative Law of Addition |
| (3) | There is an element $\mathbf{0}$ of $\mathcal{V}$ so that for every $\mathbf{y}$ in $\mathcal{V}$ we have $\mathbf{0}+\mathbf{y}=\mathbf{y}=\mathbf{y}+\mathbf{0}$. | Existence of Identity Element for Addition |
| (4) | There is an element $-\mathbf{u}$ in $\mathcal{V}$ such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}=(-\mathbf{u})+\mathbf{u}$. | Existence of Additive Inverse |
| (5) | $a(\mathbf{u}+\mathbf{v})=(a\mathbf{u})+(a\mathbf{v})$ | Distributive Laws for Scalar |
| (6) | $(a+b)\mathbf{u}=(a\mathbf{u})+(b\mathbf{u})$ | Multiplication over Addition |
| (7) | $(ab)\mathbf{u}=a(b\mathbf{u})$ | Associativity of Scalar Multiplication |
| (8) | $1\mathbf{u}=\mathbf{u}$ | Identity Property for Scalar Multiplication |

The elements of a vector space $\mathcal{V}$ are called **vectors**.

The two closure properties require that both the operations of vector addition and scalar multiplication always produce an element of the vector space as a result.

All sums indicated by "+" in properties (1) through (5) are vector sums. In property (6), the "+" on the left side of the equation represents addition of real numbers; the "+" on the right side stands for the sum of two vectors. In property (7),

[1] We actually define what are called *real vector spaces*, rather than just vector spaces. The word *real* implies that the scalars involved in the scalar multiplication are real numbers. In Chapter 7, we consider *complex vector spaces*, where the scalars are complex numbers. Other types of vector spaces involving more general sets of scalars are not considered in this book.

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the left side of the equation contains one product of real numbers, $ab$, and one instance of scalar multiplication, $(ab)$ times $\mathbf{u}$. The right side of property (7) involves two scalar multiplications—first, $b$ times $\mathbf{u}$, and then, $a$ times the vector $(b\mathbf{u})$. Usually we can tell from the context which type of operation is involved.

In any vector space, the additive identity element in property (3) is unique, and the additive inverse (property (4)) of each vector is unique (see the proof of Part (3) of Theorem 4.1 and Exercise 11).

## Examples of Vector Spaces

### Example 1

Let $\mathcal{V} = \mathbb{R}^n$, with addition and scalar multiplication of $n$-vectors as defined in Section 1.1. Since these operations always produce vectors in $\mathbb{R}^n$, the closure properties certainly hold for $\mathbb{R}^n$. By Theorem 1.3, the remaining eight properties hold as well. Thus, $\mathcal{V} = \mathbb{R}^n$ is a vector space with these operations.

Similarly, consider $\mathcal{M}_{mn}$, the set of $m \times n$ matrices. The usual operations of matrix addition and scalar multiplication on $\mathcal{M}_{mn}$ always produce $m \times n$ matrices, and so the closure properties certainly hold for $\mathcal{M}_{mn}$. By Theorem 1.12, the remaining eight properties hold as well. Hence, $\mathcal{M}_{mn}$ is a vector space with these operations.

Keep $\mathbb{R}^n$ and $\mathcal{M}_{mn}$ (with the usual operations of addition and scalar multiplication) in mind as examples, as they are representative of most of the vector spaces we consider in this chapter.

Notice that particular vector spaces could have additional operations. For example, $\mathbb{R}^n$ has the dot product, and $\mathcal{M}_{mn}$ has matrix multiplication and the transpose. But these are not shared by all vector spaces because they are not included in the definition. In an abstract vector space, we cannot assume the existence of any additional operations, such as multiplication or division of one vector by another. The only general vector space operation that combines two *vectors* is vector addition.

### Example 2

The set $\mathcal{V} = \{\mathbf{0}\}$ is a vector space with the rules for addition and scalar multiplication given by $\mathbf{0} + \mathbf{0} = \mathbf{0}$ and $a\mathbf{0} = \mathbf{0}$ for every scalar (real number) $a$. Since $\mathbf{0}$ is the only possible result of either operation, $\mathcal{V}$ must be closed under both addition and scalar multiplication. A quick check verifies that the remaining eight properties also hold for $\mathcal{V}$. This vector space is called the **trivial vector space**, and no smaller vector space is possible (why?).

### Example 3

A plane in $\mathbb{R}^3$ containing the origin can be considered as the set of all 3-vectors with initial points at the origin that lie entirely in that plane when drawn on a graph. For example, the plane $\mathcal{W}$ shown in Fig. 4.1 contains the vectors $\mathbf{u}$ and $\mathbf{v}$ as elements, while $\mathbf{q}$ is not an element of $\mathcal{W}$. We will prove that any plane $\mathcal{W}$ in $\mathbb{R}^3$ containing the origin is a vector space.

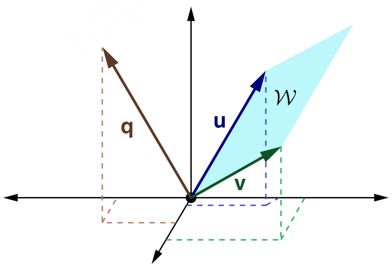


**FIGURE 4.1** A plane $\mathcal{W}$ in $\mathbb{R}^3$ containing the origin

To check the closure properties, we must show that the sum of any two vectors in $\mathcal{W}$ is a vector in $\mathcal{W}$ and that any scalar multiple of a vector in $\mathcal{W}$ also lies in $\mathcal{W}$.

**Closure under addition:** Let $\mathbf{x}$ and $\mathbf{y}$ be any two vectors in $\mathcal{W}$. Then the parallelogram they form lies entirely in the plane, because $\mathbf{x}$ and $\mathbf{y}$ do. Hence, the diagonal $\mathbf{x} + \mathbf{y}$ of this parallelogram also lies in the plane, so $\mathbf{x} + \mathbf{y}$ is in $\mathcal{W}$. This verifies that $\mathcal{W}$ is closed under addition (that is, the closure property holds for addition). (Notice that it is not enough to know that the sum $\mathbf{x} + \mathbf{y}$ is another 3-vector. We have to show that $\mathbf{x} + \mathbf{y}$ is actually in the plane $\mathcal{W}$.)

**Closure under scalar multiplication:** Let $\mathbf{x}$ be any vector in $\mathcal{W}$. Then any scalar multiple of $\mathbf{x}$, $a\mathbf{x}$, is either parallel to $\mathbf{x}$ or equal to $\mathbf{0}$. Therefore, $a\mathbf{x}$ lies in any plane through the origin that contains $\mathbf{x}$ (in particular, $\mathcal{W}$). Hence, $a\mathbf{x}$ is in $\mathcal{W}$, and $\mathcal{W}$ is closed under scalar multiplication.

**Properties (1) through (8):** Properties (1), (2), (5), (6), (7), and (8) are true for all vectors in $\mathcal{W}$ by Theorem 1.3, since $\mathcal{W} \subseteq \mathbb{R}^3$. However, properties (3) and (4) must be checked separately for $\mathcal{W}$ because they are *existence* properties. We know that the zero vector and additive inverses exist in $\mathbb{R}^3$, but are they in $\mathcal{W}$? Now, $\mathbf{0} = [0, 0, 0]$ is in $\mathcal{W}$, because the plane $\mathcal{W}$ passes through the origin, thus proving property (3). Also, the opposite (additive inverse) of any vector lying in the plane $\mathcal{W}$ also lies in $\mathcal{W}$, thus proving property (4).

Since the closure properties and all of properties (1) through (8) hold, $\mathcal{W}$ is a vector space.

### Example 4

Let $\mathcal{P}_n$ be the set of polynomials of degree $\leq n$, with real coefficients. The vectors in $\mathcal{P}_n$ have the form $\mathbf{p} = a_n x^n + \cdots + a_1 x + a_0$ for some real numbers $a_0, a_1, \ldots, a_n$. We show that $\mathcal{P}_n$ is a vector space under the usual operations of addition and scalar multiplication.

**Closure under addition:** Since addition of polynomials is performed in the usual manner here—that is, by adding corresponding coefficients—the sum of any two polynomials of degree $\leq n$ also has degree $\leq n$ and so is in $\mathcal{P}_n$. Thus, the closure property of addition holds.

**Closure under scalar multiplication:** Let $b$ be a real number and $\mathbf{p} = a_n x^n + \cdots + a_1 x + a_0$ be a vector in $\mathcal{P}_n$. We define $b\mathbf{p}$ in the usual manner to be the polynomial $(ba_n)x^n + \cdots + (ba_1)x + ba_0$, which is also in $\mathcal{P}_n$. Hence, the closure property of scalar multiplication holds.

Thus, if the remaining eight vector space properties hold, $\mathcal{P}_n$ is a vector space under these operations. We verify properties (1), (3), and (4) of the definition and leave the others for you to check.

**Property (1) (Commutative Law of Addition):** We must show that the order in which two vectors (polynomials) are added makes no difference. Now, by the commutative law of addition for real numbers,

$$\begin{aligned}(a_n x^n + \cdots + a_1 x + a_0) + \left(b_n x^n + \cdots + b_1 x + b_0\right) &= (a_n + b_n)x^n + \cdots + (a_1 + b_1)x + (a_0 + b_0) \\ &= (b_n + a_n)x^n + \cdots + (b_1 + a_1)x + (b_0 + a_0) \\ &= (b_n x^n + \cdots + b_1 x + b_0) + (a_n x^n + \cdots + a_1 x + a_0).\end{aligned}$$

**Property (3) (Existence of Additive Identity Element):** The zero-degree polynomial $\mathbf{z} = 0x^n + \cdots + 0x + 0$ acts as the additive identity element $\mathbf{0}$. That is, adding $\mathbf{z}$ to any vector $\mathbf{p} = a_n x^n + \cdots + a_1 x + a_0$ does not change the vector:

$$\mathbf{z} + \mathbf{p} = (0 + a_n)x^n + \cdots + (0 + a_1)x + (0 + a_0) = \mathbf{p}.$$

**Property (4) (Existence of Additive Inverse):** We must show that each vector $\mathbf{p} = a_n x^n + \cdots + a_1 x + a_0$ in $\mathcal{P}_n$ has an additive inverse in $\mathcal{P}_n$. But, the vector $-\mathbf{p} = -(a_n x^n + \cdots + a_1 x + a_0) = (-a_n)x^n + \cdots + (-a_1)x + (-a_0)$ has the property that $\mathbf{p} + [-\mathbf{p}] = \mathbf{z}$, the zero vector, and so $-\mathbf{p}$ acts as the additive inverse of $\mathbf{p}$. Because $-\mathbf{p}$ is also in $\mathcal{P}_n$, we are done.

The vector space in Example 4 is similar to our prototype $\mathbb{R}^n$. For any polynomial in $\mathcal{P}_n$, consider the sequence of its $n + 1$ coefficients. This sequence completely describes that polynomial and can be thought of as an $(n + 1)$-vector. For example, a polynomial $a_2 x^2 + a_1 x + a_0$ in $\mathcal{P}_2$ can be described by the 3-vector $[a_2, a_1, a_0]$. In this way, the vector space $\mathcal{P}_2$ "resembles" the vector space $\mathbb{R}^3$, and in general, $\mathcal{P}_n$ "resembles" $\mathbb{R}^{n+1}$. We will frequently capitalize on this "resemblance" in an informal way throughout the chapter. We will formalize this relationship between $\mathcal{P}_n$ and $\mathbb{R}^{n+1}$ in Section 5.5.

### Example 5

The set $\mathcal{P}$ of all polynomials (of all degrees) is a vector space under the usual (term-by-term) operations of addition and scalar multiplication (see Exercise 14).

### Example 6

Let $\mathcal{V}$ be the set of all real-valued functions defined on $\mathbb{R}$. For example, $\mathbf{f}(x) = \arctan(x)$ is in $\mathcal{V}$. We define addition of functions as usual: $\mathbf{h} = \mathbf{f} + \mathbf{g}$ is the function such that $\mathbf{h}(x) = \mathbf{f}(x) + \mathbf{g}(x)$, for every $x \in \mathbb{R}$. Similarly, if $a \in \mathbb{R}$ and $\mathbf{f}$ is in $\mathcal{V}$, we define the scalar multiple $\mathbf{h} = a\mathbf{f}$ to be the function such that $\mathbf{h}(x) = a\mathbf{f}(x)$, for every $x \in \mathbb{R}$.

**Closure Properties:** These properties hold for $\mathcal{V}$ because sums and scalar multiples of real-valued functions produce real-valued functions.

To finish verifying that $\mathcal{V}$ is a vector space, we must check that the remaining eight vector space properties hold. We verify property (2) and properties (5) through (8), and leave the others for you to check.

Suppose that $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{h}$ are in $\mathcal{V}$, and $a$ and $b$ are real numbers.

**Property (2)**: For every $x \in \mathbb{R}$, $\mathbf{f}(x) + (\mathbf{g}(x) + \mathbf{h}(x)) = (\mathbf{f}(x) + \mathbf{g}(x)) + \mathbf{h}(x)$, by the associative law of addition for real numbers. Thus, $\mathbf{f} + (\mathbf{g} + \mathbf{h}) = (\mathbf{f} + \mathbf{g}) + \mathbf{h}$.

**Properties (5) and (6)**: For every $x \in \mathbb{R}$, $a(\mathbf{f}(x) + \mathbf{g}(x)) = a\mathbf{f}(x) + a\mathbf{g}(x)$ and $(a+b)\mathbf{f}(x) = a\mathbf{f}(x) + b\mathbf{f}(x)$ by the distributive laws for real numbers of multiplication over addition. Hence, $a(\mathbf{f} + \mathbf{g}) = a\mathbf{f} + a\mathbf{g}$, and $(a+b)\mathbf{f} = a\mathbf{f} + b\mathbf{f}$.

**Property (7)**: For every $x \in \mathbb{R}$, $(ab)\mathbf{f}(x) = a(b\mathbf{f}(x))$ follows from the associative law of multiplication for real numbers. Hence, $(ab)\mathbf{f} = a(b\mathbf{f})$.

**Property (8)**: Since $1 \cdot \mathbf{f}(x) = \mathbf{f}(x)$ for every real number $x$, we have $1 \cdot \mathbf{f} = \mathbf{f}$ in $\mathcal{V}$.

## Two Unusual Vector Spaces

The next two examples place unusual operations on familiar sets to create new vector spaces. In such cases, regardless of how the operations are defined, we sometimes use the symbols $\oplus$ and $\odot$ to denote various unusual (non-regular) addition and scalar multiplication, respectively. Note that $\oplus$ is defined differently in Examples 7 and 8 (and similarly for $\odot$).

### Example 7

Let $\mathcal{V}$ be the set $\mathbb{R}^+$ of positive real numbers. This set is not a vector space under the usual operations of addition and scalar multiplication (why?). However, we can define new rules for these operations to make $\mathcal{V}$ a vector space. In what follows, we sometimes think of elements of $\mathbb{R}^+$ as abstract vectors (in which case we use boldface type, such as $\mathbf{v}$) or as the values on the positive real number line they represent (in which case we use italics, such as $v$).

To define "addition" on $\mathcal{V}$, we use *multiplication* of real numbers. That is,

$$\mathbf{v}_1 \oplus \mathbf{v}_2 = v_1 \cdot v_2$$

for every $\mathbf{v}_1$ and $\mathbf{v}_2$ in $\mathcal{V}$, where we use the symbol $\oplus$ for the "addition" operation on $\mathcal{V}$ to emphasize that this is not addition of real numbers. The definition of a vector space states only that vector addition must be a rule for combining two vectors to yield a third vector so that properties (1) through (8) hold. There is no stipulation that vector addition must be at all similar to ordinary addition of real numbers.[2] We next define "scalar multiplication," $\odot$, on $\mathcal{V}$ by

$$a \odot \mathbf{v} = v^a$$

for every $a \in \mathbb{R}$ and $\mathbf{v} \in \mathcal{V}$.

**Closure Properties:** From the given definitions, we see that if $\mathbf{v}_1$ and $\mathbf{v}_2$ are in $\mathcal{V}$ and $a$ is in $\mathbb{R}$, then $\mathbf{v}_1 \oplus \mathbf{v}_2$ is a positive real number (since it is the product of two positive real numbers), and $a \odot \mathbf{v}_1$ is a positive real number (since it is the positive, zero, or negative power of a positive real number). Therefore, both $\mathbf{v}_1 \oplus \mathbf{v}_2$ and $a \odot \mathbf{v}_1$ are in $\mathcal{V}$, thus verifying the two closure properties.

To prove the other eight properties, we assume that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathcal{V}$ and that $a, b \in \mathbb{R}$.

**Property (1)**: $\mathbf{v}_1 \oplus \mathbf{v}_2 = v_1 \cdot v_2 = v_2 \cdot v_1$ (by the commutative law of multiplication for real numbers) $= \mathbf{v}_2 \oplus \mathbf{v}_1$.

**Property (2)**: $\mathbf{v}_1 \oplus (\mathbf{v}_2 \oplus \mathbf{v}_3) = \mathbf{v}_1 \oplus (v_2 \cdot v_3) = v_1 \cdot (v_2 \cdot v_3) = (v_1 \cdot v_2) \cdot v_3$ (by the associative law of multiplication for real numbers) $= (\mathbf{v}_1 \oplus \mathbf{v}_2) \cdot v_3 = (\mathbf{v}_1 \oplus \mathbf{v}_2) \oplus \mathbf{v}_3$.

**Property (3)**: The number 1 in $\mathbb{R}^+$ acts as the zero vector $\mathbf{0}$ in $\mathcal{V}$ (why?).

**Property (4)**: The additive inverse of $\mathbf{v}$ in $\mathcal{V}$ is the positive real number $(1/v)$, because $\mathbf{v} \oplus (1/v) = v \cdot (1/v) = 1$, the zero vector in $\mathcal{V}$.

**Property (5)**: $a \odot (\mathbf{v}_1 \oplus \mathbf{v}_2) = a \odot (v_1 \cdot v_2) = (v_1 \cdot v_2)^a = v_1^a \cdot v_2^a = (a \odot \mathbf{v}_1) \cdot (a \odot \mathbf{v}_2) = (a \odot \mathbf{v}_1) \oplus (a \odot \mathbf{v}_2)$.

**Property (6)**: $(a+b) \odot \mathbf{v} = v^{a+b} = v^a \cdot v^b = (a \odot \mathbf{v}) \cdot (b \odot \mathbf{v}) = (a \odot \mathbf{v}) \oplus (b \odot \mathbf{v})$.

**Property (7)**: $(ab) \odot \mathbf{v} = v^{ab} = \left(v^b\right)^a = (b \odot \mathbf{v})^a = a \odot (b \odot \mathbf{v})$.

**Property (8)**: $1 \odot \mathbf{v} = v^1 = \mathbf{v}$.

### Example 8

Let $\mathcal{V} = \mathbb{R}^2$, with addition defined by

$$[x, y] \oplus [w, z] = [x + w + 1,\ \ y + z - 2]$$

and scalar multiplication defined by

$$a \odot [x, y] = [ax + a - 1,\ \ ay - 2a + 2]\,.$$

[2] You might expect the operation $\oplus$ to be called something other than "addition." However, most of our vector space terminology comes from the motivating example of $\mathbb{R}^n$, so the word *addition* is a natural choice for the name of the operation.

The closure properties hold for these operations (why?). In fact, $\mathcal{V}$ forms a vector space because the eight vector properties also hold. We verify properties (2), (3), (4), and (6) and leave the others for you to check.

**Property (2)**:

$$\begin{aligned}[x,y]\oplus([u,v]\oplus[w,z]) &= [x,y]\oplus[u+w+1,\ v+z-2]\\ &= [x+u+w+2,\ y+v+z-4]\\ &= [x+u+1,\ y+v-2]\oplus[w,z]\\ &= ([x,y]\oplus[u,v])\oplus[w,z].\end{aligned}$$

**Property (3)**: The vector $[-1,2]$ acts as the zero vector, since

$$[x,y]\oplus[-1,2]=[x+(-1)+1,\ y+2-2]=[x,y].$$

**Property (4)**: The additive inverse of $[x,y]$ is $[-x-2,-y+4]$, because

$$[x,y]\oplus[-x-2,-y+4]=[x-x-2+1,\ y-y+4-2]=[-1,2],$$

the zero vector in $\mathcal{V}$.

**Property (6)**:

$$\begin{aligned}(a+b)\odot[x,y] &= [(a+b)x+(a+b)-1,\ (a+b)y-2(a+b)+2]\\ &= [(ax+a-1)+(bx+b-1)+1,\ (ay-2a+2)+(by-2b+2)-2]\\ &= [ax+a-1,\ ay-2a+2]\oplus[bx+b-1,\ by-2b+2]\\ &= (a\odot[x,y])\oplus(b\odot[x,y])\,.\end{aligned}$$

## Some Elementary Properties of Vector Spaces

The next theorem contains several simple results regarding vector spaces. Although these are obviously true in the most familiar examples, we must prove them in general before we know they hold in every possible vector space.

**Theorem 4.1** *Let $\mathcal{V}$ be a vector space. Then, for every vector* $\mathbf{v}$ *in $\mathcal{V}$ and every real number $a$, we have*

| | | |
|---|---|---|
| (1) | $a\mathbf{0}=\mathbf{0}$ | *Any scalar multiple of the zero vector yields the zero vector.* |
| (2) | $0\mathbf{v}=\mathbf{0}$ | *The scalar zero multiplied by any vector yields the zero vector.* |
| (3) | $(-1)\,\mathbf{v}=-\mathbf{v}$ | *The scalar $-1$ multiplied by any vector yields the additive inverse of that vector.* |
| (4) | *If* $a\mathbf{v}=\mathbf{0}$, *then* $a=0$ *or* $\mathbf{v}=\mathbf{0}$ | *If a scalar multiplication yields the zero vector, then either the scalar is zero, or the vector is the zero vector, or both.* |

Part (3) justifies the notation for the additive inverse in property (4) of the definition of a vector space and shows we do not need to distinguish between $-\mathbf{v}$ and $(-1)\mathbf{v}$.

This theorem must be proved directly from the properties in the definition of a vector space because at this point we have no other known facts about general vector spaces. We prove parts (1), (3), and (4). The proof of part (2) is similar to the proof of part (1) and is left as Exercise 17.

*Proof.* (Abridged):

**Part (1)**: By direct proof,

$$\begin{aligned}a\mathbf{0} &= a\mathbf{0}+\mathbf{0} && \text{by property (3)}\\ &= a\mathbf{0}+(a\mathbf{0}+(-[a\mathbf{0}])) && \text{by property (4)}\\ &= (a\mathbf{0}+a\mathbf{0})+(-[a\mathbf{0}]) && \text{by property (2)}\\ &= a(\mathbf{0}+\mathbf{0})+(-[a\mathbf{0}]) && \text{by property (5)}\\ &= a\mathbf{0}+(-[a\mathbf{0}]) && \text{by property (3)}\\ &= \mathbf{0}. && \text{by property (4)}\end{aligned}$$

**Part (3)**: First, note that $\mathbf{v}+(-1)\mathbf{v}=1\mathbf{v}+(-1)\mathbf{v}$ (by property (8)) $=(1+(-1))\mathbf{v}$ (by property (6)) $=0\mathbf{v}=\mathbf{0}$ (by part (2) of Theorem 4.1). Therefore, $(-1)\mathbf{v}$ acts as *an* additive inverse for $\mathbf{v}$. We will finish the proof by showing that the additive inverse for $\mathbf{v}$ is unique. Hence $(-1)\mathbf{v}$ will be *the* additive inverse of $\mathbf{v}$.

Suppose that $\mathbf{x}$ and $\mathbf{y}$ are both additive inverses for $\mathbf{v}$. Thus, $\mathbf{x}+\mathbf{v}=\mathbf{0}$ and $\mathbf{v}+\mathbf{y}=\mathbf{0}$. Hence,

$$\mathbf{x}=\mathbf{x}+\mathbf{0}=\mathbf{x}+(\mathbf{v}+\mathbf{y})=(\mathbf{x}+\mathbf{v})+\mathbf{y}=\mathbf{0}+\mathbf{y}=\mathbf{y}.$$

Therefore, any two additive inverses of $\mathbf{v}$ are equal. (Note that this is, in essence, the same proof we gave for Theorem 2.11, the uniqueness of inverse for matrix multiplication. You should compare these proofs.)

**Part (4)**: This is an "If $A$ then $B$ or $C$" statement. Therefore, we assume that $a\mathbf{v}=\mathbf{0}$ and $a\neq 0$ and show that $\mathbf{v}=\mathbf{0}$. Now,

$$\begin{aligned}
\mathbf{v} &= 1\mathbf{v} && \text{by property (8)}\\
&= \left(\frac{1}{a}\cdot a\right)\mathbf{v} && \text{because } a\neq 0\\
&= \left(\frac{1}{a}\right)(a\mathbf{v}) && \text{by property (7)}\\
&= \left(\frac{1}{a}\right)\mathbf{0} && \text{because } a\mathbf{v}=\mathbf{0}\\
&= \mathbf{0}. && \text{by part (1) of Theorem 4.1}
\end{aligned}$$

□

Theorem 4.1 is valid even for unusual vector spaces, such as those in Examples 7 and 8. For instance, part (4) of the theorem claims that, in general, $a\mathbf{v}=\mathbf{0}$ implies $a=0$ or $\mathbf{v}=\mathbf{0}$. This statement can quickly be verified for the vector space $\mathcal{V}=\mathbb{R}^+$ with operations $\oplus$ and $\odot$ from Example 7. In this case, $a\odot\mathbf{v}=v^a$, and the zero vector $\mathbf{0}$ is the real number 1. Then, part (4) is equivalent here to the true statement that $v^a=1$ implies $a=0$ or $v=1$.

Applying parts (2) and (3) of Theorem 4.1 to an unusual vector space $\mathcal{V}$ gives a quick way of finding the zero vector $\mathbf{0}$ of $\mathcal{V}$ and the additive inverse $-\mathbf{v}$ for any vector $\mathbf{v}$ in $\mathcal{V}$. For instance, in Example 8, we have $\mathcal{V}=\mathbb{R}^2$ with scalar multiplication $a\odot[x,y]=[ax+a-1, ay-2a+2]$. To find the zero vector $\mathbf{0}$ in $\mathcal{V}$, we simply multiply the scalar 0 by any general vector $[x,y]$ in $\mathcal{V}$.

$$\mathbf{0}=0\odot[x,y]=[0x+0-1,\ 0y-2(0)+2]=[-1,2].$$

Similarly, if $[x,y]\in\mathcal{V}$, then $-1\odot[x,y]$ gives the additive inverse of $[x,y]$.

$$\begin{aligned}
-[x,y]=-1\odot[x,y]&=[-1x+(-1)-1, -1y-2(-1)+2]\\
&=[-x-2, -y+4].
\end{aligned}$$

## Failure of the Vector Space Conditions

We conclude this section by considering some sets that are not vector spaces to see what can go wrong.

### Example 9

The set $\Phi$ of real-valued functions, $f$, defined on the interval $[0,1]$ such that $f\left(\frac{1}{2}\right)=1$, is not a vector space under the usual operations of function addition and scalar multiplication because the closure properties do not hold. If $f$ and $g$ are in $\Phi$, then

$$(f+g)\left(\frac{1}{2}\right)=f\left(\frac{1}{2}\right)+g\left(\frac{1}{2}\right)=1+1=2\neq 1,$$

so $f+g$ is not in $\Phi$. Therefore, $\Phi$ is not closed under addition and cannot be a vector space. (Is $\Phi$ closed under scalar multiplication?)

### Example 10

Let $\Upsilon$ be the set $\mathbb{R}^2$ with operations

$$\mathbf{v}_1\oplus\mathbf{v}_2=\mathbf{v}_1+\mathbf{v}_2 \quad\text{and}\quad c\odot\mathbf{v}=c(\mathbf{A}\mathbf{v}),\quad \text{where } \mathbf{A}=\begin{bmatrix}-3 & 1\\ 5 & -2\end{bmatrix}.$$

With these operations, $\Upsilon$ is not a vector space. You can verify that $\Upsilon$ is closed under $\oplus$ and $\odot$, but properties (7) and (8) of the definition are not satisfied. For example, property (8) fails since

$$1 \odot \begin{bmatrix} 2 \\ 7 \end{bmatrix} = 1\left(\begin{bmatrix} -3 & 1 \\ 5 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ 7 \end{bmatrix}\right) = 1\begin{bmatrix} 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \end{bmatrix} \neq \begin{bmatrix} 2 \\ 7 \end{bmatrix}.$$

## New Vocabulary

closure properties
scalar multiplication (in a general vector space)
trivial vector space
vector addition (in a general vector space)
vectors (in a general vector space)
vector space

## Highlights

- Vector spaces have two specified operations: vector addition and scalar multiplication. A set $\mathcal{V}$ with such operations is a vector space if and only if $\mathcal{V}$ is closed under these operations and possesses the 8 additional fundamental properties stated in the definition.
- The smallest possible vector space is the trivial vector space $\{\mathbf{0}\}$.
- Familiar vector spaces (under the normal operations) include $\mathbb{R}^n$, $\mathcal{M}_{mn}$, $\mathcal{P}_n$, $\mathcal{P}$, a line through the origin, a plane through the origin, and all real-valued functions with domain $\mathbb{R}$.
- Any scalar multiple $a\mathbf{0}$ of the zero vector $\mathbf{0}$ is also equal to $\mathbf{0}$.
- The scalar 0 times any vector $\mathbf{v}$ (that is, $0\mathbf{v}$) equals the zero vector $\mathbf{0}$.
- The scalar $-1$ times any vector $\mathbf{v}$ is equal to the additive inverse $-\mathbf{v}$ of the vector.
- If a scalar multiple $a\mathbf{v}$ of a vector $\mathbf{v}$ is equal to the zero vector $\mathbf{0}$, then either $a = 0$ or $\mathbf{v} = \mathbf{0}$.

## Exercises for Section 4.1

**Remember**: To verify that a given set with its operations is a vector space, you must prove the two closure properties as well as the remaining eight properties in the definition. To show that a set with operations is *not* a vector space, you need only find an example showing that one of the closure properties or one of the remaining eight properties is not satisfied.

**1.** Rewrite properties (2), (5), (6), and (7) in the definition of a vector space using the symbols $\oplus$ for vector addition and $\odot$ for scalar multiplication. (The notations for real number addition and multiplication should not be changed.)

**2.** Prove that the set of all scalar multiples of the vector $[1, 3, 2]$ in $\mathbb{R}^3$ forms a vector space with the usual operations on 3-vectors.

**3.** Verify that the set of polynomials $\mathbf{f}$ in $\mathcal{P}_4$ such that $\mathbf{f}(3) = 0$ forms a vector space with the standard operations.

**4.** Prove that $\mathbb{R}$ is a vector space using the operations $\oplus$ and $\odot$ given by $\mathbf{x} \oplus \mathbf{y} = \left(x^3 + y^3\right)^{1/3}$ and $a \odot \mathbf{x} = (\sqrt[3]{a})x$.

★ **5.** Show that the set of singular $2 \times 2$ matrices under the usual operations is *not* a vector space.

**6.** Prove that the set of nonsingular $n \times n$ matrices under the usual operations is *not* a vector space.

**7.** Show that $\mathbb{R}$, with ordinary addition but with scalar multiplication replaced by $a \odot \mathbf{x} = \mathbf{0}$ for every real number $a$, is *not* a vector space.

★ **8.** Show that the set $\mathbb{R}$, with the usual scalar multiplication but with addition given by $x \oplus y = 2(x + y)$, is *not* a vector space.

**9.** Show that the set $\mathbb{R}^2$, with the usual scalar multiplication but with vector addition replaced by $[x, y] \oplus [w, z] = [0, y + z]$, does *not* form a vector space.

**10.** Let $\mathcal{A} = \mathbb{R}$, with the operations $\oplus$ and $\odot$ given by $\mathbf{x} \oplus \mathbf{y} = \left(x^5 + y^5\right)^{1/5}$ and $a \odot \mathbf{x} = a\mathbf{x}$. Determine whether $\mathcal{A}$ is a vector space. Prove your answer.

**11.** Let $\mathcal{V}$ be a vector space. Prove that the identity element for vector addition in $\mathcal{V}$ is unique. (Hint: Use a proof by contradiction.)

**12.** The set $\mathbb{R}^2$ with operations $[x, y] \oplus [w, z] = [x + w + 2, y + z - 5]$ and $a \odot [x, y] = [ax + 2a - 2, ay - 5a + 5]$ is a vector space. Use parts (2) and (3) of Theorem 4.1 to find the zero vector $\mathbf{0}$ and the additive inverse of each vector $\mathbf{v} = [x, y]$ for this vector space. Then check your answers.

**13.** Let $\mathcal{V}$ be a vector space. Prove the following **cancellation laws**:

**(a)** If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in $\mathcal{V}$ for which $\mathbf{u} + \mathbf{v} = \mathbf{w} + \mathbf{v}$, then $\mathbf{u} = \mathbf{w}$.

**(b)** If $a$ and $b$ are scalars and $\mathbf{v} \neq \mathbf{0}$ is a vector in $\mathcal{V}$ with $a\mathbf{v} = b\mathbf{v}$, then $a = b$.

**(c)** If $a \neq 0$ is a scalar and $\mathbf{v}, \mathbf{w} \in \mathcal{V}$ with $a\mathbf{v} = a\mathbf{w}$, then $\mathbf{v} = \mathbf{w}$.

**14.** Prove that the set $\mathcal{P}$ of all polynomials with real coefficients forms a vector space under the usual operations of polynomial (term-by-term) addition and scalar multiplication.

**15.** Let $X$ be any nonempty set, and let $\mathcal{V} = \{$all real-valued functions with domain $X\}$. Prove that $\mathcal{V}$ is a vector space using ordinary addition and scalar multiplication of real-valued functions. (Hint: Alter the proofs given in Example 6 appropriately. Note that along with the closure properties, all 8 properties in the definition must be verified, not just those shown in Example 6.)

**16.** Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be vectors in a vector space $\mathcal{V}$, and let $a_1, \ldots, a_n$ be any real numbers. Use induction to prove that $\sum_{i=1}^{n} a_i \mathbf{v}_i$ is in $\mathcal{V}$.

**17.** Prove part (2) of Theorem 4.1.

**18.** Prove that every nontrivial vector space has an infinite number of distinct elements.

**19.** Suppose $\mathcal{V} = \{(x, y) \mid x, y \in \mathbb{R}\}$. Let $\mathbf{u}$ and $\mathbf{v}$ be the vectors whose initial points are both $(-1, 2)$ and whose terminal points are $(x, y)$ and $(w, z)$, respectively.

**(a)** Define an operation $\oplus$ of addition on $\mathcal{V}$ as follows: Let $(x, y) \oplus (w, z)$ be the terminal point of the vector $\mathbf{u} + \mathbf{v}$, assuming that its initial point is $(-1, 2)$. Find a direct formula for $(x, y) \oplus (w, z)$.

**(b)** Define an operation $\odot$ of scalar multiplication on $\mathcal{V}$ as follows: Let $a \odot (x, y)$ be the terminal point of the vector $a\mathbf{u}$, assuming that its initial point is $(-1, 2)$. Find a direct formula for $a \odot (x, y)$.

**(c)** Compare your formulas from parts (a) and (b) with the operations for the unusual vector space of Example 8. (This exercise shows that for the vector space in Example 8, the vectors simply represent the terminal points of vectors with initial point at $(-1, 2)$, and the operations represent "regular" addition and scalar multiplication of vectors, using $(-1, 2)$ as the initial point instead of the origin.)

★ **20.** True or False:

**(a)** The set $\mathbb{R}^n$ under any operations of "addition" and "scalar multiplication" is a vector space.

**(b)** The set of all polynomials of degree 7 is a vector space under the usual operations of addition and scalar multiplication.

**(c)** The set of all polynomials of degree $\leq 7$ is a vector space under the usual operations of addition and scalar multiplication.

**(d)** If $\mathbf{x}$ is a vector in a vector space $\mathcal{V}$, and $c$ is a nonzero scalar, then $c\mathbf{x} = \mathbf{0}$ implies $\mathbf{x} = \mathbf{0}$.

**(e)** In a vector space, scalar multiplication by the zero scalar always results in the zero scalar.

**(f)** In a vector space, scalar multiplication of a vector $\mathbf{x}$ by $-1$ always results in the additive inverse of $\mathbf{x}$.

**(g)** The set of all real valued functions $f$ on $\mathbb{R}$ such that $f(1) = 0$ is a vector space under the usual operations of addition and scalar multiplication.

## 4.2 Subspaces

When a vector space is a subset of a known vector space and has the same operations, it becomes easier to handle. These subsets, called **subspaces**, also provide additional information about the larger vector space, as we will see.

### Definition of a Subspace and Examples

**Definition** Let $\mathcal{V}$ be a vector space. Then $\mathcal{W}$ is a **subspace** of $\mathcal{V}$ if and only if $\mathcal{W}$ is a subset of $\mathcal{V}$, and $\mathcal{W}$ is itself a vector space with the same operations as $\mathcal{V}$.

That is, $\mathcal{W}$ is a subspace of $\mathcal{V}$ if and only if $\mathcal{W}$ is a vector space inside $\mathcal{V}$ such that for every $a$ in $\mathbb{R}$ and every $\mathbf{v}$ and $\mathbf{w}$ in $\mathcal{W}$, $\mathbf{v} + \mathbf{w}$ and $a\mathbf{v}$ yield the same vectors when the operations are performed in $\mathcal{W}$ as when they are performed in $\mathcal{V}$.

**Example 1**

Consider the subset $\mathcal{W}$ of $\mathbb{R}^3$ of all points lying on a plane through the origin. Example 3 of Section 4.1 showed that $\mathcal{W}$ is a vector space under the usual addition and scalar multiplication in $\mathbb{R}^3$. Therefore, $\mathcal{W}$ is a subspace of $\mathbb{R}^3$.

**Example 2**

Consider the subset $\mathcal{S}$ of all scalar multiples of the vector $[1,3,2]$ in $\mathbb{R}^3$. $\mathcal{S}$ forms a vector space under the usual addition and scalar multiplication in $\mathbb{R}^3$ (see Exercise 2 in Section 4.1). Hence, $\mathcal{S}$ is a subspace of $\mathbb{R}^3$.

Notice that $\mathcal{S}$ corresponds geometrically to the set of points lying on the line through the origin in $\mathbb{R}^3$ in the direction of the vector $[1,3,2]$ (see Fig. 4.2). In the same manner, every line through the origin determines a subspace of $\mathbb{R}^3$—namely, the set of scalar multiples of a nonzero vector in the direction of that line.

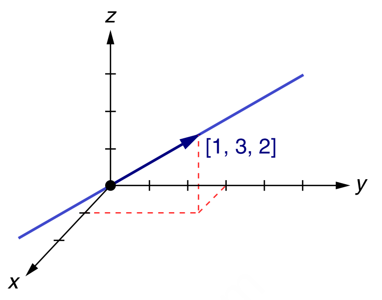


**FIGURE 4.2** Line containing all scalar multiples of $[1,3,2]$

**Example 3**

Let $\mathcal{V}$ be any vector space. Then $\mathcal{V}$ is a subspace of itself (why?). Also, if $\mathcal{W}$ is the subset $\{\mathbf{0}\}$ of $\mathcal{V}$, then $\mathcal{W}$ is a vector space under the same operations as $\mathcal{V}$ (see Example 2 of Section 4.1). Therefore, $\mathcal{W}=\{\mathbf{0}\}$ is a subspace of $\mathcal{V}$.

Although the subspaces $\mathcal{V}$ and $\{\mathbf{0}\}$ of a vector space $\mathcal{V}$ are important, they occasionally complicate matters because they must be considered as special cases in proofs. All subspaces of $\mathcal{V}$ other than $\mathcal{V}$ itself are called **proper subspaces** of $\mathcal{V}$. The subspace $\mathcal{W}=\{\mathbf{0}\}$ is called the **trivial subspace** of $\mathcal{V}$. A vector space containing at least one nonzero vector has at least two distinct subspaces, the trivial subspace and the vector space itself. In fact, under the usual operations, $\mathbb{R}$ has only these two subspaces (see Exercise 16).

If we consider Examples 1 to 3 in the context of $\mathbb{R}^3$, we find at least four different types of subspaces of $\mathbb{R}^3$. These are the trivial subspace $\{[0,0,0]\}=\{\mathbf{0}\}$, subspaces like Example 2 that can be geometrically represented as a line (thus "resembling" $\mathbb{R}$), subspaces like Example 1 that can be represented as a plane (thus "resembling" $\mathbb{R}^2$), and the subspace $\mathbb{R}^3$ itself.[3] All but the last are proper subspaces. Later we will show that each subspace of $\mathbb{R}^3$ is, in fact, one of these four types. Similarly, we will show later that all subspaces of $\mathbb{R}^n$ "resemble" $\{\mathbf{0}\}, \mathbb{R}, \mathbb{R}^2, \mathbb{R}^3, \ldots, \mathbb{R}^{n-1}$, or $\mathbb{R}^n$.

**Example 4**

Consider the vector spaces (using ordinary function addition and scalar multiplication) in the following chain:

$$\begin{aligned}\mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{P}_2 \subset \cdots \subset \mathcal{P} & \\ & \subset \{\text{differentiable real-valued functions on } \mathbb{R}\} \\ & \subset \{\text{continuous real-valued functions on } \mathbb{R}\} \\ & \subset \{\text{all real-valued functions on } \mathbb{R}\}.\end{aligned}$$

Some of these we encountered in Section 4.1, and the rest are discussed in Exercise 7 of this section. Each of these vector spaces is a proper subspace of every vector space after it in the chain (why?).

## When Is a Subset a Subspace?

It is important to note that not every subset of a vector space is a subspace. A subset $\mathcal{S}$ of a vector space $\mathcal{V}$ fails to be a subspace of $\mathcal{V}$ if $\mathcal{S}$ does not satisfy the properties of a vector space in its own right or if $\mathcal{S}$ does not use the same operations as $\mathcal{V}$.

[3] Although some subspaces of $\mathbb{R}^3$ "resemble" $\mathbb{R}$ and $\mathbb{R}^2$ geometrically, note that $\mathbb{R}$ and $\mathbb{R}^2$ are not actually subspaces of $\mathbb{R}^3$ because they are not subsets of $\mathbb{R}^3$.

**Example 5**

Consider the first quadrant in $\mathbb{R}^2$—that is, the set $\Omega$ of all 2-vectors of the form $[x, y]$ where $x \geq 0$ and $y \geq 0$. This subset $\Omega$ of $\mathbb{R}^2$ is not a vector space under the normal operations of $\mathbb{R}^2$ because it is not closed under scalar multiplication. (For example, $[3, 4]$ is in $\Omega$, but $-2 \cdot [3, 4] = [-6, -8]$ is not in $\Omega$.) Therefore, $\Omega$ cannot be a subspace of $\mathbb{R}^2$.

**Example 6**

Consider the vector space $\mathbb{R}$ under the usual operations. Let $\mathcal{W}$ be the subset $\mathbb{R}^+$. By Example 7 of Section 4.1, we know that $\mathcal{W}$ is a vector space under the unusual operations $\oplus$ and $\odot$, where $\oplus$ represents multiplication and $\odot$ represents exponentiation. Although $\mathcal{W}$ is a nonempty subset of $\mathbb{R}$ and is itself a vector space, $\mathcal{W}$ is not a subspace of $\mathbb{R}$ because $\mathcal{W}$ and $\mathbb{R}$ do not share the same operations.

The following theorem provides a shortcut for verifying that a (nonempty) subset $\mathcal{W}$ of a vector space is a subspace; if the closure properties hold for $\mathcal{W}$, then the remaining eight vector space properties automatically follow as well.

**Theorem 4.2** *Let $\mathcal{V}$ be a vector space, and let $\mathcal{W}$ be a nonempty subset of $\mathcal{V}$ using the same operations. Then $\mathcal{W}$ is a subspace of $\mathcal{V}$ if and only if $\mathcal{W}$ is closed under vector addition and scalar multiplication in $\mathcal{V}$.*

Notice that this theorem applies only to *nonempty subsets* of a vector space. Even though the empty set is a subset of every vector space, it is not a subspace of any vector space because it does not contain an additive identity.

*Proof.* Since this is an "if and only if" statement, the proof has two parts.

**Part (1):** Suppose $\mathcal{W}$ is a subspace of $\mathcal{V}$. We must show $\mathcal{W}$ is closed under the two operations. Now, as a subspace, $\mathcal{W}$ is itself a vector space. Hence, the closure properties hold for $\mathcal{W}$ as they do for any vector space.

**Part (2):** Suppose the closure properties hold for a nonempty subset $\mathcal{W}$ of $\mathcal{V}$. We must show $\mathcal{W}$ itself is a vector space under the operations in $\mathcal{V}$. That is, we must prove the remaining eight vector space properties hold for $\mathcal{W}$.

Properties (1), (2), (5), (6), (7), and (8) are all true in $\mathcal{W}$ because they are true in $\mathcal{V}$, a known vector space. That is, since these properties hold for all vectors in $\mathcal{V}$, they must be true for all vectors in its subset, $\mathcal{W}$. For example, to prove property (1) for $\mathcal{W}$, let $\mathbf{u}, \mathbf{v} \in \mathcal{W}$. Then,

$$\begin{aligned}
\underbrace{\mathbf{u}+\mathbf{v}}_{\text{addition in } \mathcal{W}} &= \underbrace{\mathbf{u}+\mathbf{v}}_{\text{addition in } \mathcal{V}} && \text{because } \mathcal{W} \text{ and } \mathcal{V} \text{ share the same operations} \\
&= \underbrace{\mathbf{v}+\mathbf{u}}_{\text{addition in } \mathcal{V}} && \text{because } \mathcal{V} \text{ is a vector space and property (1) holds} \\
&= \underbrace{\mathbf{v}+\mathbf{u}}_{\text{addition in } \mathcal{W}} && \text{because } \mathcal{W} \text{ and } \mathcal{V} \text{ share the same operations.}
\end{aligned}$$

Next we prove property (3), the existence of an additive identity in $\mathcal{W}$. Because $\mathcal{W}$ is nonempty, we can choose an element $\mathbf{w}_1$ from $\mathcal{W}$. Now $\mathcal{W}$ is closed under scalar multiplication, so $0\mathbf{w}_1$ is in $\mathcal{W}$. However, since this is the same operation as in $\mathcal{V}$, a known vector space, part (2) of Theorem 4.1 implies that $0\mathbf{w}_1 = \mathbf{0}$. Hence, $\mathbf{0}$ is in $\mathcal{W}$. Because $\mathbf{0} + \mathbf{v} = \mathbf{v}$ for all $\mathbf{v}$ in $\mathcal{V}$, it follows that $\mathbf{0} + \mathbf{w} = \mathbf{w}$ for all $\mathbf{w}$ in $\mathcal{W}$. Therefore, $\mathcal{W}$ contains the same additive identity that $\mathcal{V}$ has.

Finally we must prove that property (4), the existence of additive inverses, holds for $\mathcal{W}$. Let $\mathbf{w} \in \mathcal{W}$. Then $\mathbf{w} \in \mathcal{V}$. Part (3) of Theorem 4.1 shows $(-1)\mathbf{w}$ is the additive inverse of $\mathbf{w}$ in $\mathcal{V}$. If we can show that this additive inverse is also in $\mathcal{W}$, we will be done. But since $\mathcal{W}$ is closed under scalar multiplication, $(-1)\mathbf{w} \in \mathcal{W}$. □

## Verifying Subspaces in $\mathcal{M}_{nn}$ and $\mathbb{R}^n$

In the next three examples, we apply Theorem 4.2 to determine whether several subsets of $\mathcal{M}_{nn}$ and $\mathbb{R}^n$ are subspaces. Assume that $\mathcal{M}_{nn}$ and $\mathbb{R}^n$ have the usual operations.

**Example 7**

Consider $\mathcal{U}_n$, the set of upper triangular $n \times n$ matrices. Since $\mathcal{U}_n$ is nonempty, we may apply Theorem 4.2 to see whether $\mathcal{U}_n$ is a subspace of $\mathcal{M}_{nn}$. Closure of $\mathcal{U}_n$ under vector addition holds because the sum of any two $n \times n$ upper triangular matrices is again upper triangular. The closure property in $\mathcal{U}_n$ for scalar multiplication also holds, since any scalar multiple of an upper triangular matrix is again upper triangular. Hence, $\mathcal{U}_n$ is a subspace of $\mathcal{M}_{nn}$.

Similar arguments show that $\mathcal{L}_n$ (lower triangular $n \times n$ matrices) and $\mathcal{D}_n$ (diagonal $n \times n$ matrices) are also subspaces of $\mathcal{M}_{nn}$.

The subspace $\mathcal{D}_n$ of $\mathcal{M}_{nn}$ in Example 7 is the intersection of the subspaces $\mathcal{U}_n$ and $\mathcal{L}_n$. In fact, the intersection of subspaces of a vector space always produces a subspace under the same operations (see Exercise 18).

**Example 8**

Let $\mathcal{Y}$ be the set of vectors in $\mathbb{R}^4$ of the form $[a, 0, b, 0]$, that is, 4-vectors whose second and fourth coordinates are zero. The set $\mathcal{Y}$ is clearly nonempty, since $[0, 0, 0, 0] \in \mathcal{Y}$. We prove that $\mathcal{Y}$ is a subspace of $\mathbb{R}^4$ by checking the closure properties.

To prove closure under vector addition, we must add two arbitrary elements of $\mathcal{Y}$ and check that the result has the correct form for a vector in $\mathcal{Y}$. Now, $[a, 0, b, 0] + [c, 0, d, 0] = [(a + c), 0, (b + d), 0]$. The second and fourth coordinates of the sum are zero, so $\mathcal{Y}$ is closed under addition. Similarly, we must prove closure under scalar multiplication. Now, $k[a, 0, b, 0] = [ka, 0, kb, 0]$. Since the second and fourth coordinates of the product are zero, $\mathcal{Y}$ is closed under scalar multiplication. Hence, by Theorem 4.2, $\mathcal{Y}$ is a subspace of $\mathbb{R}^4$.

**Example 9**

Let $\mathcal{W}$ be the set of vectors in $\mathbb{R}^3$ of the form $[4a + 2b,\ a,\ b]$. Note that

$$[4a + 2b,\ a,\ b] = a[4, 1, 0] + b[2, 0, 1].$$

Thus, $\mathcal{W}$ is the set of all linear combinations of $[4, 1, 0]$ and $[2, 0, 1]$, which is clearly nonempty. We show that $\mathcal{W}$ is a subspace of $\mathbb{R}^3$ by checking the closure properties.

To check closure under addition, we must verify that the sum of two such linear combinations also has this same form. But,

$$(a[4, 1, 0] + b[2, 0, 1]) + (c[4, 1, 0] + d[2, 0, 1]) = (a + c)[4, 1, 0] + (b + d)[2, 0, 1],$$

which clearly is a linear combination of this same type (since $a + c$ and $b + d$ are scalars). Thus $\mathcal{W}$ is closed under addition.

For closure under scalar multiplication, we note that

$$k\,(a[4, 1, 0] + b[2, 0, 1]) = (ka)[4, 1, 0] + (kb)[2, 0, 1],$$

thus verifying that a scalar multiple of such a linear combination is another linear combination of this same type (since $ka$, $kb$ are scalars). Thus, $\mathcal{W}$ is closed under scalar multiplication as well.

Note that $\mathcal{W}$ is the type of subspace of $\mathbb{R}^3$ discussed in Example 1, since it is a plane through the origin containing the nonparallel vectors $[4, 1, 0]$ and $[2, 0, 1]$ (see Fig. 4.3). In other words, from the origin, it is not possible to reach endpoints lying outside this plane by using a GEaS for $\mathbb{R}^3$ with dials corresponding to $[4, 1, 0]$ and $[2, 0, 1]$.

## Subsets That Are Not Subspaces

If either closure property fails to hold for a subset of a vector space, the subset cannot be a subspace. For example, the following subsets of $\mathbb{R}^n$ are not subspaces. In each case, at least one of the two closure properties fails. (Can you determine which properties?)

$S_1$: The set of $n$-vectors whose first coordinate is nonnegative (in $\mathbb{R}^2$, this set is a half-plane)
$S_2$: The set of unit $n$-vectors (in $\mathbb{R}^3$, this set is a sphere)
$S_3$: For $n \geq 2$, the set of $n$-vectors with a zero in at least one coordinate (in $\mathbb{R}^3$, this set is the union of three planes)
$S_4$: The set of $n$-vectors having all integer coordinates
$S_5$: For $n \geq 2$, the set of all $n$-vectors whose first two coordinates add up to 3 (in $\mathbb{R}^2$, this is the line $x + y = 3$)

The subsets $S_2$ and $S_5$, which do not contain the additive identity $\mathbf{0}$ of $\mathbb{R}^n$, can quickly be disqualified as subspaces. In general,

> If a subset $\mathcal{S}$ of a vector space $\mathcal{V}$ does not contain the zero vector $\mathbf{0}$ of $\mathcal{V}$, then $\mathcal{S}$ is not a subspace of $\mathcal{V}$.

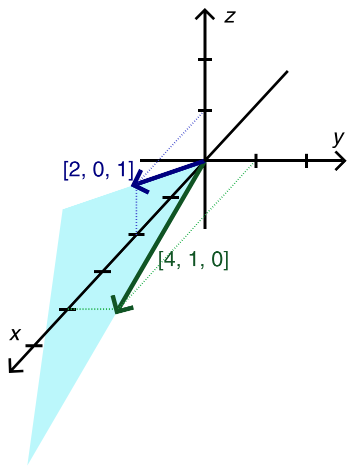


**FIGURE 4.3** The plane through the origin containing [4, 1, 0] and [2, 0, 1]

Checking for the presence of the additive identity is usually easy and thus is a fast way to show that certain subsets are not subspaces.

Similarly, none of the following subsets of $\mathcal{M}_{nn}$, $n \geq 2$, is a subspace because, in each case, at least one of the two closure properties fails:

$T_1$: the set of nonsingular $n \times n$ matrices
$T_2$: the set of singular $n \times n$ matrices
$T_3$: the set of $n \times n$ matrices in reduced row echelon form.

You should check that the closure property for addition fails in each case and that the closure property for scalar multiplication fails in $T_1$ and $T_3$. Another quick way to realize that $T_1$ is not a subspace is to notice that it does not contain $\mathbf{O}_n$ (the zero vector of $\mathcal{M}_{nn}$).

## Linear Combinations Remain in a Subspace

As in Chapter 1, we define a **linear combination** of vectors in a general vector space to be a sum of scalar multiples of the vectors. The next theorem asserts that if a finite set of vectors is in a given subspace of a vector space, then so is any linear combination of those vectors.

**Theorem 4.3** *Let $\mathcal{W}$ be a subspace of a vector space $\mathcal{V}$, and let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be vectors in $\mathcal{W}$. Then, for any scalars $a_1, a_2, \ldots, a_n$, we have $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n \in \mathcal{W}$.*

Essentially, this theorem points out that a subspace is "closed under linear combinations." In other words, any GEaS with dials corresponding to vectors in the subspace can only produce vectors within the subspace. The proof of Theorem 4.3 amounts to repeatedly applying the two closure properties as each new term is added into the sum until we have shown the entire linear combination is in $\mathcal{W}$. You are asked to provide such a proof in part (a) of Exercise 20.

**Example 10**

In Example 9, we found that the set $\mathcal{W}$ of all vectors of the form $[4a + 2b,\ a,\ b]$ is a subspace of $\mathbb{R}^3$. Now, $[6, 1, 1]$, $[2, 2, -3]$, and $[0, 3, -6]$ are all in $\mathcal{W}$. (Each vector has the correct form $[4a + 2b,\ a,\ b]$.) Then, by Theorem 4.3, any linear combination of these vectors is also in $\mathcal{W}$. For example, $3[6, 1, 1] - 5[2, 2, -3] + 4[0, 3, -6] = [8, 5, -6]$ is also in $\mathcal{W}$ (since it has the form $[4a + 2b,\ a,\ b]$). Of course, this makes sense geometrically, since we saw in Example 9 that $\mathcal{W}$ is a plane through the origin, and we would expect any linear combination of vectors in such a plane to remain in that plane.

## An Eigenspace Is a Subspace

We conclude this section by noting that any eigenspace of an $n \times n$ matrix is a subspace of $\mathbb{R}^n$. (In fact, this is why the word "space" appears in the term "eigenspace.")

**Theorem 4.4** *Let* $\mathbf{A}$ *be an* $n \times n$ *matrix, and let* $\lambda$ *be an eigenvalue for* $\mathbf{A}$*, having eigenspace* $E_\lambda$*. Then* $E_\lambda$ *is a subspace of* $\mathbb{R}^n$.

*Proof.* Let $\lambda$ be an eigenvalue for an $n \times n$ matrix $\mathbf{A}$. By definition, the eigenspace $E_\lambda$ of $\lambda$ is the set of all $n$-vectors $\mathbf{X}$ having the property that $\mathbf{AX} = \lambda\mathbf{X}$, including the zero $n$-vector. We will use Theorem 4.2 to show that $E_\lambda$ is a subspace of $\mathbb{R}^n$.

Since $\mathbf{0} \in E_\lambda$, $E_\lambda$ is a nonempty subset of $\mathbb{R}^n$. We must show that $E_\lambda$ is closed under addition and scalar multiplication.

Let $\mathbf{X}_1, \mathbf{X}_2$ be any two vectors in $E_\lambda$. To show that $\mathbf{X}_1 + \mathbf{X}_2 \in E_\lambda$, we need to verify that $\mathbf{A}(\mathbf{X}_1 + \mathbf{X}_2) = \lambda(\mathbf{X}_1 + \mathbf{X}_2)$. But, $\mathbf{A}(\mathbf{X}_1 + \mathbf{X}_2) = \mathbf{AX}_1 + \mathbf{AX}_2 = \lambda\mathbf{X}_1 + \lambda\mathbf{X}_2 = \lambda(\mathbf{X}_1 + \mathbf{X}_2)$.

Similarly, let $\mathbf{X}$ be a vector in $E_\lambda$, and let $c$ be a scalar. We must show that $c\mathbf{X} \in E_\lambda$. But, $\mathbf{A}(c\mathbf{X}) = c(\mathbf{AX}) = c(\lambda\mathbf{X}) = \lambda(c\mathbf{X})$, and so $c\mathbf{X} \in E_\lambda$. Hence, $E_\lambda$ is a subspace of $\mathbb{R}^n$. □

### Example 11

Consider

$$\mathbf{A} = \begin{bmatrix} 16 & -4 & -2 \\ 3 & 3 & -6 \\ 2 & -8 & 11 \end{bmatrix}.$$

Computing $|x\mathbf{I}_3 - \mathbf{A}|$ produces $p_{\mathbf{A}}(x) = x^3 - 30x^2 + 225x = x(x-15)^2$. Solving $(0\mathbf{I}_3 - \mathbf{A})\mathbf{X} = \mathbf{0}$ yields $E_0 = \{c[1,3,2]\,|\,c \in \mathbb{R}\}$, the subspace of $\mathbb{R}^3$ from Example 2. Similarly, solving $(15\mathbf{I}_3 - \mathbf{A}) = \mathbf{0}$ gives $E_{15} = \{a[4,1,0] + b[2,0,1]\,|\,a, b \in \mathbb{R}\}$, the same subspace of $\mathbb{R}^3$ as in Examples 9 and 10.

## New Vocabulary

linear combination (of vectors in a vector space)
proper subspace(s)
subspace
trivial subspace

## Highlights

- A subset $\mathcal{W}$ of a vector space $\mathcal{V}$ is a subspace of $\mathcal{V}$ if $\mathcal{W}$ is a vector space itself under the same operations.
- Any subspace of a vector space $\mathcal{V}$ other than $\mathcal{V}$ itself is considered a proper subspace of $\mathcal{V}$.
- The subset $\{\mathbf{0}\}$ is a trivial subspace of any vector space.
- Familiar proper nontrivial subspaces of $\mathbb{R}^3$ are: any line through the origin, any plane through the origin.
- Familiar proper subspaces of the real-valued functions on $\mathbb{R}$ are: $\mathcal{P}_n$, $\mathcal{P}$, all differentiable real-valued functions on $\mathbb{R}$, all continuous real-valued functions on $\mathbb{R}$.
- Familiar proper subspaces of $\mathcal{M}_{nn}$ $(n \geq 2)$ are: $\mathcal{U}_n$, $\mathcal{L}_n$, $\mathcal{D}_n$, the symmetric $n \times n$ matrices, the skew-symmetric $n \times n$ matrices.
- A nonempty subset $\mathcal{W}$ of a vector space $\mathcal{V}$ is a subspace of $\mathcal{V}$ if $\mathcal{W}$ is closed under addition and scalar multiplication.
- If a subset $S$ of a vector space $\mathcal{V}$ does not contain the zero vector $\mathbf{0}$, then $S$ cannot be a subspace of $\mathcal{V}$.
- If $T$ is any finite set of vectors in a subspace $\mathcal{W}$, then any linear combination of the vectors in $T$ is also in $\mathcal{W}$.
- If $\lambda$ is an eigenvalue for an $n \times n$ matrix $\mathbf{A}$, then $E_\lambda$ (the eigenspace for $\lambda$) is a subspace of $\mathbb{R}^n$.
- If $\mathcal{W}_1$ and $\mathcal{W}_2$ are subspaces of a vector space $\mathcal{V}$, then $\mathcal{W}_1 \cap \mathcal{W}_2$, the intersection of these subspaces, is also a subspace of $\mathcal{V}$.

## Exercises for Section 4.2

1. Prove or disprove that each given subset of $\mathbb{R}^2$ is a subspace of $\mathbb{R}^2$ under the usual vector operations. (In these problems, $a$ and $b$ represent arbitrary real numbers. Assume all vectors have their initial point at the origin.)
   ★ **(a)** The set of unit 2-vectors
   **(b)** The set of 2-vectors of the form $[1, a]$

★ **(c)** The set of 2-vectors of the form $[a, 2a]$
**(d)** The set of 2-vectors whose second coordinate is zero
★ **(e)** The set $\{[1, 2]\}$
**(f)** The set of 2-vectors having a zero in at least one coordinate
★ **(g)** The set of 2-vectors of the form $[a, b]$, where $|a| = |b|$
**(h)** The set of vectors in the plane whose terminal point lies on the line $y = -6x$
**(i)** The set of vectors in the plane whose terminal point lies above the line $y = 2x$
★ **(j)** The set of vectors in the plane whose terminal point lies on the parabola $y = x^2$
**(k)** The set of vectors in the plane whose terminal point lies on the line $y = 4x + 7$
★ **(l)** The set of vectors in the plane whose terminal point lies inside the circle of radius 1 centered at the origin

**2.** Prove or disprove that each given subset of $\mathcal{M}_{22}$ is a subspace of $\mathcal{M}_{22}$ under the usual matrix operations. (In these problems, $a$ and $b$ represent arbitrary real numbers.)
★ **(a)** The set of matrices of the form $\begin{bmatrix} a & -a \\ b & 0 \end{bmatrix}$
**(b)** The set of $2 \times 2$ matrices that have at least one row of zeroes
★ **(c)** The set of symmetric $2 \times 2$ matrices
**(d)** The set of nonsingular $2 \times 2$ matrices
★ **(e)** The set of $2 \times 2$ matrices having the sum of all entries zero
**(f)** The set of $2 \times 2$ matrices having trace zero (Recall that the *trace* of a square matrix is the sum of the main diagonal entries.)
★ **(g)** The set of $2 \times 2$ matrices $\mathbf{A}$ such that $\mathbf{A}\begin{bmatrix} 1 & 3 \\ -2 & -6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
★ **(h)** The set of $2 \times 2$ matrices having the product of all entries zero
**(i)** The set of $2 \times 2$ matrices $\mathbf{A}$ such that $\mathbf{A}^2 = \mathbf{O}_2$

**3.** Prove or disprove that each given subset of $\mathcal{P}_5$ is a subspace of $\mathcal{P}_5$ under the usual operations.
★ **(a)** $\{\mathbf{p} \in \mathcal{P}_5 \mid$ the coefficient of the first-degree term of $\mathbf{p}$ equals the coefficient of the fifth-degree term of $\mathbf{p}\}$
★ **(b)** $\{\mathbf{p} \in \mathcal{P}_5 \mid \mathbf{p}(3) = 0\}$
**(c)** $\{\mathbf{p} \in \mathcal{P}_5 \mid$ the sum of the coefficients of $\mathbf{p}$ is zero$\}$
**(d)** $\{\mathbf{p} \in \mathcal{P}_5 \mid \mathbf{p}(3) = \mathbf{p}(5)\}$
★ **(e)** $\{\mathbf{p} \in \mathcal{P}_5 \mid \mathbf{p}$ is an odd-degree polynomial (highest-order nonzero term has odd degree)$\}$
**(f)** $\{\mathbf{p} \in \mathcal{P}_5 \mid \mathbf{p}$ has a relative maximum at $x = 0\}$
★ **(g)** $\{\mathbf{p} \in \mathcal{P}_5 \mid \mathbf{p}'(4) = 0$, where $\mathbf{p}'$ is the derivative of $\mathbf{p}\}$
**(h)** $\{\mathbf{p} \in \mathcal{P}_5 \mid \mathbf{p}'(4) = 1$, where $\mathbf{p}'$ is the derivative of $\mathbf{p}\}$
**(i)** $\{\mathbf{p} \in \mathcal{P}_5 \mid \mathbf{p}$ has at least 6 distinct roots$\}$
**(j)** $\{\mathbf{p} \in \mathcal{P}_5 \mid \mathbf{p}$ has at least 1 root$\}$

**4.** Show that the set of vectors of the form $[a, b, a + b, c, a - 2b + c]$ in $\mathbb{R}^5$ forms a subspace of $\mathbb{R}^5$ under the usual operations.

**5.** Show that the set of vectors of the form $[2a - 5c, b + 4c, a - b, 7c - 2b, c + 6a]$ in $\mathbb{R}^5$ forms a subspace of $\mathbb{R}^5$ under the usual operations.

**6.** This exercise explores a particular type of subspace of $\mathbb{R}^3$.
**(a)** Prove that the set of all 3-vectors orthogonal to $[2, -3, 7]$ forms a subspace of $\mathbb{R}^3$.
**(b)** Is the subspace from part (a) all of $\mathbb{R}^3$, a plane passing through the origin in $\mathbb{R}^3$, or a line passing through the origin in $\mathbb{R}^3$?

**7.** Show that each of the following sets is a subspace of the vector space of all real-valued functions on the given domain, under the usual operations of function addition and scalar multiplication:
**(a)** The set of continuous real-valued functions with domain $\mathbb{R}$
**(b)** The set of differentiable real-valued functions with domain $\mathbb{R}$
**(c)** The set of all real-valued functions $\mathbf{f}$ defined on the interval $[0, 1]$ such that $\mathbf{f}\left(\frac{1}{2}\right) = 0$ (Compare this vector space with the set in Example 9 of Section 4.1.)
**(d)** The set of all continuous real-valued functions $\mathbf{f}$ defined on the interval $[0, 1]$ such that $\int_0^1 \mathbf{f}(x)\, dx = 0$

**8.** Let $\mathcal{W}$ be the set of differentiable real-valued functions $y = \mathbf{f}(x)$ defined on $\mathbb{R}$ that satisfy the differential equation $3(dy/dx) - 2y = 0$. Show that, under the usual function operations, $\mathcal{W}$ is a subspace of the vector space of all differentiable real-valued functions. (Do not forget to show $\mathcal{W}$ is nonempty.)

**9.** Show that the set $\mathcal{W}$ of solutions to the differential equation $y'' + 2y' - 9y = 0$ is a subspace of the vector space of all twice-differentiable real-valued functions defined on $\mathbb{R}$. (Do not forget to show that $\mathcal{W}$ is nonempty.)

**10.** Prove that the set of discontinuous real-valued functions defined on $\mathbb{R}$ (for example, $\mathbf{f}(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 & \text{if } x > 0 \end{cases}$) with the usual function operations is not a subspace of the vector space of all real-valued functions with domain $\mathbb{R}$.

**11.** Let $\mathbf{A}$ be a fixed $n \times n$ matrix, and let $\mathcal{W}$ be the subset of $\mathcal{M}_{nn}$ of all $n \times n$ matrices that commute with $\mathbf{A}$ under multiplication (that is, $\mathbf{B} \in \mathcal{W}$ if and only if $\mathbf{AB} = \mathbf{BA}$). Show that $\mathcal{W}$ is a subspace of $\mathcal{M}_{nn}$ under the usual vector space operations. (Do not forget to show that $\mathcal{W}$ is nonempty.)

**12.** This exercise concerns closure with singular and nonsingular matrices in $\mathcal{M}_{nn}$.

- **(a)** A careful reading of the proof of Theorem 4.2 reveals that only closure under scalar multiplication (not closure under addition) is sufficient to prove the remaining 8 vector space properties for $\mathcal{W}$. Explain, nevertheless, why closure under addition is a necessary condition for $\mathcal{W}$ to be a subspace of $\mathcal{V}$.
- **(b)** Show that the set of singular $n \times n$ matrices is closed under scalar multiplication in $\mathcal{M}_{nn}$.
- **(c)** Use parts (a) and (b) to determine which of the 8 vector space properties are true for the set of singular $n \times n$ matrices.
- **(d)** Show that the set of singular $n \times n$ matrices is not closed under vector addition and hence is not a subspace of $\mathcal{M}_{nn}$ $(n \ge 2)$.
- ★ **(e)** Is the set of nonsingular $n \times n$ matrices closed under scalar multiplication? Why or why not?

**13.** Assume in this problem that all vectors have their initial point at the origin.

- **(a)** Prove that the set of all vectors in $\mathbb{R}^2$ whose terminal point lies on a common line passing through the origin is a subspace of $\mathbb{R}^2$ (under the usual operations).
- **(b)** Prove that the set of all vectors in $\mathbb{R}^2$ whose terminal point lies on a common line not passing through the origin does not form a subspace of $\mathbb{R}^2$ (under the usual operations).

**14.** Let $\mathbf{A}$ be a fixed $m \times n$ matrix. Let $\mathcal{V}$ be the set of solutions $\mathbf{X}$ (in $\mathbb{R}^n$) of the homogeneous system $\mathbf{AX} = \mathbf{0}$. Show that $\mathcal{V}$ is a subspace of $\mathbb{R}^n$ (under the usual $n$-vector operations).

★ **15.** Suppose $\mathbf{A}$ is an $n \times n$ matrix and $\lambda \in \mathbb{R}$ is *not* an eigenvalue for $\mathbf{A}$. Determine exactly which vectors are in $S = \{\mathbf{X} \in \mathbb{R}^n \mid \mathbf{AX} = \lambda\mathbf{X}\}$. Is this set a subspace of $\mathbb{R}^n$? Why or why not?

**16.** Prove that $\mathbb{R}$ (under the usual operations) has no subspaces except $\mathbb{R}$ and $\{\mathbf{0}\}$. (Hint: Let $\mathcal{V}$ be a nontrivial subspace of $\mathbb{R}$, and show that $\mathcal{V} = \mathbb{R}$.)

**17.** Let $\mathcal{W}$ be a subspace of a vector space $\mathcal{V}$. Consider $\mathcal{W}' = \{\mathbf{v} \in \mathcal{V} \mid \mathbf{v} \notin \mathcal{W}\}$. Show that $\mathcal{W}'$ is not a subspace of $\mathcal{V}$.

**18.** Let $\mathcal{V}$ be a vector space, and let $\mathcal{W}_1$ and $\mathcal{W}_2$ be subspaces of $\mathcal{V}$. Prove that $\mathcal{W}_1 \cap \mathcal{W}_2$ is a subspace of $\mathcal{V}$. (Do not forget to show $\mathcal{W}_1 \cap \mathcal{W}_2$ is nonempty.)

**19.** Let $\mathcal{V}$ be any vector space, and let $\mathcal{W}$ be a nonempty subset of $\mathcal{V}$. Prove that $\mathcal{W}$ is a subspace of $\mathcal{V}$ if and only if $a\mathbf{w}_1 + b\mathbf{w}_2$ is an element of $\mathcal{W}$ for every $a, b \in \mathbb{R}$ and every $\mathbf{w}_1, \mathbf{w}_2 \in \mathcal{W}$. (Hint: For one half of the proof, first consider the case where $a = b = 1$ and then the case where $b = 0$ and $a$ is arbitrary.)

**20.** This exercise asks for proofs for Theorem 4.3 and its converse.

- ► **(a)** Prove Theorem 4.3. (Hint: Use induction.)
- **(b)** Let $\mathcal{W}$ be a nonempty subset of a vector space $\mathcal{V}$, and suppose every linear combination of vectors in $\mathcal{W}$ is also in $\mathcal{W}$. Prove that $\mathcal{W}$ is a subspace of $\mathcal{V}$. (This is the converse of Theorem 4.3.)

**21.** Let $\lambda$ be an eigenvalue for an $n \times n$ matrix $\mathbf{A}$. Show that if $\mathbf{X}_1, \ldots, \mathbf{X}_k$ are eigenvectors for $\mathbf{A}$ corresponding to $\lambda$, then any linear combination of $\mathbf{X}_1, \ldots, \mathbf{X}_k$ is in $E_\lambda$.

★ **22.** True or False:

- **(a)** A nonempty subset $\mathcal{W}$ of a vector space $\mathcal{V}$ is always a subspace of $\mathcal{V}$ under the same operations as those in $\mathcal{V}$.
- **(b)** Every vector space has at least one subspace.
- **(c)** Any plane $\mathcal{W}$ in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$ (under the usual operations).
- **(d)** The set of all lower triangular $5 \times 5$ matrices is a subspace of $\mathcal{M}_{55}$ (under the usual operations).
- **(e)** The set of all vectors of the form $[0, a, b, 0]$ is a subspace of $\mathbb{R}^4$ (under the usual operations).
- **(f)** If a subset $\mathcal{W}$ of a vector space $\mathcal{V}$ contains the zero vector $\mathbf{0}$ of $\mathcal{V}$, then $\mathcal{W}$ must be a subspace of $\mathcal{V}$ (under the same operations).
- **(g)** Any linear combination of vectors from a subspace $\mathcal{W}$ of a vector space $\mathcal{V}$ must also be in $\mathcal{W}$.
- **(h)** If $\lambda$ is an eigenvalue for a $4 \times 4$ matrix $\mathbf{A}$, then $E_\lambda$ is a subspace of $\mathbb{R}^4$.

## 4.3 Span

In this section, we study the concept of linear combinations in more depth. We show that the set of all linear combinations of the vectors in a subset $S$ of $\mathcal{V}$ forms an important subspace of $\mathcal{V}$, called the span of $S$ in $\mathcal{V}$.

### Finite Linear Combinations

In Section 4.2, we introduced linear combinations of vectors in a general vector space. We now extend the concept of linear combination to allow a finite sum of scalar multiples from infinite, as well as finite, sets.

**Definition** Let $S$ be a nonempty (possibly infinite) subset of a vector space $\mathcal{V}$. Then a vector $\mathbf{v}$ in $\mathcal{V}$ is a **(finite) linear combination of the vectors in** $S$ if and only if there exists some *finite* subset $S' = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of $S$ such that $\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$ for some real numbers $a_1, \ldots, a_n$.

Examples 1 and 2 below involve a finite set $S$, while Example 3 uses an infinite set $S$. In all these examples, however, only a *finite* number of vectors from $S$ are used at any given time to form linear combinations.

**Example 1**

Consider the subset $S = \{[1,-1,0],[1,0,2],[0,-2,5]\}$ of $\mathbb{R}^3$. The vector $[1,-2,-2]$ is a linear combination of the vectors in $S$ according to the definition, because $[1,-2,-2] = 2[1,-1,0] + (-1)[1,0,2]$. Here we are using $S' = \{[1,-1,0],[1,0,2]\}$ as the (finite) subset of $S$ to form the linear combination. However, we could have also chosen $S'$ to be $S$ itself, by placing a zero coefficient in front of the remaining vector $[0,-2,5]$. That is, $[1,-2,-2] = 2[1,-1,0] + (-1)[1,0,2] + 0[0,-2,5]$.

We see from Example 1 that if $S$ is a *finite* subset of a vector space $\mathcal{V}$, any linear combination $\mathbf{v}$ formed using *some* of the vectors in $S$ can always be formed using *all* the vectors in $S$ by placing zero coefficients on the remaining vectors.

A linear combination formed from a set $\{\mathbf{v}\}$ containing a single vector is just a scalar multiple $a\mathbf{v}$ of $\mathbf{v}$, as we see in the next example.

**Example 2**

Let $S = \{[1,-2,7]\}$, a subset of $\mathbb{R}^3$ containing a single element. Then the only linear combinations that can be formed from $S$ are scalar multiples of $[1,-2,7]$, such as $[3,-6,21]$ and $[-4,8,-28]$.

**Example 3**

Consider $\mathcal{P}$, the vector space of polynomials with real coefficients, and let $S = \{1, x^2, x^4, \ldots\}$, the infinite subset of $\mathcal{P}$ consisting of all nonnegative even powers of $x$ (since $x^0 = 1$). We can form linear combinations of vectors in $S$ using any finite subset $S'$ of $S$. For example, $\mathbf{p}(x) = 7x^8 - (1/4)x^4 + 10$ is a linear combination formed from $S$ because it is a sum of scalar multiples of elements of the finite subset $S' = \{x^8, x^4, 1\}$ of $S$. Similarly, $\mathbf{q}(x) = 8x^{10} + 3x^6$ is a linear combination of the vectors in $S$ using the finite subset $S'' = \{x^{10}, x^6\}$. In fact, the possible linear combinations of vectors in $S$ are precisely the polynomials involving only even powers of $x$.

Notice that we cannot use all of the elements in an infinite set $S$ when forming a linear combination because an "infinite" sum would result. This is why a linear combination is frequently called a *finite* linear combination in order to stress that only a finite number of vectors are combined at any time.

### Definition of the Span of a Set

**Definition** Let $S$ be a nonempty subset of a vector space $\mathcal{V}$. Then, span($S$), the **span** of $S$ in $\mathcal{V}$, is the set of all possible (finite) linear combinations of the vectors in $S$.

We now consider some examples of the span of a subset.

**Example 4**

Consider the subset $S = \{\mathbf{j}, \mathbf{k}\}$ of $\mathbb{R}^3$. Then, span$(S)$ is the set of all finite linear combinations of the vectors $\mathbf{j} = [0, 1, 0]$ and $\mathbf{k} = [0, 0, 1]$ in $S$. That is,

$$\text{span}(S) = \{a[0, 1, 0] + b[0, 0, 1] \mid a, b \in \mathbb{R}\} = \{[0, a, b] \mid a, b \in \mathbb{R}\}.$$

**Example 5**

Consider the subset $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ of $\mathcal{M}_{22}$. Then, span$(S)$ is the set of all finite linear combinations of the matrices in $S$. That is,

$$\begin{aligned} \text{span}(S) &= \left\{ a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\} \\ &= \left\{ \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\}. \end{aligned}$$

**Example 6**

From our work in Example 3, we see that for $S = \{1, x^2, x^4, \ldots\}$ in $\mathcal{P}$, span$(S)$ is the set of all polynomials containing only even-degree terms. Imagine a GEaS for polynomials, analogous to a GEaS for vectors in $\mathbb{R}^n$. However, this polynomial GEaS has an infinite number of dials—one for each polynomial in $S$. Then, in this case, span$(S)$ is the set of all of the polynomials that can be obtained using this GEaS. However, even though the number of dials is infinite, only finitely many of them can be used to create a specific polynomial. A different set of dials could be used for different polynomials in span$(S)$. Thus, span$(S)$ only contains polynomials, not infinite series.

Occasionally, for a given subset $S$ of a vector space $\mathcal{V}$, every vector in $\mathcal{V}$ is a finite linear combination of the vectors in $S$. That is, span$(S) = \mathcal{V}$ itself. When this happens, we say that $\mathcal{V}$ is **spanned by** $S$ or that $S$ **spans** $\mathcal{V}$. Here, we are using span as a *verb* to indicate that the span (*noun*) of a set $S$ equals $\mathcal{V}$.

**Example 7**

Note that $\mathbb{R}^3$ is spanned (*verb*) by $S_1 = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, since the span (*noun*) of $S_1$ is $\mathbb{R}^3$. That is, every 3-vector can be expressed as a linear combination of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ (why?). However, as we saw in Example 4, $\mathbb{R}^3$ is not spanned by the smaller set $S_2 = \{\mathbf{j}, \mathbf{k}\}$, since span$(S_2)$ is the $yz$-plane in $\mathbb{R}^3$ (why?). More generally, $\mathbb{R}^n$ is spanned by the set of standard unit vectors $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$. Note that no proper subset of $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ will span $\mathbb{R}^n$.

Similarly, $\mathcal{P}_n$ is spanned by $\{1, x, x^2, \ldots, x^n\}$ because every polynomial $a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0(1)$ in $\mathcal{P}_n$ is clearly a linear combination of $1, x, x^2, \ldots, x^n$.

Finally, consider the set $\{\Psi_{ij}\}$ of $m \times n$ matrices, for $1 \le i \le m$, and $1 \le j \le n$, where each $\Psi_{ij}$ has 1 for its $(i, j)$ entry, and zeroes elsewhere. Notice that $\mathcal{M}_{mn}$ is spanned by the set $\{\Psi_{ij}\}$ because any matrix $\mathbf{A}$ in $\mathcal{M}_{mn}$ can be expressed as a linear combination of these $\Psi_{ij}$ matrices. For example, in $\mathcal{M}_{32}$, the matrix $\begin{bmatrix} 8 & -2 \\ -1 & 5 \\ 6 & -3 \end{bmatrix}$ can be expressed as a linear combination of such matrices as follows:

$$8\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + (-2)\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + (-1)\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} + 5\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} + 6\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} + (-3)\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

## Span($S$) Is the Minimal Subspace Containing $S$

The next theorem completely characterizes the span.

**Theorem 4.5** *Let $S$ be a nonempty subset of a vector space $\mathcal{V}$. Then:*

**(1)** $S \subseteq \text{span}(S)$.

**(2)** Span$(S)$ *is a subspace of $\mathcal{V}$ (under the same operations as $\mathcal{V}$).*

**(3)** *If $\mathcal{W}$ is a subspace of $\mathcal{V}$ with $S \subseteq \mathcal{W}$, then* span$(S) \subseteq \mathcal{W}$.

**(4)** Span$(S)$ *is the smallest subspace of $\mathcal{V}$ containing $S$.*

*Proof.*
**Part (1)**: We must show that each vector $\mathbf{w} \in S$ is also in span($S$). But if $\mathbf{w} \in S$, then $\mathbf{w} = 1\mathbf{w}$ is a linear combination from the subset $\{\mathbf{w}\}$ of $S$. Hence, $\mathbf{w} \in \text{span}(S)$.

**Part (2)**: Since $S$ is nonempty, part (1) shows that span($S$) is nonempty. Therefore, by Theorem 4.2, span($S$) is a subspace of $\mathcal{V}$ if we can prove the closure properties hold for span($S$).

First, let us verify closure under scalar multiplication. Let $\mathbf{v}$ be in span($S$), and let $c$ be a scalar. We must show that $c\mathbf{v} \in \text{span}(S)$. Now, since $\mathbf{v} \in \text{span}(S)$, there is a finite subset $S' = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of $S$ and real numbers $a_1, \ldots, a_n$ such that $\mathbf{v} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$. Then,

$$c\mathbf{v} = c(a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n) = (ca_1)\mathbf{v}_1 + \cdots + (ca_n)\mathbf{v}_n.$$

Hence, $c\mathbf{v}$ is a linear combination of the finite subset $S'$ of $S$, and so $c\mathbf{v} \in \text{span}(S)$.

Finally, we need to prove that span($S$) is closed under vector addition. We must show that adding two linear combinations of vectors in $S$ yields a linear combination of vectors in $S$. This is proved in a manner analogous to the proof of Lemma 2.8, in which we merely combine the like terms in the linear combinations involved to create a single linear combination. The possibility that $S$ could have an infinite number of elements makes the notation in this proof a bit cumbersome, but basically we are simply combining like terms. You are asked to complete the details for this part of the proof in Exercise 28.

**Part (3)**: This part asserts that if $S$ is a subset of a subspace $\mathcal{W}$, then any (finite) linear combination from $S$ is also in $\mathcal{W}$. This is just a rewording of Theorem 4.3 using the "span" concept. The fact that span($S$) cannot contain vectors outside of $\mathcal{W}$ is illustrated in Fig. 4.4.

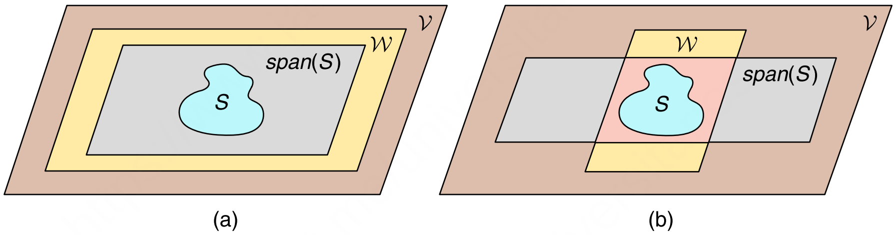


**FIGURE 4.4** (a) Situation that *must* occur if $\mathcal{W}$ is a subspace containing $S$; (b) situation that *cannot* occur if $\mathcal{W}$ is a subspace containing $S$

**Part (4)**: This is merely a summary of the other three parts. Parts (1) and (2) assert that span($S$) is a subspace of $\mathcal{V}$ containing $S$. But part (3) shows that span($S$) is the smallest such subspace because span($S$) must be a subset of, and hence smaller than, any other subspace of $\mathcal{V}$ that contains $S$. □

Theorem 4.5 implies that span($S$) is created by appending to $S$ precisely those vectors needed to make the closure properties hold. In fact, the whole idea behind span is to "close up" a subset of a vector space to create a subspace.

**Example 8**

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be any two vectors in $\mathbb{R}^4$. Then, by Theorem 4.5, span($\{\mathbf{v}_1, \mathbf{v}_2\}$) is the smallest subspace of $\mathbb{R}^4$ containing $\mathbf{v}_1$ and $\mathbf{v}_2$.
In particular, if $\mathbf{v}_1 = [1, 3, -2, 5]$ and $\mathbf{v}_2 = [0, -4, 3, -1]$, then span($\{\mathbf{v}_1, \mathbf{v}_2\}$) is the subspace of $\mathbb{R}^4$ consisting of all vectors of the form

$$a[1, 3, -2, 5] + b[0, -4, 3, -1] = [a,\ 3a - 4b,\ -2a + 3b,\ 5a - b].$$

Theorem 4.5 assures us that no smaller subspace of $\mathbb{R}^4$ contains $[1, 3, -2, 5]$ and $[0, -4, 3, -1]$.

The following useful result is left for you to prove in Exercise 22.

**Corollary 4.6** *Let $\mathcal{V}$ be a vector space, and let $S_1$ and $S_2$ be subsets of $\mathcal{V}$ with $S_1 \subseteq S_2$. Then* $\text{span}(S_1) \subseteq \text{span}(S_2)$.

## Simplifying Span($S$) Using Row Reduction

Recall that the row space of a matrix is the set of all linear combinations of the vectors forming the rows of the matrix. The span of a set $S$ is a generalization of this concept, since it is the set of all (finite) linear combinations of the vectors in $S$. That is,

> The span of the set of the rows of a matrix is precisely the row space of the matrix.

Notice that if we form the matrix $\mathbf{A}$ whose rows are the vectors in $S$, the reduced row echelon form of $\mathbf{A}$ has the same row space as $\mathbf{A}$ by Theorem 2.9. This suggests that we can often get a simpler expression for span($S$) by using the rows of the reduced row echelon form of $\mathbf{A}$. Hence, we have the following:

> **Method for Simplifying Span($S$) Using Row Reduction (Simplified Span Method)**
> Suppose that $S$ is a finite subset of $\mathbb{R}^n$ containing $k$ vectors, with $k \geq 2$.
>
> **Step 1**: Form a $k \times n$ matrix $\mathbf{A}$ by using the vectors in $S$ as the rows of $\mathbf{A}$. (Thus, span($S$) is the row space of $\mathbf{A}$.)
>
> **Step 2**: Let $\mathbf{C}$ be the reduced row echelon form matrix for $\mathbf{A}$.
>
> **Step 3**: Then, a simplified form for span($S$) is given by the set of all linear combinations of the *nonzero* rows of $\mathbf{C}$.

---

### Example 9

Let $S$ be the subset $\{[1,4,-1,-5],[2,8,5,4],[-1,-4,2,7],[6,24,-1,-20]\}$ of $\mathbb{R}^4$. By definition, span($S$) is the set of all vectors of the form

$$a[1,4,-1,-5]+b[2,8,5,4]+c[-1,-4,2,7]+d[6,24,-1,-20]$$

for $a,b,c,d \in \mathbb{R}$. We want to use the Simplified Span Method to find a simplified form for the vectors in span($S$).

**Step 1**: We create

$$\mathbf{A}=\begin{bmatrix} 1 & 4 & -1 & -5 \\ 2 & 8 & 5 & 4 \\ -1 & -4 & 2 & 7 \\ 6 & 24 & -1 & -20 \end{bmatrix},$$

whose rows are the vectors in $S$. Then, span($S$) is the row space of $\mathbf{A}$; that is, the set of all linear combinations of the rows of $\mathbf{A}$.

**Step 2**: We simplify the form of the row space of $\mathbf{A}$ by obtaining its reduced row echelon form matrix

$$\mathbf{C}=\begin{bmatrix} 1 & 4 & 0 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

**Step 3**: By Theorem 2.9, the row space of $\mathbf{A}$ is the same as the row space of $\mathbf{C}$, which is the set of all 4-vectors of the form

$$a[1,4,0,-3]+b[0,0,1,2]=[a,\ 4a,\ b,\ -3a+2b].$$

Therefore,

$$\text{span}(S)=\{[a,4a,b,-3a+2b] \mid a,b \in \mathbb{R}\},$$

a subspace of $\mathbb{R}^4$.

---

Notice in Example 9 that the vector $[3,12,-2,-13]$ is in span($S$) ($a=3, b=-2$). However, the vector $[-2,-8,4,6]$ is not in span($S$) because the following system has no solutions:

$$\begin{cases} a = -2 \\ 4a = -8 \\ b = 4 \\ -3a+2b = 6 \end{cases}.$$

The method used in Example 9 works in vector spaces other than $\mathbb{R}^n$, as we see in the next example. This fact will follow from the discussion of isomorphism in Section 5.5. (However, we will not use this fact in proofs of theorems until after Section 5.5.)

**Example 10**

Let $S$ be the subset $\{5x^3 + 2x^2 + 4x - 3, -x^2 + 3x - 7, 2x^3 + 4x^2 - 8x + 5, x^3 + 2x + 5\}$ of $\mathcal{P}_3$. We use the Simplified Span Method to find a simplified form for the vectors in span($S$).

Consider the coefficients of each polynomial as the coordinates of a vector in $\mathbb{R}^4$, yielding the corresponding set of vectors $T = \{[5, 2, 4, -3], [0, -1, 3, -7], [2, 4, -8, 5], [1, 0, 2, 5]\}$. We now use the Simplified Span Method on $T$.

**Step 1**: We create the matrix whose rows are the vectors in $T$:

$$\mathbf{A} = \begin{bmatrix} 5 & 2 & 4 & -3 \\ 0 & -1 & 3 & -7 \\ 2 & 4 & -8 & 5 \\ 1 & 0 & 2 & 5 \end{bmatrix}.$$

**Step 2**: We compute $\mathbf{C}$, the reduced row echelon form of $\mathbf{A}$, which is

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

**Step 3**: The row space of $\mathbf{A}$ is the same as the row space of $\mathbf{C}$, which is the set of all 4-vectors of the form

$$a\,[1, 0, 2, 0] + b\,[0, 1, -3, 0] + c\,[0, 0, 0, 1].$$

Finally, converting the nonzero rows of $\mathbf{C}$ to polynomials in $\mathcal{P}_3$ by using the coordinates as coefficients, we see that

$$\begin{aligned} \text{span}(S) &= \{a(x^3 + 2x) + b(x^2 - 3x) + c(1) \mid a, b, c \in \mathbb{R}\} \\ &= \{ax^3 + bx^2 + (2a - 3b)x + c \mid a, b, c \in \mathbb{R}\}. \end{aligned}$$

## Special Case: The Span of the Empty Set

Until now, our results involving span have specified that the subset $S$ of the vector space $\mathcal{V}$ be nonempty. However, our understanding of span($S$) as the smallest subspace of $\mathcal{V}$ containing $S$ allows us to give a meaningful definition for the span of the empty set.

**Definition** $\text{Span}(\{\}) = \{\mathbf{0}\}$.

This definition makes sense because the trivial subspace is the smallest subspace of $\mathcal{V}$, hence the smallest one containing the empty set. Thus, Theorem 4.5 is also true when the set $S$ is empty. Similarly, to maintain consistency, we *define* any linear combination of the empty set of vectors to be $\mathbf{0}$. This ensures that the span of the empty set is the set of all linear combinations of vectors taken from this set.

## New Vocabulary

finite linear combination (of vectors in a vector space)
Simplified Span Method
span (of a set of vectors)
spanned by (as in "$\mathcal{V}$ is spanned by $S$")
span of the empty set

## Highlights

- The span of a set $S$ is the collection of all finite linear combinations of vectors from the set $S$.
- A set $S$ spans a vector space $\mathcal{V}$ (i.e., $\mathcal{V}$ is spanned by $S$) if every vector in $\mathcal{V}$ is a (finite) linear combination of vectors in $S$.

- The span of the rows of a matrix $\mathbf{A}$ is the row space of $\mathbf{A}$.
- $\mathbb{R}^3$ is spanned by $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$; $\mathbb{R}^n$ is spanned by $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$; $\mathcal{P}_n$ is spanned by $\{1, x, x^2, \ldots, x^n\}$; $\mathcal{M}_{mn}$ is spanned by $\{\Psi_{ij}\}$, where each $\Psi_{ij}$ has 1 for its $(i, j)$ entry, and zeroes elsewhere.
- If $S$ is a set of vectors in a vector space $\mathcal{V}$, then span$(S)$ is a subspace of $\mathcal{V}$, and is, in fact, the smallest subspace of $\mathcal{V}$ that contains $S$.
- If $S_1 \subseteq S_2$, then span$(S_1) \subseteq$ span$(S_2)$.
- The Simplified Span Method generally produces a more simplified form of the span of a set $S$ of vectors by calculating the reduced row echelon form of the matrix whose *rows* are the vectors in $S$.
- The span of the empty set is $\{\mathbf{0}\}$.

## Exercises for Section 4.3

1. In each of the following cases, use the Simplified Span Method to find a simplified general form for all the vectors in span$(S)$, where $S$ is the given subset of $\mathbb{R}^n$:
   ★ (a) $S = \{[1, 1, 0], [2, -3, -5]\}$
   (b) $S = \{[2, -2, -2, 1], [3, 1, -1, 0], [0, -3, -1, 1], [-3, 0, 1, 0]\}$
   ★ (c) $S = \{[1, -1, 1], [2, -3, 3], [0, 1, -1]\}$
   (d) $S = \{[1, -4, -5], [-2, 5, 4], [1, -6, -9], [3, 5, 19]\}$
   ★ (e) $S = \{[1, 3, 0, 1], [0, 0, 1, 1], [0, 1, 0, 1], [1, 5, 1, 4]\}$
   (f) $S = \{[21, -14, 35], [6, -4, 10], [-9, 6, -15], [15, -10, 25]\}$
2. In each case, use the Simplified Span Method to find a simplified general form for all the vectors in span$(S)$, where $S$ is the given subset of $\mathcal{P}_3$:
   ★ (a) $S = \{x^3 - 1, x^2 - x, x - 1\}$
   (b) $S = \{x^3 + 2x^2 - 8x + 15, 2x^3 + 3x^2 - 11x + 26, x^3 - x^2 + 7x, -2x^2 + 10x - 10\}$
   ★ (c) $S = \{x^3 - x + 5, 3x^3 - 3x + 10, 5x^3 - 5x - 6, 6x - 6x^3 - 13\}$
   (d) $S = \left\{x^3 - 2x^2 - 2x - 6,\ x^3 - 7x^2 - 2x - 21,\ -x^3 + 4x^2 + 2x + 12,\ 2x^2 + 6\right\}$
3. In each case, use the Simplified Span Method to find a simplified general form for all the vectors in span$(S)$, where $S$ is the given subset of $\mathcal{M}_{22}$. (Hint: Rewrite each matrix as a 4-vector.)
   ★ (a) $S = \left\{\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$
   (b) $S = \left\{\begin{bmatrix} 6 & 3 \\ 0 & 15 \end{bmatrix}, \begin{bmatrix} -3 & 4 \\ 11 & 9 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 2 & 8 \end{bmatrix}\right\}$
   ★ (c) $S = \left\{\begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ 8 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 4 \\ 4 & -1 \end{bmatrix}, \begin{bmatrix} 3 & -4 \\ 5 & 6 \end{bmatrix}\right\}$
   (d) $S = \left\{\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} -6 & 1 \\ 2 & 16 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} -2 & 0 \\ 1 & 5 \end{bmatrix}\right\}$

★ 4. This exercise involves simplifying the form of a particular subspace of $\mathbb{R}^4$.
   (a) Express the subspace $\mathcal{W}$ of $\mathbb{R}^4$ of all 4-vectors of the form $[a + b, a + c, b + c, c]$ as the row space of a matrix $\mathbf{A}$.
   (b) Find the reduced row echelon form matrix $\mathbf{B}$ for $\mathbf{A}$.
   (c) Use the matrix $\mathbf{B}$ from part (b) to find a simplified form for the vectors in $\mathcal{W}$.
5. This exercise involves simplifying the form of a particular subspace of $\mathbb{R}^5$.
   (a) Express the subspace $\mathcal{W}$ of $\mathbb{R}^5$ of all 5-vectors of the form $[2a + 3c, a + 2b + c, 5a + 2b + 7c, a + b + c, 11b - 4c]$ as the row space of a matrix $\mathbf{A}$.
   (b) Find the reduced row echelon form matrix $\mathbf{B}$ for $\mathbf{A}$.
   (c) Use the matrix $\mathbf{B}$ from part (b) to find a simplified form for the vectors in $\mathcal{W}$.
6. Prove that the set $S = \{[2, 2, 1], [4, 3, 1], [5, 1, -2]\}$ spans $\mathbb{R}^3$.
7. Prove that the set $S = \{[1, 1, 8], [-3, 2, -9], [2, 0, 10], [2, -1, 7]\}$ does not span $\mathbb{R}^3$.
8. Show that the set $\{x^2 - x + 1, x - 1, 1\}$ spans $\mathcal{P}_2$.
9. Prove that the set $\{2x^2 - 5x + 47, x^2 - 2x + 20, x^2 - 3x + 27\}$ does not span $\mathcal{P}_2$.

**10.** This exercise concerns the span of a particular set of vectors in $\mathbb{R}^3$.

**(a)** Let $S = \{[-4, 18, -6], [1, -8, 6], [-1, -13, 21]\}$. Show that $[-8, 15, 15] \in \text{span}(S)$ by expressing it as a linear combination of the vectors in $S$.

**(b)** Prove that the set $S$ in part (a) does not span $\mathbb{R}^3$.

★ **11.** Consider the subset $S = \{x^3 - 2x^2 + x - 3, 2x^3 - 3x^2 + 2x + 5, 4x^2 + x - 3, 4x^3 - 7x^2 + 4x - 1\}$ of $\mathcal{P}$. Show that $3x^3 - 8x^2 + 2x + 16$ is in span$(S)$ by expressing it as a linear combination of the elements of $S$.

**12.** Prove that the set $S$ of all vectors in $\mathbb{R}^4$ that have zeroes in exactly two coordinates spans $\mathbb{R}^4$. (Hint: Find a subset of $S$ that spans $\mathbb{R}^4$.)

**13.** Let $\mathbf{a}$ be any nonzero element of $\mathbb{R}$. Prove that $\text{span}(\{\mathbf{a}\}) = \mathbb{R}$.

**14.** Consider the subset $S = \{1+x^2, x+x^3, 3-2x+3x^2-12x^3\}$ of $\mathcal{P}$, and let $\mathcal{W} = \left\{ax^3 + bx^2 + cx + b \,\middle|\, a, b, c \in \mathbb{R}\right\}$. Show that $\mathcal{W} = \text{span}(S)$.

**15.** Let $\mathbf{A} = \begin{bmatrix} -9 & -15 & 8 \\ -10 & -14 & 8 \\ -30 & -45 & 25 \end{bmatrix}$.

★ **(a)** Find a set $S$ of two fundamental eigenvectors for $\mathbf{A}$ corresponding to the eigenvalue $\lambda = 1$. Multiply by a scalar to eliminate any fractions in your answers.

**(b)** Verify that the set $S$ from part (a) spans $E_1$.

**16.** Let $S_1 = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a nonempty subset of a vector space $\mathcal{V}$. Let $S_2 = \{-\mathbf{v}_1, -\mathbf{v}_2, \ldots, -\mathbf{v}_n\}$. Show that $\text{span}(S_1) = \text{span}(S_2)$.

**17.** Let $\mathbf{u}$ and $\mathbf{v}$ be two nonzero vectors in $\mathbb{R}^3$, and let $S = \{\mathbf{u}, \mathbf{v}\}$. Show that span$(S)$ is a line through the origin if $\mathbf{u} = a\mathbf{v}$ for some real number $a$, but otherwise span$(S)$ is a plane through the origin.

**18.** Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be three vectors in $\mathbb{R}^3$ and let $\mathbf{A}$ be the matrix whose rows are $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. Show that $S = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ spans $\mathbb{R}^3$ if and only if $|\mathbf{A}| \neq 0$. (Hint: To prove that $\text{span}(S) = \mathbb{R}^3$ implies $|\mathbf{A}| \neq 0$, suppose $\mathbf{x} \in \mathbb{R}^3$ such that $\mathbf{Ax} = \mathbf{0}$. First, show that $\mathbf{x}$ is orthogonal to $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. Then, express $\mathbf{x}$ as a linear combination of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. Prove that $\mathbf{x} \cdot \mathbf{x} = 0$, and then use Theorem 2.7 and Corollary 3.6. To prove that $|\mathbf{A}| \neq 0$ implies $\text{span}(S) = \mathbb{R}^3$, show that $\mathbf{A}$ is row equivalent to $\mathbf{I}_3$ and apply Theorem 2.9.)

**19.** Let $S = \{\mathbf{p}_1, \ldots, \mathbf{p}_k\}$ be a finite subset of $\mathcal{P}$. Prove that there is some positive integer $n$ such that $\text{span}(S) \subseteq \mathcal{P}_n$.

★ **20.** Suppose that $S_1$ is the set of symmetric $n \times n$ matrices and that $S_2$ is the set of skew-symmetric $n \times n$ matrices. Prove that $\text{span}(S_1 \cup S_2) = \mathcal{M}_{nn}$.

**21.** Recall the set $\mathcal{U}_n$ of upper triangular $n \times n$ matrices and the set $\mathcal{L}_n$ of lower triangular $n \times n$ matrices. Prove that $\text{span}(\mathcal{U}_n \cup \mathcal{L}_n) = \mathcal{M}_{nn}$. (Hint: Consider $\{\Psi_{ij}\}$, where each $\Psi_{ij}$ has 1 for its $(i, j)$ entry, and zeroes elsewhere. Each such matrix is upper or lower triangular or both.)

► **22.** Prove Corollary 4.6.

**23.** This exercise concerns a condition for a subset to be a subspace.

**(a)** Prove that if $S$ is a nonempty subset of a vector space $\mathcal{V}$, then $S$ is a subspace of $\mathcal{V}$ if and only if $\text{span}(S) = S$.

**(b)** Describe the span of the set of the skew-symmetric matrices in $\mathcal{M}_{33}$.

**24.** Let $S_1$ and $S_2$ be subsets of a vector space $\mathcal{V}$. Prove that $\text{span}(S_1) = \text{span}(S_2)$ if and only if $S_1 \subseteq \text{span}(S_2)$ and $S_2 \subseteq \text{span}(S_1)$.

**25.** Let $S_1$ and $S_2$ be two subsets of a vector space $\mathcal{V}$.

**(a)** Prove that $\text{span}(S_1 \cap S_2) \subseteq \text{span}(S_1) \cap \text{span}(S_2)$.

★ **(b)** Give an example of distinct subsets $S_1$ and $S_2$ of $\mathbb{R}^3$ for which the inclusion in part (a) is actually an equality.

★ **(c)** Give an example of subsets $S_1$ and $S_2$ of $\mathbb{R}^3$ for which the inclusion in part (a) is not an equality.

**26.** Let $S_1$ and $S_2$ be subsets of a vector space $\mathcal{V}$.

**(a)** Show that $\text{span}(S_1) \cup \text{span}(S_2) \subseteq \text{span}(S_1 \cup S_2)$.

**(b)** Prove that if $S_1 \subseteq S_2$, then the inclusion in part (a) is an equality.

★ **(c)** Give an example of subsets $S_1$ and $S_2$ in $\mathcal{P}_5$ for which the inclusion in part (a) is not an equality.

**27.** Let $\mathbf{A}$ be an $n \times n$ matrix and $\lambda$ be an eigenvalue for $\mathbf{A}$. Suppose $S$ is a complete set of fundamental eigenvectors (as computed using the Diagonalization Method) for $\mathbf{A}$ corresponding to $\lambda$. Prove that $S$ spans $E_\lambda$.

► **28.** The purpose of this exercise is to finish the proof of Theorem 4.5 by providing the details necessary to show that span$(S)$ is closed under addition. Suppose $\mathbf{v}_1, \mathbf{v}_2 \in \text{span}(S)$. Our goal is to show that $\mathbf{v}_1 + \mathbf{v}_2 \in \text{span}(S)$.

**(a)** Explain why $\mathbf{v}_1$ and $\mathbf{v}_2$ can be expressed as linear combinations of the vectors in finite subsets $S_1$ and $S_2$ of $S$, respectively.

**(b)** Let $S_3 = S_1 \cup S_2$. By adding terms with zero coefficients, if necessary, show how $\mathbf{v}_1$ and $\mathbf{v}_2$ can each be expressed as a linear combination of the vectors in the subset $S_3$ of $S$.

**(c)** Compute $\mathbf{v}_1 + \mathbf{v}_2$ by adding the linear combinations from part (b) and combining like terms, and explain why this proves that $\mathbf{v}_1 + \mathbf{v}_2 \in \text{span}(S)$.

**29.** Prove that if a subset $S$ of a vector space contains at least one nonzero vector, then span($S$) contains an infinite number of vectors.

★ **30.** True or False:

**(a)** Span($S$) is only defined if $S$ is a finite subset of a vector space.

**(b)** If $S$ is a subset of a vector space $\mathcal{V}$, then span($S$) contains every finite linear combination of vectors in $S$.

**(c)** If $S$ is a subset of a vector space $\mathcal{V}$, then span($S$) is the smallest set in $\mathcal{V}$ containing $S$.

**(d)** If $S$ is a subset of a vector space $\mathcal{V}$, and $\mathcal{W}$ is a subspace of $\mathcal{V}$ containing $S$, then we must have $\mathcal{W} \subseteq \text{span}(S)$.

**(e)** The row space of a $4 \times 5$ matrix $\mathbf{A}$ is a subspace of $\mathbb{R}^4$.

**(f)** A simplified form for the span of a finite set $S$ of vectors in $\mathbb{R}^n$ can be found by row reducing the matrix whose rows are the vectors of $S$.

## 4.4 Linear Independence

In this section, we explore the concept of a linearly independent set of vectors and examine methods for determining whether or not a given set of vectors is linearly independent. We will also see that there are important connections between the concepts of span and linear independence.

### Linear Independence and Dependence

At first, we define linear independence and linear dependence only for finite sets of vectors. We extend the definition to infinite sets near the end of this section.

**Definition** Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a finite nonempty subset of a vector space $\mathcal{V}$.

$S$ is **linearly dependent** if and only if there exist real numbers $a_1, \ldots, a_n$, not all zero, such that $a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = \mathbf{0}$. That is, $S$ is linearly dependent if and only if the zero vector can be expressed as a nontrivial linear combination of the vectors in $S$.

$S$ is **linearly independent** if and only if it is *not* linearly dependent. In other words, $S$ is linearly independent if and only if for any set of real numbers $a_1, \ldots, a_n$, the equation $a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = \mathbf{0}$ implies $a_1 = a_2 = \cdots = a_n = 0$. That is, the only linear combination of the vectors in $S$ that is equal to the zero vector is the *trivial* linear combination.

The empty set, { }, is linearly independent.

#### Example 1

Let $S = \{[2,5], [3,-2], [4,-9]\}$. Notice that $(-1)[2,5] + 2[3,-2] + (-1)[4,-9] = [0,0]$. This shows that there is a nontrivial linear combination (that is, a linear combination in which not all of the scalars are zero) of the vectors in $S$ that is equal to the zero vector. Therefore, by definition, $S$ is a *linearly dependent* set.

Notice in Example 1 that it is possible to express one of the vectors in $S$ as a linear combination of the others: for example, we have $[4,-9] = (-1)[2,5] + 2[3,-2]$. We will show shortly that all linearly dependent sets of two or more elements have this property.

#### Example 2

The set of vectors $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ in $\mathbb{R}^3$ is *linearly independent* because if we form a linear combination of these vectors that equals the zero vector—that is, if

$$a[1,0,0] + b[0,1,0] + c[0,0,1] = [0,0,0],$$

we must have $[a,b,c] = [0,0,0]$, and thus $a = b = c = 0$. In other words, the only way to create a linear combination of these vectors that equals the zero vector is for all of the coefficients involved to be zero. For a similar reason, we can assert more generally that the set $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ in $\mathbb{R}^n$ is *linearly independent*.

Similarly, $\{x^n, \ldots, x^2, x, 1\}$ is *linearly independent* in $\mathcal{P}_n$ because a polynomial $a_nx^n + \cdots + a_2x^2 + a_1x + a_0(1)$ can only equal the zero polynomial if $a_n = \cdots = a_2 = a_1 = a_0 = 0$.

Finally, recall the set $\{\Psi_{ij}\}$ of $m \times n$ matrices $(1 \le i \le m,\ 1 \le j \le n)$ from Example 7 of Section 4.3. Because each of these matrices has 1 as its $(i,j)$ entry while all the other matrices have 0 as their $(i,j)$ entry, the only way that a linear combination of such matrices can equal

$\mathbf{O}_{mn}$ is for the $(i, j)$th coefficient in that linear combination to be zero. (Verify!) But this means that all of the coefficients must be zero, and therefore, the set $\{\Psi_{ij}\}$ is *linearly independent* in $\mathcal{M}_{mn}$.

Notice that it is not possible to express one of the vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in Example 2 as a linear combination of the others. We will show shortly that all linearly independent sets of two or more vectors have this property.

**Example 3**

The set $S = \{x^2+1, 2x-1, 2x^2, 4x-3\}$ in $\mathcal{P}_2$ is *linearly dependent* because

$$2(x^2+1) + (-4)(2x-1) + (-1)(2x^2) + (2)(4x-3) = 0.$$

That is, there is a way to form a linear combination of the polynomials in $S$ that equals the zero polynomial so that the linear combination contains at least one nonzero coefficient. However, notice that the subset $S' = \{x^2+1, 2x-1, 2x^2\}$ of $S$ is a *linearly independent* set, because any equation of the form

$$a(x^2+1) + b(2x-1) + c(2x^2) = 0$$

has only the trivial solution $a = b = c = 0$. (The equation is equivalent to $(a+2c)x^2 + (2b)x + (a-b) = 0$, and from the coefficient of $x$, we must have $b = 0$. Then, from the constant term, we find $a = 0$, and from the coefficient of $x^2$, we have $c = 0$.)

## Linear Dependence and Independence With One- and Two-Element Sets

Suppose $S = \{\mathbf{v}\}$, a one-element set. Now, $S$ is linearly dependent if $a\mathbf{v} = \mathbf{0}$ for some $a \neq 0$. But then, by part (4) of Theorem 4.1, $\mathbf{v} = \mathbf{0}$. Conversely, if $\mathbf{v} = \mathbf{0}$, then $a\mathbf{v} = \mathbf{0}$ for any $a \neq 0$. Thus, we conclude:

If $S = \{\mathbf{v}\}$, a one-element set, then $S$ is linearly dependent if and only if $\mathbf{v} = \mathbf{0}$.
Equivalently, $S = \{\mathbf{v}\}$ is linearly independent if and only if $\mathbf{v} \neq \mathbf{0}$.

**Example 4**

Let $S_1 = \{[3, -1, 4]\}$. Since $S_1$ contains a single nonzero vector, $S_1$ is a *linearly independent* subset of $\mathbb{R}^3$. On the other hand, $S_2 = \{[0, 0, 0, 0]\}$ is a *linearly dependent* subset of $\mathbb{R}^4$.

Next, suppose $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ is a two-element set. If $\mathbf{v}_1 = \mathbf{0}$, $S$ is linearly dependent since $1\mathbf{v}_1 + 0\mathbf{v}_2 = \mathbf{0}$ is a linear combination of these vectors that is equal to $\mathbf{0}$ using a nonzero coefficient. Similarly, if $\mathbf{v}_2 = \mathbf{0}$, the equation $0\mathbf{v}_1 + 1\mathbf{v}_2 = \mathbf{0}$ shows that $S$ is linearly dependent. Hence, we conclude:

If either vector in $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ is the zero vector, $S$ is linearly dependent.

Finally, suppose $\mathbf{v}_1 \neq \mathbf{0}$ and $\mathbf{v}_2 \neq \mathbf{0}$. Then the following principle applies:

A set of two nonzero vectors is linearly dependent if and only if one of the vectors is a scalar multiple of the other.

To see this, first suppose $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent, with $\mathbf{v}_1 \neq \mathbf{0}$ and $\mathbf{v}_2 \neq \mathbf{0}$. Then there exist real numbers $a_1$ and $a_2$, not both zero, such that $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 = \mathbf{0}$. If $a_1 \neq 0$ and $a_2 = 0$, then $a_1\mathbf{v}_1 = \mathbf{0}$, which means $\mathbf{v}_1 = \mathbf{0}$, a contradiction. We get an analogous contradiction if $a_2 \neq 0$ and $a_1 = 0$. Therefore, *both* $a_1$ and $a_2$ must be nonzero. Now, since $a_1 \neq 0$, we get $\mathbf{v}_1 = -\frac{a_2}{a_1}\mathbf{v}_2$. That is, $\mathbf{v}_1$ is a scalar multiple of $\mathbf{v}_2$. Similarly, because $a_2 \neq 0$, we find that $\mathbf{v}_2$ is a scalar multiple of $\mathbf{v}_1$. In summary, if $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent, with $\mathbf{v}_1 \neq \mathbf{0}$ and $\mathbf{v}_2 \neq \mathbf{0}$, then $\mathbf{v}_1$ and $\mathbf{v}_2$ are, in fact, scalar multiples of each other.

Conversely, if $\mathbf{v}_1 \neq \mathbf{0}$, $\mathbf{v}_2 \neq \mathbf{0}$, and one of $\mathbf{v}_1$ and $\mathbf{v}_2$ is a scalar multiple of the other, then, without loss of generality, $\mathbf{v}_1 = c\mathbf{v}_2$, for some scalar $c$. Then, $1\mathbf{v}_1 + (-c)\mathbf{v}_2 = 0$, with not all coefficients equal to zero. Thus, $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent.

Equivalently, we have:

> A set of two nonzero vectors is linearly independent if and only if neither of the vectors is a scalar multiple of the other.

Notice that in the argument above, we showed that if one of two nonzero vectors is a scalar multiple of the other, then the second is also a scalar multiple of the first. Therefore, in practice, when checking a two-element set for linear dependence or independence, we only need to determine whether *one* of the vectors is a scalar multiple of the other.

**Example 5**

The set of vectors $S_1 = \{[1, -1, 2], [-3, 3, -6]\}$ in $\mathbb{R}^3$ is *linearly dependent* since one of the vectors is a scalar multiple (and hence a linear combination) of the other. For example, $[1, -1, 2] = (-\frac{1}{3})[-3, 3, -6]$.

Also, the set $S_2 = \{[3, -8], [2, 5]\}$ is a *linearly independent* subset of $\mathbb{R}^2$ because neither of these vectors is a scalar multiple of the other.

We have noted that if $S$ is a one- or two-element subset of a vector space $\mathcal{V}$ containing the zero vector $\mathbf{0}$, then $S$ is linearly dependent. Similarly, if $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is any finite subset of $\mathcal{V}$ containing the zero vector $\mathbf{0}$, then $S$ is linearly dependent: if $\mathbf{v}_k = \mathbf{0}$, then $0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 1\mathbf{v}_k + \cdots + 0\mathbf{v}_n = \mathbf{0}$, so $\mathbf{0}$ can be expressed as a nontrivial linear combination of the vectors in $S$. We therefore have:

> Any finite subset of a vector space that contains the zero vector $\mathbf{0}$ is linearly dependent.

## Using Row Reduction to Test for Linear Independence

**Example 6**

Consider the subset $S = \{[1, -1, 0, 2], [0, -2, 1, 0], [2, 0, -1, 1]\}$ of $\mathbb{R}^4$. We will investigate whether $S$ is linearly independent.

We proceed by assuming that

$$a[1, -1, 0, 2] + b[0, -2, 1, 0] + c[2, 0, -1, 1] = [0, 0, 0, 0]$$

and solve for $a$, $b$, and $c$ to see whether all these coefficients must be zero. That is, we solve

$$[a + 2c, -a - 2b, b - c, 2a + c] = [0, 0, 0, 0],$$

or alternately,

$$\begin{cases} a \phantom{- 2b} + 2c = 0 \\ -a - 2b \phantom{+ 2c} = 0 \\ \phantom{-a -} b - \phantom{2}c = 0 \\ 2a \phantom{- 2b} + \phantom{2}c = 0 \end{cases}.$$

Row reducing

$$\begin{array}{c} \begin{matrix} a & b & c \end{matrix} \\ \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ -1 & -2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 2 & 0 & 1 & 0 \end{array}\right] \end{array}, \quad \text{we obtain} \quad \begin{array}{c} \begin{matrix} a & b & c \end{matrix} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}.$$

Thus, this system has only the trivial solution $a = b = c = 0$. Hence, $S$ is linearly independent.

Notice that in Example 6, the columns of the matrix to the left of the augmentation bar are just the vectors in $S$. In general, to test a finite set of vectors in $\mathbb{R}^n$ for linear independence, we simply row reduce the matrix whose *columns* are the vectors in the set, and then check whether the associated homogeneous system has only the trivial solution. (In practice it is not necessary to include the augmentation bar and the column of zeroes to its right, since this column never changes in the row reduction process.) Thus, we have:

**Method to Test for Linear Independence Using Row Reduction (Independence Test Method)**
Let $S$ be a finite nonempty set of vectors in $\mathbb{R}^n$. To determine whether $S$ is linearly independent, perform the following steps:

**Step 1**: Create the matrix $\mathbf{A}$ whose *columns* are the vectors in $S$.

**Step 2:** Find $\mathbf{B}$, the reduced row echelon form of $\mathbf{A}$.

**Step 3:** If there is a pivot in every column of $\mathbf{B}$, then $S$ is linearly independent. Otherwise, $S$ is linearly dependent.

### Example 7

Consider the subset $S = \{[3, 1, -1], [-5, -2, 2], [2, 2, -1]\}$ of $\mathbb{R}^3$. Using the Independence Test Method, we row reduce

$$\begin{bmatrix} 3 & -5 & 2 \\ 1 & -2 & 2 \\ -1 & 2 & -1 \end{bmatrix} \text{ to obtain } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since we found a pivot in every column, the set $S$ is linearly independent.

### Example 8

Consider the subset $S = \{[1, 0, 2], [-1, -5, -12], [5, 10, 30], [-3, 0, -11], [6, -25, -18]\}$ of $\mathbb{R}^3$. Using the Independence Test Method, we row reduce

$$\begin{bmatrix} 1 & -1 & 5 & -3 & 6 \\ 0 & -5 & 10 & 0 & -25 \\ 2 & -12 & 30 & -11 & -18 \end{bmatrix} \text{ to obtain } \begin{bmatrix} 1 & 0 & 3 & 0 & -1 \\ 0 & 1 & -2 & 0 & 5 \\ 0 & 0 & 0 & 1 & -4 \end{bmatrix}.$$

Since we have no pivots in columns 3 and 5, the set $S$ is linearly dependent.

In Example 8, notice that the entries 3 and $-2$ in the final third column indicate how to express the original third vector of $S$ as a linear combination of the previous columns (both containing pivots): that is,

$$[5, 10, 30] = 3[1, 0, 2] + (-2)[-1, -5, -12].$$

Similarly, the final fifth column $[-1, 5, -4]$ indicates how to express the original fifth vector of $S$ as a linear combination of the previous columns having pivots:

$$[6, -25, -18] = (-1)[1, 0, 2] + 5[-1, -5, -12] + (-4)[-3, 0, -11].$$

Notice also in Example 8 that there are more columns than rows in the matrix to be row reduced. Since each pivot must be in a different row, there must ultimately be some column without a pivot. In such cases, the original set of vectors must be linearly dependent by the Independence Test Method. Therefore, we have:

**Theorem 4.7** *If $S$ is any set in $\mathbb{R}^n$ containing $k$ distinct vectors, where $k > n$, then $S$ is linearly dependent.*

The Independence Test Method can be adapted for use on vector spaces other than $\mathbb{R}^n$, as in the next example. We will justify that the Independence Test Method is actually valid in such cases in Section 5.5.

### Example 9

Consider the following subset of $\mathcal{M}_{22}$:

$$S = \left\{ \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 6 & -1 \\ 3 & 2 \end{bmatrix}, \begin{bmatrix} -11 & 3 \\ -2 & 2 \end{bmatrix} \right\}.$$

We determine whether $S$ is linearly independent using the Independence Test Method. First, we represent the $2 \times 2$ matrices in $S$ as 4-vectors. Placing them in a matrix, using each 4-vector as a column, we get

$$\begin{bmatrix} 2 & -1 & 6 & -11 \\ 3 & 0 & -1 & 3 \\ -1 & 1 & 3 & -2 \\ 4 & 1 & 2 & 2 \end{bmatrix}, \text{ which reduces to } \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

There is no pivot in column 4. Hence, $S$ is linearly dependent.

Notice in Example 9 that the entries of the final fourth column represent the coefficients for a linear combination of the first three matrices in $S$ that produces the fourth matrix; that is,

$$\begin{bmatrix} -11 & 3 \\ -2 & 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} + 3\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} - \frac{3}{2}\begin{bmatrix} 6 & -1 \\ 3 & 2 \end{bmatrix}.$$

## Alternate Characterizations of Linear Independence

In Examples 8 and 9, where the given sets of vectors are linearly dependent, the Independence Test Method indicated how to express some of the vectors as linear combinations of the others. More generally, we have:

**Theorem 4.8** *Suppose $S$ is a finite set of vectors having at least two elements. Then $S$ is linearly dependent if and only if some vector in $S$ can be expressed as a linear combination of the other vectors in $S$.*

Notice that the "if and only if" statement of Theorem 4.8 is formed by combining two "if...then" statements. Hence, we can use the contrapositive of each of these "if...then" statements to obtain the following result equivalent to Theorem 4.8:

A finite set $S$ of vectors is *linearly independent* if and only if *no* vector in $S$ can be expressed as a linear combination of the other vectors in $S$.

*Proof.* We start by assuming that $S$ is linearly dependent. Therefore, we have coefficients $a_1, \ldots, a_n$ such that $a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = \mathbf{0}$, with $a_i \neq 0$ for some $i$. Then,

$$\mathbf{v}_i = \left(-\frac{a_1}{a_i}\right)\mathbf{v}_1 + \cdots + \left(-\frac{a_{i-1}}{a_i}\right)\mathbf{v}_{i-1} + \left(-\frac{a_{i+1}}{a_i}\right)\mathbf{v}_{i+1} + \cdots + \left(-\frac{a_n}{a_i}\right)\mathbf{v}_n,$$

which expresses $\mathbf{v}_i$ as a linear combination of the other vectors in $S$.

For the second half of the proof, we assume that there is a vector $\mathbf{v}_i$ in $S$ that is a linear combination of the other vectors in $S$. Without loss of generality, assume $\mathbf{v}_i = \mathbf{v}_1$; that is, $i = 1$. Therefore, there are real numbers $a_2, \ldots, a_n$ such that

$$\mathbf{v}_1 = a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + \cdots + a_n\mathbf{v}_n.$$

Letting $a_1 = -1$, we get $a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = \mathbf{0}$. Since $a_1 \neq 0$, this shows that $S$ is linearly dependent, completing the proof of the theorem. □

Notice that the comments after both Examples 1 and 2 illustrate Theorem 4.8 for the given vectors in those examples.

### Example 10

The set of vectors $S = \{[1, 2, -1], [0, 1, 2], [2, 7, 4]\}$ in $\mathbb{R}^3$ is linearly dependent because it is possible to express some vector in the set $S$ as a linear combination of the others. For example, $[2, 7, 4] = 2[1, 2, -1] + 3[0, 1, 2]$. From a geometric point of view, the fact that $[2, 7, 4]$ can be expressed as a linear combination of the vectors $[1, 2, -1]$ and $[0, 1, 2]$ means that $[2, 7, 4]$ lies in the plane spanned by $[1, 2, -1]$ and $[0, 1, 2]$, assuming that all three vectors have their initial points at the origin (see Fig. 4.5).

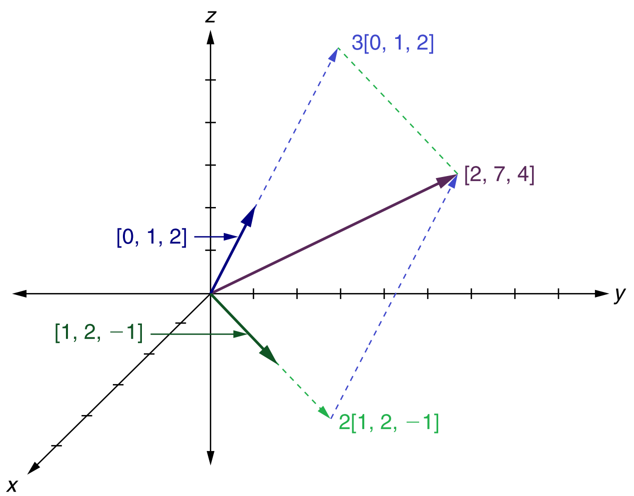


FIGURE 4.5 The vector [2, 7, 4] in the plane spanned by [1, 2, −1] and [0, 1, 2]

For a set $S$ containing a vector $\mathbf{v}$, we introduce the notation $S - \{\mathbf{v}\}$ to represent the set of all (other) vectors in $S$ except $\mathbf{v}$. Now consider a GEaS with at least two dials, where the dials correspond to the vectors in a set $S$. Theorem 4.8 suggests that $S$ is linearly dependent if and only if there is a dial for some vector $\mathbf{v}$ such that the movement resulting from that one dial can be accomplished using a combination of other dials (representing the vectors in $S - \{\mathbf{v}\}$). In other words, the dial for $\mathbf{v}$ would not be needed at all! When we recall that the result of all possible movements on the GEaS is the span of the set of vectors corresponding to the dials, this means that we would obtain the same span with or without the vector $\mathbf{v}$.

Motivated by this observation, we define a **redundant vector** $\mathbf{v}$ in a set $S$ to be a vector in $S$ such that $\text{span}(S) = \text{span}(S - \{\mathbf{v}\})$. It is not hard to show that $\mathbf{v}$ is redundant if and only if it can be expressed as a linear combination of other vectors in $S$. Then, Theorem 4.8 asserts that a set is linearly dependent if and only if it contains at least one redundant vector (see Exercise 12). In particular, notice that in Example 8, the vectors $[5, 10, 30]$ and $[6, -25, -18]$ are redundant vectors for the given set $S$ since each of them is a linear combination of previous vectors. If these two vectors are removed, the resulting set $\{[1, 0, 2], [-1, -5, -12], [-3, 0, -11]\}$ of vectors is linearly independent. Similarly, in Example 9, the fourth matrix is a redundant vector since it is a linear combination of previous vectors, and removing this matrix results in a linearly independent set consisting of the first three matrices.

The characterizations of linear dependence and linear independence from Theorem 4.8 can be expressed in alternate notation using the concept of span. Theorem 4.8 implies that a subset $S$ of two or more vectors in a vector space $\mathcal{V}$ is linearly independent precisely when no vector $\mathbf{v}$ in $S$ is in the span of the remaining vectors. Thus we have:

A set $S$ in a vector space $\mathcal{V}$ is *linearly independent* if and only if there is no vector $\mathbf{v} \in S$ such that $\mathbf{v} \in \text{span}(S - \{\mathbf{v}\})$.

You can easily verify that this statement also holds in the special cases when $S = \{\mathbf{v}\}$ or $S = \{\,\}$.[4] Equivalently, we have:

A set $S$ in a vector space $\mathcal{V}$ is *linearly dependent* if and only if there is some vector $\mathbf{v} \in S$ such that $\mathbf{v} \in \text{span}(S - \{\mathbf{v}\})$.

Another useful characterization of linear independence is the following:

A nonempty set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is *linearly independent* if and only if

**(1)** $\mathbf{v}_1 \neq \mathbf{0}$, and

**(2)** for each $k$, $2 \le k \le n$, $\mathbf{v}_k \notin \text{span}(\{\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}\})$.

That is, $S$ is linearly independent if each vector in $S$ can not be expressed as a linear combination of those vectors listed before it. You are asked to prove this in Exercise 19.

[4] In the former case, $S - \{\mathbf{v}\} = \{\,\}$, the empty set. In the latter case, there is no vector $\mathbf{v} \in S$ that can be chosen.

**Example 11**

Consider $S = \{x+1,\ x^2+1,\ x^3+1,\ x^4+1\}$ in $\mathcal{P}_4$. The first vector in $S$ is nonzero, and each remaining vector in $S$ cannot be a linear combination of earlier vectors since it has a higher degree than those listed before it. Thus, by the preceding principle, $S$ is a linearly independent set.

## Linear Independence of Infinite Sets

Typically, when investigating linear independence we are concerned with a *finite* set $S$ of vectors. However, we occasionally want to discuss linear independence for infinite sets of vectors.

**Definition** An infinite subset $S$ of a vector space $\mathcal{V}$ is **linearly dependent** if and only if there is some finite subset $T$ of $S$ such that $T$ is linearly dependent. Otherwise, $S$ is **linearly independent**.

**Example 12**

Consider the subset $S$ of $\mathcal{M}_{22}$ consisting of all nonsingular $2 \times 2$ matrices. We will show that $S$ is linearly dependent.

Let $T = \{\mathbf{I}_2, 2\mathbf{I}_2\}$, a subset of $S$. Clearly, since the second element of $T$ is a scalar multiple of the first element of $T$, $T$ is a linearly dependent set. Hence, $S$ is linearly dependent, since one of its finite subsets is linearly dependent.

The above definition of linear independence is equivalent to:

An infinite subset $S$ of a vector space $\mathcal{V}$ is linearly independent if and only if every finite subset $T$ of $S$ is linearly independent.

From this, Theorem 4.8 implies that an infinite subset $S$ of a vector space $\mathcal{V}$ is linearly independent if and only if no vector in $S$ is a finite linear combination of other vectors in $S$. (These characterizations of linear independence are obviously valid as well when $S$ is a finite set.)

**Example 13**

Let $S = \left\{1,\ 1+x,\ 1+x+x^2,\ 1+x+x^2+x^3, \ldots\right\}$, an infinite subset of $\mathcal{P}$. We will show that $S$ is linearly independent.

Suppose $T = \{\mathbf{p}_1, \ldots, \mathbf{p}_n\}$ is a finite subset of $S$, with the polynomials written in order of increasing degree. Also suppose that

$$a_1\mathbf{p}_1 + \cdots + a_n\mathbf{p}_n = \mathbf{0}.$$

We need to show that $a_1 = a_2 = \cdots = a_n = 0$. We prove this by contradiction.

Suppose at least one $a_i$ is nonzero. Let $a_k$ be the last nonzero coefficient in the sum. Then,

$$a_1\mathbf{p}_1 + \cdots + a_k\mathbf{p}_k = \mathbf{0}, \quad \text{with } a_k \neq 0.$$

Hence,

$$\mathbf{p}_k = -\frac{a_1}{a_k}\mathbf{p}_1 - \frac{a_2}{a_k}\mathbf{p}_2 - \cdots - \frac{a_{k-1}}{a_k}\mathbf{p}_{k-1}.$$

Because the degrees of all the polynomials in $T$ are different and they were listed in order of increasing degree, this equation expresses $\mathbf{p}_k$ as a linear combination of polynomials whose degrees are lower than that of $\mathbf{p}_k$, giving the desired contradiction.

## Uniqueness of Expression of a Vector as a Linear Combination

The next theorem serves as the foundation for the rest of this chapter because it gives an even more powerful connection between the concepts of span and linear independence.

**Theorem 4.9** *Let $S$ be a nonempty subset of a vector space $\mathcal{V}$. Then $S$ is linearly independent if and only if every vector* $\mathbf{v} \in \operatorname{span}(S)$ *can be expressed* uniquely *as a linear combination of the elements of $S$ (assuming terms with zero coefficients are ignored).*

The phrase "assuming terms with zero coefficients are ignored" means that two finite linear combinations from a set $S$ are considered the same when all their terms with nonzero coefficients agree. (When more terms with zero coefficients are added to a linear combination, it is not considered a different linear combination.) Remember that a *finite* linear combination from an infinite set $S$ involves only a finite number of vectors from $S$.

We prove this theorem for the case where $S$ is finite and ask you to generalize the proof to the infinite case in Exercise 24.

*Proof.* (Abridged) Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.

Suppose first that $S$ is linearly independent. Assume that $\mathbf{v} \in \text{span}(S)$ can be expressed both as $\mathbf{v} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$ and $\mathbf{v} = b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n$. In order to show that the linear combination for $\mathbf{v}$ is unique, we need to prove that $a_i = b_i$ for all $i$. But $\mathbf{0} = \mathbf{v} - \mathbf{v} = (a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n) - (b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n) = (a_1 - b_1)\mathbf{v}_1 + \cdots + (a_n - b_n)\mathbf{v}_n$. Since $S$ is a linearly independent set, each $a_i - b_i = 0$, by the definition of linear independence, and thus $a_i = b_i$ for all $i$.

Conversely, assume every vector in span($S$) can be uniquely expressed as a linear combination of elements of $S$. Suppose that $a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = \mathbf{0}$. Notice also that $0\mathbf{v}_1 + \cdots + 0\mathbf{v}_n = \mathbf{0}$. Now since $\mathbf{0} \in \text{span}(S)$, $\mathbf{0}$ must have a unique expression as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Hence, these two linear combinations must be the same. That is, $a_1 = a_2 = \cdots = a_n = 0$. Therefore, by the definition of linear independence, $S$ is linearly independent. $\square$

If we consider a GEaS whose dials correspond to the vectors in $S$, then Theorem 4.9 asserts that $S$ is linearly independent if and only if there is precisely one way of turning the dials to reach any vector in span($S$)!

**Example 14**

Recall the linearly independent subset $S = \{[1, -1, 0, 2], [0, -2, 1, 0], [2, 0, -1, 1]\}$ of $\mathbb{R}^4$ from Example 6. Now

$$[11, 1, -6, 10] = 3[1, -1, 0, 2] + (-2)[0, -2, 1, 0] + 4[2, 0, -1, 1]$$

so $[11, 1, -6, 10] \in \text{span}(S)$. Then by Theorem 4.9, this is the *only* possible way to express $[11, 1, -6, 10]$ as a linear combination of the elements in $S$.

**Example 15**

Recall the linearly dependent subset $S = \{[1, 0, 2], [-1, -5, -12], [5, 10, 30], [-3, 0, -11], [6, -25, -18]\}$ of $\mathbb{R}^3$ from Example 8. Just after that example, we showed that $[5, 10, 30] = 3[1, 0, 2] + (-2)[-1, -5, -12]$. Thus,

$$[5, 10, 30] = 3[1, 0, 2] + (-2)[-1, -5, -12] + 0[5, 10, 30] + 0[6, -25, -18],$$

but notice that we can also express this vector as

$$[5, 10, 30] = 0[1, 0, 2] + 0[-1, -5, -12] + 1[5, 10, 30] + 0[6, -25, -18].$$

Since $[5, 10, 30]$ is obviously in span($S$), we have found a vector in span($S$) for which the linear combination of elements in $S$ is not unique, just as Theorem 4.9 asserts.

**Example 16**

Recall the set $S$ of nonsingular $2 \times 2$ matrices discussed in Example 12. Because $S$ is linearly dependent, some vector in span($S$) can be expressed in more than one way as a linear combination of vectors in $S$. For example,

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 1\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 1\begin{bmatrix} 3 & 0 \\ -1 & 3 \end{bmatrix} + (-1)\begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}.$$

## Summary of Results

This section includes several different, but equivalent, descriptions of linearly independent and linearly dependent sets of vectors. Several additional characterizations are described in the exercises. The most important results from both the section and the exercises are summarized in Table 4.1.

**TABLE 4.1 Equivalent conditions for a subset $S$ of a vector space to be linearly independent or linearly dependent**

| $S$ is linearly independent | $S$ is linearly dependent | Source |
|---|---|---|
| If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subseteq S$ and $a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = \mathbf{0}$, then $a_1 = a_2 = \cdots = a_n = 0$. (The zero vector requires zero coefficients.) | There is a subset $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of $S$ such that $a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = \mathbf{0}$, with some $a_i \neq 0$. (The zero vector does not require all coefficients to be 0.) | Definition |
| *No* vector in $S$ is a finite linear combination of *other* vectors in $S$. | *Some* vector in $S$ is a finite linear combination of *other* vectors in $S$. | Theorem 4.8 and Remarks after Example 12 |
| For every $\mathbf{v} \in S$, we have $\mathbf{v} \notin \text{span}(S - \{\mathbf{v}\})$. | There is a $\mathbf{v} \in S$ such that $\mathbf{v} \in \text{span}(S - \{\mathbf{v}\})$. | Alternate Characterization |
| For every $\mathbf{v} \in S$, $\text{span}(S - \{\mathbf{v}\})$ does *not* contain all the vectors of $\text{span}(S)$. | There is some $\mathbf{v} \in S$ such that $\text{span}(S - \{\mathbf{v}\}) = \text{span}(S)$. | Exercise 12 |
| If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, then $\mathbf{v}_1 \neq \mathbf{0}$, and, for each $k \geq 2$, $\mathbf{v}_k \notin \text{span}(\{\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}\})$. (*Each* $\mathbf{v}_k$ is *not* a linear combination of the *previous* vectors in $S$.) | If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, then $\mathbf{v}_1 = \mathbf{0}$, or, for some $k \geq 2$, $\mathbf{v}_k = a_1\mathbf{v}_1 + \cdots + a_{k-1}\mathbf{v}_{k-1}$. (*Some* $\mathbf{v}_k$ is a linear combination of the *previous* vectors in $S$.) | Exercise 19 |
| *Every* finite subset of $S$ is linearly independent. | *Some* finite subset of $S$ is linearly dependent. | Definition when $S$ is infinite |
| *Every* vector in $\text{span}(S)$ can be *uniquely* expressed as a linear combination of the vectors in $S$. | *Some* vector in $\text{span}(S)$ can be expressed in more than one way as a linear combination of vectors in $S$. | Theorem 4.9 |

## New Vocabulary

Independence Test Method
linearly dependent (set of vectors)
linearly independent (set of vectors)
redundant vector
trivial linear combination

## Highlights

- A finite set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly dependent if there is a nontrivial linear combination of the vectors that equals $\mathbf{0}$; that is, if $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n = \mathbf{0}$ with some $a_i \neq 0$.
- A finite set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly independent if the only linear combination of the vectors that equals $\mathbf{0}$ is the trivial linear combination (i.e., all coefficients $= 0$); that is, if $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n = \mathbf{0}$, then $a_1 = a_2 = \cdots = a_n = \mathbf{0}$.
- A single element set $\{\mathbf{v}\}$ is linearly independent if and only if $\mathbf{v} \neq \mathbf{0}$.
- A two-element set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent if and only if neither vector is a scalar multiple of the other.
- The set $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is linearly independent in $\mathbb{R}^n$; the set $\{1, x, x^2, \ldots, x^n\}$ is linearly independent in $\mathcal{P}_n$; the set $\{\Psi_{ij} \mid 1 \leq i \leq m,\ 1 \leq j \leq n\}$ is linearly independent in $\mathcal{M}_{mn}$, where each $\Psi_{ij}$ has 1 for its $(i, j)$ entry, and zeroes elsewhere.
- Any set containing the zero vector is linearly dependent.
- The Independence Test Method determines whether a finite subset $S$ of $\mathbb{R}^n$ is linearly independent by calculating the reduced row echelon form of the matrix whose *columns* are the vectors in $S$. The set $S$ is linearly independent if and only if a pivot is obtained in every column.
- If a subset of $\mathbb{R}^n$ contains more than $n$ vectors, then the subset is linearly dependent.
- A finite set of vectors is linearly dependent if *some* vector can be expressed as a linear combination of the others (i.e., some vector is in the span of the other vectors). (Such a vector is said to be redundant.)
- A finite set of vectors is linearly independent if *no* vector can be expressed as a linear combination of the others (i.e., no vector is in the span of the other vectors).
- A finite set of vectors is linearly independent if the first vector is nonzero and *no* vector can be expressed as a linear combination of those listed before it in the set.
- An infinite set of vectors is linearly dependent if *some* finite subset is linearly dependent.
- An infinite set of vectors is linearly independent if *every* finite subset is linearly independent.
- A set $S$ of vectors is linearly independent if and only if every vector in $\text{span}(S)$ is produced by a unique linear combination of the vectors in $S$.

## Exercises for Section 4.4

★ **1.** In each part, determine by quick inspection whether the given set of vectors is linearly independent. State a reason for your conclusion.

**(a)** $\{[0,1,1]\}$

**(b)** $\{[1,2,-1],[3,1,-1]\}$

**(c)** $\{[1,2,-5],[-2,-4,10]\}$

**(d)** $\{[4,2,1],[-1,3,7],[0,0,0]\}$

**(e)** $\{[2,-5,1],[1,1,-1],[0,2,-3],[2,2,6]\}$

**2.** Use the Independence Test Method to determine which of the following sets of vectors are linearly independent:

★ **(a)** $\{[1,9,-2],[3,4,5],[-2,5,-7]\}$

★ **(b)** $\{[2,-1,3],[4,-1,6],[-2,0,2]\}$

**(c)** $\{[5,5,0],[2,1,2],[8,5,6]\}$

**(d)** $\{[5,7,15],[1,1,3],[3,2,10]\}$

★ **(e)** $\{[2,5,-1,6],[4,3,1,4],[1,-1,1,-1]\}$

**(f)** $\{[2,4,22,63],[2,2,1,-7],[1,3,17,67],[1,1,5,-4]\}$

**(g)** $\{[4,-1,0,7],[8,2,3,-7],[5,1,2,-5],[-2,-1,-1,4]\}$

**3.** Use the Independence Test Method to determine which of the following subsets of $\mathcal{P}_2$ are linearly independent:

★ **(a)** $\left\{x^2+x+1,\ x^2-1,\ x^2+1\right\}$

**(b)** $\{x^2+2x+6, -2x^2+x+6, x^2+x-5\}$

★ **(c)** $\{4x-2, 60x-25, -24x+9\}$

**(d)** $\left\{x^2+ax+b \,\middle|\, |a|=1,\ |b|=2\right\}$

**4.** Determine which of the following subsets of $\mathcal{P}$ are linearly independent:

★ **(a)** $\{x^2-1, x^2+1, x^2+x\}$

**(b)** $\{5x^3+6x^2+2, 2x^3+x^2, 2x^2+1, 4x^3+19x^2+9\}$

★ **(c)** $\{4x^2+2, x^2+x-1, x, x^2-5x-3\}$

**(d)** $\{2x^5+x^3, x^5-x-1, x^3+x+1, -x^5+x^3+1\}$

★ **(e)** $\{1, x, x^2, x^3, \ldots\}$

**(f)** $\{1, 1-2x, 1-2x+3x^2, 1-2x+3x^2-4x^3, \ldots\}$

**5.** Show that the following is a linearly dependent subset of $\mathcal{M}_{22}$:

$$\left\{\begin{bmatrix}1&-3\\-1&-4\end{bmatrix},\begin{bmatrix}5&-3\\1&1\end{bmatrix},\begin{bmatrix}2&3\\1&3\end{bmatrix},\begin{bmatrix}15&1\\5&11\end{bmatrix}\right\}.$$

**6.** Prove that the following is linearly independent in $\mathcal{M}_{32}$:

$$\left\{\begin{bmatrix}-2&8\\3&3\\6&-8\end{bmatrix},\begin{bmatrix}0&2\\0&-1\\0&-2\end{bmatrix},\begin{bmatrix}-2&-1\\-1&6\\6&-9\end{bmatrix},\begin{bmatrix}-2&0\\-3&3\\4&-5\end{bmatrix}\right\}.$$

**7.** Let $S=\{[1,1,0],[-2,0,1]\}$.

**(a)** Show that $S$ is linearly independent.

★ **(b)** Find two different vectors $\mathbf{v}$ in $\mathbb{R}^3$ such that $S\cup\{\mathbf{v}\}$ is also linearly independent.

★ **(c)** Find a nonzero vector $\mathbf{u}$ in $\mathbb{R}^3$ such that $S\cup\{\mathbf{u}\}$ is linearly dependent.

**8.** Suppose that $S$ is the subset $\{[4,3,-1,5],[4,-1,2,3],[1,-1,1,4]\}$ of $\mathbb{R}^4$.

**(a)** Show that $S$ is linearly independent.

**(b)** Find a linear combination of vectors in $S$ that produces $[9,6,-2,29]$ (an element of span$(S)$).

**(c)** Is there a different linear combination of the elements of $S$ that yields $[9,6,-2,29]$? If so, find one. If not, why not?

**9.** Consider $S=\{2x^3+5x^2-x+3, 3x^3-4x^2+2x+1, 7x^3+6x^2+7, 3x^3+19x^2-5x+8\}\subseteq\mathcal{P}_3$.

**(a)** Show that $S$ is linearly dependent.

**(b)** Show that every three-element subset of $S$ is linearly dependent.

**(c)** Explain why every subset of $S$ containing exactly two vectors is linearly independent. (Note: There are six possible two-element subsets.)

**10.** Let $\mathbf{u}=[u_1,u_2,u_3]$, $\mathbf{v}=[v_1,v_2,v_3]$, $\mathbf{w}=[w_1,w_2,w_3]$ be three vectors in $\mathbb{R}^3$. Show that $S=\{\mathbf{u},\mathbf{v},\mathbf{w}\}$ is linearly independent if and only if

$$\begin{vmatrix}u_1&u_2&u_3\\v_1&v_2&v_3\\w_1&w_2&w_3\end{vmatrix}\neq 0.$$

(Hint: Consider the transpose and use the Independence Test Method.) (Compare this exercise with Exercise 18 in Section 4.3.)

**11.** For each of the following vector spaces, find a linearly independent subset $S$ containing exactly four elements:

★ **(a)** $\mathbb{R}^4$

**(b)** $\mathbb{R}^5$

★ **(c)** $\mathcal{P}_3$

**(d)** $\mathcal{M}_{23}$

★ **(e)** $\mathcal{V}$ = set of all symmetric matrices in $\mathcal{M}_{33}$.

**12.** Let $S$ be a (possibly infinite) subset of a vector space $\mathcal{V}$.

**(a)** Prove that $S$ is linearly dependent if and only if there is a redundant vector $\mathbf{v} \in S$; that is, a vector $\mathbf{v}$ such that $\text{span}(S - \{\mathbf{v}\}) = \text{span}(S)$.

**(b)** Prove that $\mathbf{v}$ is a redundant vector in $S$ if and only if $\mathbf{v}$ is a linear combination of other vectors in $S$.

**13.** For each of the following linearly dependent sets, find a redundant vector $\mathbf{v}$ in $S$ and verify that $\text{span}(S - \{\mathbf{v}\}) = \text{span}(S)$.

**(a)** $S = \{[4, -2, 6, 1], [1, 0, -1, 2], [0, 0, 0, 0], [6, -2, 5, 5]\}$

★ **(b)** $S = \{[1, 1, 0, 0], [1, 1, 1, 0], [0, 0, -6, 0]\}$

**(c)** $S = \{\, [x_1, x_2, x_3, x_4] \in \mathbb{R}^4 \,|\, x_i = \pm 1, \text{ for each } i\}$

**14.** Let $S_1 = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a subset of a vector space $\mathcal{V}$, let $c$ be a nonzero real number, and let $S_2 = \{c\mathbf{v}_1, \ldots, c\mathbf{v}_n\}$. Show that $S_1$ is linearly independent if and only if $S_2$ is linearly independent.

**15.** Let $\mathbf{f}$ be a polynomial with at least two nonzero terms having different degrees. Prove that the set $\{\mathbf{f}(x), x\mathbf{f}'(x)\}$ (where $\mathbf{f}'$ is the derivative of $\mathbf{f}$) is linearly independent in $\mathcal{P}$.

**16.** Let $\mathcal{V}$ be a vector space, $\mathcal{W}$ a subspace of $\mathcal{V}$, $S$ a linearly independent subset of $\mathcal{W}$, and $\mathbf{v} \in \mathcal{V} - \mathcal{W}$. Prove that $S \cup \{\mathbf{v}\}$ is linearly independent.

**17.** Let $\mathbf{A}$ be an $n \times m$ matrix, let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be a finite subset of $\mathbb{R}^m$, and let $T = \{\mathbf{A}\mathbf{v}_1, \ldots, \mathbf{A}\mathbf{v}_k\}$, a subset of $\mathbb{R}^n$.

**(a)** Prove that if $T$ is a linearly independent subset of $\mathbb{R}^n$ containing $k$ distinct vectors, then $S$ is a linearly independent subset of $\mathbb{R}^m$.

★ **(b)** Find a matrix $\mathbf{A}$ and a set $S$ for which the converse to part (a) is false.

**(c)** Show that the converse to part (a) is true if $\mathbf{A}$ is square and nonsingular.

**18.** Prove that every subset of a linearly independent set is linearly independent.

**19.** Suppose $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a finite subset of a vector space $\mathcal{V}$. Prove that $S$ is linearly independent if and only if $\mathbf{v}_1 \neq \mathbf{0}$ and, for each $k$ with $2 \le k \le n$, $\mathbf{v}_k \notin \text{span}(\{\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}\})$. (Hint: Half of the proof is done by contrapositive. For this half, assume that $S$ is linearly dependent, and use an argument similar to the first half of the proof of Theorem 4.8 to show some $\mathbf{v}_k$ is in $\text{span}(\{\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}\})$. For the other half, assume $S$ is linearly independent and show $\mathbf{v}_1 \neq \mathbf{0}$ and each $\mathbf{v}_k \notin \text{span}(\{\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}\})$.)

**20.** This exercise further explores the result in Exercise 19.

**(a)** Show that $\{3x^4 - 5x^3 + 4x - 8, 12x^3 - 15x^2 + 4, 36x^2 - 30x, 72x - 30, 72\}$ is a linearly independent subset of $\mathcal{P}_4$. (Hint: Reverse the order of the elements, and use Exercise 19.)

**(b)** Prove the following generalization of part (a): Let $\mathbf{f}$ be an $n$th degree polynomial in $\mathcal{P}$, and let $\mathbf{f}^{(i)}$ be the $i$th derivative of $\mathbf{f}$. Show that $\left\{\mathbf{f}, \mathbf{f}^{(1)}, \mathbf{f}^{(2)}, \ldots, \mathbf{f}^{(n)}\right\}$ is a linearly independent subset of $\mathcal{P}$.

**21.** Let $S$ be a nonempty (possibly infinite) subset of a vector space $\mathcal{V}$.

**(a)** Prove that $S$ is linearly independent if and only if *some* vector $\mathbf{v}$ in $\text{span}(S)$ has a unique expression as a linear combination of the vectors in $S$ (ignoring zero coefficients).

★ **(b)** The contrapositives of both halves of the "if and only if" statement in part (a), when combined, give a necessary and sufficient condition for $S$ to be linearly dependent. What is this condition?

**22.** Suppose $\mathbf{A}$ is an $n \times n$ matrix and that $\lambda$ is an eigenvalue for $\mathbf{A}$. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be a set of fundamental eigenvectors for $\mathbf{A}$ corresponding to $\lambda$. Prove that $S$ is linearly independent. (Hint: Consider that each $\mathbf{v}_i$ has a 1 in a coordinate in which all the other vectors in $S$ have a 0.)

**23.** Suppose $T$ is a linearly independent subset of a vector space $\mathcal{V}$ and that $\mathbf{v} \in \mathcal{V}$.

**(a)** Prove that if $T \cup \{\mathbf{v}\}$ is linearly dependent, then $\mathbf{v} \in \text{span}(T)$.

**(b)** Prove that if $\mathbf{v} \notin \text{span}(T)$, then $T \cup \{\mathbf{v}\}$ is linearly independent. (Compare this to Exercise 16.)

▶ **24.** Prove Theorem 4.9 for the case where $S$ is an infinite set. (Hint: Generalize the proof of Theorem 4.9 given for the finite case. In the first half of the proof, suppose that $\mathbf{v} \in \text{span}(S)$ and that $\mathbf{v}$ can be expressed both as $a_1\mathbf{u}_1 + \cdots + a_k\mathbf{u}_k$ and $b_1\mathbf{v}_1 + \cdots + b_l\mathbf{v}_l$ for distinct $\mathbf{u}_1, \ldots, \mathbf{u}_k$ and distinct $\mathbf{v}_1, \ldots, \mathbf{v}_l$ in $S$. Consider the union $W = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\} \cup \{\mathbf{v}_1, \ldots, \mathbf{v}_l\}$, and label the distinct vectors in the union as $\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$. Then use the given linear combinations to express $\mathbf{v}$ in two ways as a linear combination of the vectors in $W$. Finally, use the fact that $W$ is a linearly independent set.)

★ **25.** True or False:

**(a)** The set $\{[2, -3, 1], [-8, 12, -4]\}$ is a linearly independent subset of $\mathbb{R}^3$.

**(b)** A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ in a vector space $\mathcal{V}$ is linearly dependent if $\mathbf{v}_2$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_3$.

**(c)** A subset $S = \{\mathbf{v}\}$ of a vector space $\mathcal{V}$ is linearly dependent if $\mathbf{v} = \mathbf{0}$.

**(d)** A subset $S$ of a vector space $\mathcal{V}$ is linearly independent if there is a vector $\mathbf{v} \in S$ such that $\mathbf{v} \in \text{span}(S - \{\mathbf{v}\})$.

**(e)** If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a linearly independent set of vectors in a vector space $\mathcal{V}$, and $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n = \mathbf{0}$, then $a_1 = a_2 = \cdots = a_n = 0$.

**(f)** If $S$ is a subset of $\mathbb{R}^4$ containing 6 vectors, then $S$ is linearly dependent.

**(g)** Let $S$ be a finite nonempty set of vectors in $\mathbb{R}^n$. If the matrix $\mathbf{A}$ whose rows are the vectors in $S$ has $n$ pivots after row reduction, then $S$ is linearly independent.

**(h)** If $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set of a vector space $\mathcal{V}$, then no vector in $\text{span}(S)$ can be expressed as two different linear combinations of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

**(i)** If $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ is a subset of a vector space $\mathcal{V}$, and $\mathbf{v}_3 = 5\mathbf{v}_1 - 3\mathbf{v}_2$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent.

## 4.5 Basis and Dimension

Suppose that $S$ is a subset of a vector space $\mathcal{V}$ and that $\mathbf{v}$ is some vector in $\mathcal{V}$. Theorem 4.9 prompts the following two questions about $S$ and $\mathbf{v}$:

**Existence:** Is there a linear combination of vectors in $S$ equal to $\mathbf{v}$? (i.e., is $\mathbf{v} \in \text{span}(S)$?)
**Uniqueness:** If so, is this the only such linear combination? (i.e., is $S$ linearly independent?)

The interplay between existence and uniqueness questions is a pervasive theme throughout mathematics. In this section, we tie these concepts together by examining those subsets of vector spaces that simultaneously span and are linearly independent. Such a subset is called a **basis**.

### Definition of Basis

**Definition** Let $\mathcal{V}$ be a vector space, and let $B$ be a subset of $\mathcal{V}$. Then $B$ is a **basis** for $\mathcal{V}$ if and only if both of the following are true:

**(1)** $B$ spans $\mathcal{V}$.

**(2)** $B$ is linearly independent.

#### Example 1

We show that $B = \{[1, 2, 1], [2, 3, 1], [-1, 2, -3]\}$ is a basis for $\mathbb{R}^3$ by showing that it both spans $\mathbb{R}^3$ and is linearly independent.

First, we use the Simplified Span Method from Section 4.3 to show that $B$ spans $\mathbb{R}^3$. Expressing the vectors in $B$ as rows and row reducing the matrix

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ -1 & 2 & -3 \end{bmatrix} \quad \text{yields} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which proves that $\text{span}(B) = \{a[1, 0, 0] + b[0, 1, 0] + c[0, 0, 1] \mid a, b, c \in \mathbb{R}\} = \mathbb{R}^3$.

Next, we must show that $B$ is linearly independent. Expressing the vectors in $B$ as columns, and using the Independence Test Method from Section 4.4, we row reduce

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 2 \\ 1 & 1 & -3 \end{bmatrix} \quad \text{to obtain} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence, $B$ is also linearly independent.

Since $B$ spans $\mathbb{R}^3$ and is linearly independent, $B$ is a basis for $\mathbb{R}^3$. ($B$ is not the only basis for $\mathbb{R}^3$, as we show in the next example.)

#### Example 2

The vector space $\mathbb{R}^n$ has $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ as a basis. Although $\mathbb{R}^n$ has other bases as well, the basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is the most useful for general applications and is therefore referred to as the **standard basis** for $\mathbb{R}^n$. Thus, we refer to $\{\mathbf{i}, \mathbf{j}\}$ and $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ as the standard bases for $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively.

Each of our fundamental examples of vector spaces also has a "standard basis."

**Example 3**

The standard basis in $\mathcal{M}_{32}$ is defined as the set

$$\left\{\begin{bmatrix}1&0\\0&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&0\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&0\\0&1\end{bmatrix}\right\}.$$

More generally, we define the **standard basis** in $\mathcal{M}_{mn}$ to be the set of $m \cdot n$ distinct matrices

$$\left\{ \boldsymbol{\Psi}_{ij} \,\middle|\, 1 \le i \le m, 1 \le j \le n \right\},$$

where $\boldsymbol{\Psi}_{ij}$ is the $m \times n$ matrix with 1 in the $(i, j)$ position and zeroes elsewhere. You should check that these $m \cdot n$ matrices are linearly independent and span $\mathcal{M}_{mn}$. In addition to the standard basis, $\mathcal{M}_{mn}$ has many other bases as well.

**Example 4**

We define $\{1, x, x^2, x^3\}$ to be the standard basis for $\mathcal{P}_3$. More generally, the **standard basis** for $\mathcal{P}_n$ is defined to be the set $\{1, x, x^2, \ldots, x^n\}$, containing $n + 1$ elements. Similarly, we define the infinite set $\{1, x, x^2, \ldots\}$ to be the **standard basis** for $\mathcal{P}$. Again, note that in each case these sets both span and are linearly independent.

Of course, the polynomial spaces have other bases. For example, the following is also a basis for $\mathcal{P}_4$:

$$\left\{x^4,\ x^4 - x^3,\ x^4 - x^3 + x^2,\ x^4 - x^3 + x^2 - x,\ x^3 - 1\right\}.$$

In Exercise 3, you are asked to verify that this is a basis.

**Example 5**

The empty set, $\{\ \}$, is a basis for the trivial vector space, $\{\mathbf{0}\}$. At the end of Section 4.3, we defined the span of the empty set to be the trivial vector space. That is, $\{\ \}$ spans $\{\mathbf{0}\}$. Similarly, at the beginning of Section 4.4, we defined $\{\ \}$ to be linearly independent.

## A Technical Lemma

In Examples 1 through 4 we saw that $\mathbb{R}^n$, $\mathcal{P}_n$, and $\mathcal{M}_{mn}$ each have some *finite* set for a basis, while $\mathcal{P}$ has an infinite basis. We will mostly be concerned with those vector spaces that have finite bases. To begin our study of such vector spaces, we need to show that if a vector space has *one* basis that is finite, then *all* of its bases are finite, and all have the same size. Proving this requires some effort. We begin with Lemma 4.10.

In Lemma 4.10, and throughout the remainder of the text, we use the notation $|S|$ to represent the number of elements in a finite set $S$. For example, if $B$ is the standard basis for $\mathbb{R}^3$, $|B| = 3$. Note that $|\{\ \}| = 0$.

**Lemma 4.10** *Let $S$ and $T$ be subsets of a vector space $\mathcal{V}$ such that $S$ spans $\mathcal{V}$, $S$ is finite, and $T$ is linearly independent. Then $T$ is finite and $|T| \le |S|$.*

*Proof.* If $S$ is empty, then $\mathcal{V} = \{\mathbf{0}\}$. Since $\{\mathbf{0}\}$ is not linearly independent, $T$ is also empty, and so $|T| = |S| = 0$.

Assume that $|S| = n \ge 1$. We will proceed with a proof by contradiction. Suppose that either $T$ is infinite or $|T| > |S| = n$. Then, since every finite subset of $T$ is also linearly independent (see Table 4.1), there is a linearly independent set $Y \subseteq T$ such that $|Y| = n + 1$. Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and let $Y = \{\mathbf{w}_1, \ldots, \mathbf{w}_n, \mathbf{w}_{n+1}\}$. We will obtain a contradiction by showing that $Y$ is linearly dependent.

Now, there are scalars $c_{ij}$, for $1 \le i \le n + 1$ and $1 \le j \le n$ such that

$$\mathbf{w}_1 = c_{11}\mathbf{v}_1 + c_{12}\mathbf{v}_2 + \cdots + c_{1n}\mathbf{v}_n$$
$$\mathbf{w}_2 = c_{21}\mathbf{v}_1 + c_{22}\mathbf{v}_2 + \cdots + c_{2n}\mathbf{v}_n$$

$$\begin{aligned} &\vdots \qquad\qquad \vdots \\ \mathbf{w}_n &= c_{n1}\mathbf{v}_1 + c_{n2}\mathbf{v}_2 + \cdots + c_{nn}\mathbf{v}_n \\ \mathbf{w}_{n+1} &= c_{n+1,1}\mathbf{v}_1 + c_{n+1,2}\mathbf{v}_2 + \cdots + c_{n+1,n}\mathbf{v}_n. \end{aligned}$$

Let $\mathbf{C}$ be the $(n+1)\times n$ matrix whose $(i, j)$ entry is $c_{ij}$. Now, the homogeneous system $\mathbf{C}^T\mathbf{x} = \mathbf{0}$ has $n+1$ variables with only $n$ equations, and so has a nontrivial solution $\mathbf{u} = [u_1, \ldots, u_{n+1}]$. Then,

$$\begin{aligned} u_1\mathbf{w}_1 + \cdots + u_{n+1}\mathbf{w}_{n+1} &= u_1\left(c_{11}\mathbf{v}_1 + c_{12}\mathbf{v}_2 + \cdots + c_{1n}\mathbf{v}_n\right) \\ &\quad + u_2\left(c_{21}\mathbf{v}_1 + c_{22}\mathbf{v}_2 + \cdots + c_{2n}\mathbf{v}_n\right) \\ &\qquad\vdots \\ &\quad + u_{n+1}\left(c_{n+1,1}\mathbf{v}_1 + c_{n+1,2}\mathbf{v}_2 + \cdots + c_{n+1,n}\mathbf{v}_n\right) \\ &= \left(c_{11}u_1 + c_{21}u_2 + \cdots + c_{n+1,1}u_{n+1}\right)\mathbf{v}_1 \\ &\quad + \left(c_{12}u_1 + c_{22}u_2 + \cdots + c_{n+1,2}u_{n+1}\right)\mathbf{v}_2 \\ &\qquad\vdots \\ &\quad + \left(c_{1n}u_1 + c_{2n}u_2 + \cdots + c_{n+1,n}u_{n+1}\right)\mathbf{v}_n. \end{aligned}$$

But the coefficient of each $\mathbf{v}_i$ in the last expression is just the $i$th entry of $\mathbf{C}^T\mathbf{u}$. However, because $\mathbf{C}^T\mathbf{u} = \mathbf{0}$, the coefficient of each $\mathbf{v}_i$ equals 0. Therefore,

$$u_1\mathbf{w}_1 + u_2\mathbf{w}_2 + \cdots + u_{n+1}\mathbf{w}_{n+1} = 0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_n = \mathbf{0}.$$

But, since $\mathbf{u}$ is a nontrivial solution to $\mathbf{C}^T\mathbf{x} = \mathbf{0}$, at least one $u_i$ is nonzero. Hence, $Y = \{\mathbf{w}_1, \ldots, \mathbf{w}_{n+1}\}$ is linearly dependent, since there is a nontrivial linear combination of the vectors in $Y$ that produces the zero vector. This contradiction completes the proof of the lemma. □

**Example 6**

Let $T = \{[1,4,3], [2,-7,6], [5,5,-5], [0,3,19]\}$, a subset of $\mathbb{R}^3$. Because $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ is a spanning set for $\mathbb{R}^3$ containing 3 elements, and $|T| > 3$, Lemma 4.10 tells us that $T$ is linearly dependent. (We could also apply Theorem 4.7 here to show that $T$ is linearly dependent.)

On the other hand, let $S = \{[4,-2,5,1], [-3,-1,4,0], [3,8,0,-7]\}$, a subset of $\mathbb{R}^4$. Now $T = \{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4\}$ is a linearly independent set in $\mathbb{R}^4$ with $|T| = 4$, so Lemma 4.10 implies that any spanning set for $\mathbb{R}^4$ must have at least 4 elements. But $|S| = 3$, which means $S$ cannot span $\mathbb{R}^4$. (We can also verify that $S$ does not span $\mathbb{R}^4$ using the Simplified Span Method.)

## Dimension

We can now prove the main result of this section.

**Theorem 4.11** *Let $\mathcal{V}$ be a vector space, and let $B_1$ and $B_2$ be bases for $\mathcal{V}$ such that $B_1$ has finitely many elements. Then $B_2$ also has finitely many elements, and $|B_1| = |B_2|$.*

*Proof.* Because $B_1$ and $B_2$ are bases for $\mathcal{V}$, $B_1$ spans $\mathcal{V}$ and $B_2$ is linearly independent. Hence, Lemma 4.10 shows that $B_2$ has finitely many elements and $|B_2| \le |B_1|$. Now, since $B_2$ is finite, we can reverse the roles of $B_1$ and $B_2$ in this argument to show that $|B_1| \le |B_2|$. Therefore, $|B_1| = |B_2|$. □

It follows from Theorem 4.11 that if a vector space $\mathcal{V}$ has *one* basis containing a finite number of elements, then *every* basis for $\mathcal{V}$ is finite, and all bases for $\mathcal{V}$ have the same number of elements. This allows us to unambiguously define the **dimension** of such a vector space, as follows:

**Definition** Let $\mathcal{V}$ be a vector space. If $\mathcal{V}$ has a basis $B$ containing a finite number of elements, then $\mathcal{V}$ is said to be **finite dimensional**. In this case, the **dimension** of $\mathcal{V}$, $\dim(\mathcal{V})$, is the number of elements in any basis for $\mathcal{V}$. In particular, $\dim(\mathcal{V}) = |B|$.

If $\mathcal{V}$ does not have a finite basis, then $\mathcal{V}$ is **infinite dimensional**.

### Example 7

Because $\mathbb{R}^3$ has the (standard) basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, the dimension of $\mathbb{R}^3$ is 3. Theorem 4.11 then implies that every other basis for $\mathbb{R}^3$ also has exactly three elements. More generally, $\dim(\mathbb{R}^n) = n$, since $\mathbb{R}^n$ has the basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$.

### Example 8

Because the standard basis $\{1, x, x^2, x^3\}$ for $\mathcal{P}_3$ has 4 elements (see Example 4), $\dim(\mathcal{P}_3) = 4$. Every other basis for $\mathcal{P}_3$, such as $\{x^3 - x,\ x^2 + x + 1,\ x^3 + x - 5,\ 2x^3 + x^2 + x - 3\}$, also has 4 elements. (Verify that the latter set is a basis for $\mathcal{P}_3$.)

Also, $\dim(\mathcal{P}_n) = n + 1$, since $\mathcal{P}_n$ has the basis $\{1, x, x^2, \ldots, x^n\}$, containing $n + 1$ elements. Be careful! Many students *erroneously* believe that the dimension of $\mathcal{P}_n$ is $n$ because of the subscript $n$.

### Example 9

The standard basis for $\mathcal{M}_{32}$ contains 6 elements (see Example 3). Hence, $\dim(\mathcal{M}_{32}) = 6$. In general, from the size of the standard basis for $\mathcal{M}_{mn}$, we see that $\dim(\mathcal{M}_{mn}) = m \cdot n$.

### Example 10

Let $\mathcal{V} = \{\mathbf{0}\}$ be the trivial vector space. Then $\dim(\mathcal{V}) = 0$ because the empty set, which contains no elements, is a basis for $\mathcal{V}$.

### Example 11

Consider the following subsets of $\mathbb{R}^4$:

$$S_1 = \{[1,3,1,2], [3,11,5,10], [-2,4,4,4]\} \quad \text{and}$$
$$S_2 = \{[1,5,-2,3], [-2,-8,8,8], [1,1,-10,-2], [0,2,4,-9], [3,13,-10,-8]\}.$$

Since $|S_1| = 3$, $|S_2| = 5$, and $\dim(\mathbb{R}^4) = 4$, Theorem 4.11 shows us that neither $S_1$ nor $S_2$ is a basis for $\mathbb{R}^4$. In particular, $S_1$ cannot span $\mathbb{R}^4$ because the standard basis for $\mathbb{R}^4$ would then be a linearly independent set that is larger than $S_1$, contradicting Lemma 4.10. Similarly, $S_2$ cannot be linearly independent because the standard basis would be a spanning set that is smaller than $S_2$, again contradicting Lemma 4.10.

Notice, however, that in this case we can make no conclusions regarding whether $S_1$ is linearly independent or whether $S_2$ spans $\mathbb{R}^4$ based solely on the size of these sets. We must check for these properties separately using the techniques of Sections 4.3 and 4.4.

## Sizes of Spanning Sets and Linearly Independent Sets

Example 11 motivates the next result, which summarizes much of what we have learned regarding the sizes of spanning sets and linearly independent sets.

**Theorem 4.12** *Let $\mathcal{V}$ be a finite dimensional vector space.*

**(1)** *Suppose $S$ is a finite subset of $\mathcal{V}$ that spans $\mathcal{V}$. Then $\dim(\mathcal{V}) \le |S|$. Moreover, $|S| = \dim(\mathcal{V})$ if and only if $S$ is a basis for $\mathcal{V}$.*

**(2)** *Suppose $T$ is a linearly independent subset of $\mathcal{V}$. Then $T$ is finite and $|T| \le \dim(\mathcal{V})$. Moreover, $|T| = \dim(\mathcal{V})$ if and only if $T$ is a basis for $\mathcal{V}$.*

*Proof.* Let $B$ be a basis for $\mathcal{V}$ with $|B| = n$. Then $\dim(\mathcal{V}) = |B|$, by definition.

**Part (1)**: Since $S$ is a finite spanning set and $B$ is linearly independent, Lemma 4.10 implies that $|B| \leq |S|$, and so $\dim(\mathcal{V}) \leq |S|$.

If $|S| = \dim(\mathcal{V})$, we prove that $S$ is a basis for $\mathcal{V}$ by contradiction. If $S$ is not a basis, then it is not linearly independent (because it spans). So, by Exercise 12 in Section 4.4 (see Table 4.1), there is a redundant vector in $S$—that is, a vector $\mathbf{v}$ such that $\text{span}(S - \{\mathbf{v}\}) = \text{span}(S) = \mathcal{V}$. But then $S - \{\mathbf{v}\}$ is a spanning set for $\mathcal{V}$ having fewer than $n$ elements, contradicting the fact that we just proved that the size of a spanning set is never less than the dimension.

Finally, suppose $S$ is a basis for $\mathcal{V}$. By Theorem 4.11, $S$ is finite, and $|S| = \dim(\mathcal{V})$ by the definition of dimension.

**Part (2)**: Using $B$ as the spanning set $S$ in Lemma 4.10 proves that $T$ is finite and $|T| \leq \dim(\mathcal{V})$.

If $|T| = \dim(\mathcal{V})$, we prove that $T$ is a basis for $\mathcal{V}$ by contradiction. If $T$ is not a basis for $\mathcal{V}$, then $T$ does not span $\mathcal{V}$ (because it is linearly independent). Therefore, there is a vector $\mathbf{v} \in \mathcal{V}$ such that $\mathbf{v} \notin \text{span}(T)$. Hence, by part (b) of Exercise 23 in Section 4.4, $T \cup \{\mathbf{v}\}$ is also linearly independent. But $T \cup \{\mathbf{v}\}$ has $n + 1$ elements, contradicting the fact we just proved—that a linearly independent subset must have size $\leq \dim(\mathcal{V})$.

Finally, if $T$ is a basis for $\mathcal{V}$, then $|T| = \dim(\mathcal{V})$, by the definition of dimension. □

---

**Example 12**

Recall the subset $B = \{[1, 2, 1], [2, 3, 1], [-1, 2, -3]\}$ of $\mathbb{R}^3$ from Example 1. In that example, after showing that $B$ spans $\mathbb{R}^3$, we could have immediately concluded that $B$ is a basis for $\mathbb{R}^3$ without having proved linear independence by using part (1) of Theorem 4.12 because $B$ is a spanning set with $\dim(\mathbb{R}^3) = 3$ elements.

Similarly, consider $T = \{3, x + 5, x^2 - 7x + 12, x^3 + 4\}$, a subset of $\mathcal{P}_3$. $T$ is linearly independent from Exercise 19 in Section 4.4 (see Table 4.1) because each vector in $T$ is not in the span of those before it. Since $|T| = 4 = \dim(\mathcal{P}_3)$, part (2) of Theorem 4.12 shows that $T$ is a basis for $\mathcal{P}_3$.

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## Dimension of a Subspace

We next show that every subspace of a finite dimensional vector space is also finite dimensional. This is important because it tells us that the theorems we have developed about finite dimension apply to all *subspaces* of our basic examples $\mathbb{R}^n$, $\mathcal{M}_{mn}$, and $\mathcal{P}_n$.

**Theorem 4.13** *Let $\mathcal{V}$ be a finite dimensional vector space, and let $\mathcal{W}$ be a subspace of $\mathcal{V}$. Then $\mathcal{W}$ is also finite dimensional with* $\dim(\mathcal{W}) \leq \dim(\mathcal{V})$. *Moreover,* $\dim(\mathcal{W}) = \dim(\mathcal{V})$ *if and only if* $\mathcal{W} = \mathcal{V}$.

The proof of Theorem 4.13 is left for you to do, with hints, in Exercise 21. The only subtle part of this proof involves showing that $\mathcal{W}$ actually has a basis.[5]

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**Example 13**

Consider the nested sequence of subspaces of $\mathbb{R}^3$ given by $\{\mathbf{0}\} \subset \{\text{scalar multiples of } [4, -7, 0]\} \subset xy\text{-plane} \subset \mathbb{R}^3$. Their respective dimensions are 0, 1, 2, and 3 (why?). Hence, the dimensions of each successive pair of these subspaces satisfy the inequality given in Theorem 4.13.

---

**Example 14**

It can be shown that $B = \{x^3 + 2x^2 - 4x + 18, 3x^2 + 4x - 4, x^3 + 5x^2 - 3, 3x + 2\}$ is a linearly independent subset of $\mathcal{P}_3$. Consider $\mathcal{W} = \text{span}(B)$. Notice that $B$ is a basis for $\mathcal{W}$, so $\dim(\mathcal{W}) = 4$. But then, by the third sentence of Theorem 4.13, $\dim(\mathcal{W}) = \dim(\mathcal{P}_3)$, so $\mathcal{W} = \mathcal{P}_3$. (Of course, we can also see that $B$ is a basis for $\mathcal{P}_3$ from part (2) of Theorem 4.12.)

---

[5] Although it is true that *every* vector space has a basis, we must be careful here, because we have not proven this. In fact, Theorem 4.13 establishes that every subspace of a finite dimensional vector space *does* have a basis and that this basis is finite. Although every finite dimensional vector space has a finite basis by definition, the proof that every infinite dimensional vector space has a basis requires advanced set theory and is beyond the scope of this text.

## Diagonalization and Bases

If an $n \times n$ matrix $\mathbf{A}$ is diagonalizable, we illustrated a method in Section 3.4 for diagonalizing $\mathbf{A}$. In fact, a set $S$ of fundamental eigenvectors produced by the Diagonalization Method for a given eigenvalue $\lambda$ for $\mathbf{A}$ spans the eigenspace $E_\lambda$ (see Exercise 27 in Section 4.3). Also, any set $S$ of fundamental eigenvectors for $\lambda$ is linearly independent (see Exercise 22 in Section 4.4).

In Section 5.6, we will show that when the fundamental eigenvectors produced by the Diagonalization Method for *all* of the eigenvalues of $\mathbf{A}$ are combined together into a set, that set also is linearly independent. Thus, if a total of $n$ fundamental eigenvectors are produced altogether for $\mathbf{A}$, then these fundamental eigenvectors actually form a basis for $\mathbb{R}^n$. Also, the matrix $\mathbf{P}$ whose columns consist of these eigenvectors must row reduce to $\mathbf{I}_n$, by the Independence Test Method. (This will establish the claim in Section 3.4 that $\mathbf{P}$ is nonsingular.) We illustrate all of this in the following example:

**Example 15**

Consider the $3 \times 3$ matrix

$$\mathbf{A} = \begin{bmatrix} -2 & 12 & -4 \\ -2 & 8 & -2 \\ -3 & 9 & -1 \end{bmatrix}.$$

You can verify that solving for fundamental eigenvectors for the eigenvalue $\lambda_1 = 1$ yields $[4, 2, 3]$. Thus, $E_1 = \text{span}(\{[4, 2, 3]\})$. Similarly, you can check that $[3, 1, 0]$ and $[-1, 0, 1]$ are fundamental eigenvectors for the eigenvalue $\lambda_2 = 2$. Thus, $E_2 = \text{span}(\{[3, 1, 0], [-1, 0, 1]\})$. Since we obtained a total of 3 fundamental eigenvectors for $\mathbf{A}$, the matrix $\mathbf{A}$ is diagonalizable. By row reducing

$$\mathbf{P} = \begin{bmatrix} 4 & 3 & -1 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \quad \text{to obtain} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

we illustrate that this set of fundamental eigenvectors is indeed linearly independent (*and* that $\mathbf{P}$ is nonsingular). Notice that this set $\{[4, 2, 3], [3, 1, 0], [-1, 0, 1]\}$ of fundamental eigenvectors is a basis for $\mathbb{R}^3$.

## New Vocabulary

basis
dimension
finite dimensional (vector space)
infinite dimensional (vector space)
standard basis (for $\mathbb{R}^n$, $\mathcal{M}_{mn}$, $\mathcal{P}_n$)

## Highlights

- A basis for a vector space $\mathcal{V}$ is a subset that both spans $\mathcal{V}$ and is linearly independent.
- If a finite basis exists for a vector space $\mathcal{V}$, then $\mathcal{V}$ is said to be finite dimensional.
- For a finite dimensional vector space $\mathcal{V}$, all bases for $\mathcal{V}$ have the same number of vectors, and this number is known as the dimension of $\mathcal{V}$.
- The standard basis for $\mathbb{R}^n$ is $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$; $\dim(\mathbb{R}^n) = n$.
- The standard basis for $\mathcal{P}_n$ is $\{1, x, x^2, \ldots, x^n\}$; $\dim(\mathcal{P}_n) = n + 1$.
- The standard basis for $\mathcal{M}_{mn}$ is $\{\Psi_{ij}\}$, where each $\Psi_{ij}$ has a 1 in the $(i, j)$ entry, and zeroes elsewhere; $\dim(\mathcal{M}_{mn}) = m \cdot n$.
- The trivial vector space $\{\mathbf{0}\}$ is spanned by the empty set, $\{\ \}$.
- If no finite basis exists for a vector space $\mathcal{V}$, then $\mathcal{V}$ is infinite dimensional. $\mathcal{P}$ is an infinite dimensional vector space, as is the set of all real-valued functions with domain $\mathbb{R}$ (under normal operations).
- In a vector space $\mathcal{V}$ with dimension $n$, the size of a spanning set $S$ is always $\geq n$. If $|S| = n$, then $S$ is a basis for $\mathcal{V}$.
- In a vector space $\mathcal{V}$ with dimension $n$, the size of a linearly independent set $T$ is always $\leq n$. If $|T| = n$, then $T$ is a basis for $\mathcal{V}$.
- In a vector space $\mathcal{V}$ with dimension $n$, the dimension of a subspace $\mathcal{W}$ is always $\leq n$. If $\dim(\mathcal{W}) = n$, then $\mathcal{W} = \mathcal{V}$.
- If an $n \times n$ matrix $\mathbf{A}$ is diagonalizable, then any set of $n$ fundamental eigenvectors for $\mathbf{A}$ produced by the Diagonalization Method forms a basis for $\mathbb{R}^n$.

## Exercises for Section 4.5

1. Prove that each of the following subsets of $\mathbb{R}^4$ is a basis for $\mathbb{R}^4$ by showing both that it spans $\mathbb{R}^4$ and is linearly independent:
   (a) $\{[1,5,6,-7],[1,3,9,-8],[-6,2,6,4],[5,2,1,0]\}$
   (b) $\{[5,1,3,-4],[9,2,-1,3],[4,-1,0,4],[2,8,1,7]\}$
   (c) $\{[1,1,1,1],[1,1,1,-2],[1,1,-2,-3],[1,-2,-3,-4]\}$
   (d) $\left\{\left[\frac{15}{2},5,\frac{12}{5},1\right],\left[2,\frac{1}{2},\frac{3}{4},1\right],\left[-\frac{13}{2},1,0,4\right],\left[\frac{18}{5},0,\frac{1}{5},-\frac{1}{5}\right]\right\}$
2. Prove that the following set is a basis for $\mathcal{M}_{22}$ by showing that it spans $\mathcal{M}_{22}$ and is linearly independent:

$$\left\{\begin{bmatrix}4&3\\-7&9\end{bmatrix},\begin{bmatrix}8&-1\\7&4\end{bmatrix},\begin{bmatrix}0&5\\3&1\end{bmatrix},\begin{bmatrix}2&5\\4&-3\end{bmatrix}\right\}.$$

3. Show that the subset $\{x^4,\ x^4-x^3,\ x^4-x^3+x^2,\ x^4-x^3+x^2-x,\ x^3-1\}$ of $\mathcal{P}_4$ is a basis for $\mathcal{P}_4$.
4. Determine which of the following subsets of $\mathbb{R}^4$ form a basis for $\mathbb{R}^4$:
   ★ (a) $S=\{[7,1,2,0],[8,0,1,-1],[1,0,0,-2]\}$
   (b) $S=\{[2,0,1,6],[2,1,3,18],[6,2,-1,2],[-4,-1,1,1]\}$
   ★ (c) $S=\{[7,1,2,0],[8,0,1,-1],[1,0,0,-2],[3,0,1,-1]\}$
   (d) $S=\{[2,3,-2,-5],[3,5,-1,-4],[1,1,-3,-6]\}$
   ★ (e) $S=\{[1,2,3,2],[1,4,9,3],[6,-2,1,4],[3,1,2,1],[10,-9,-15,6]\}$
   (f) $S=\{[2,1,-6,8],[2,-7,2,8],[-4,5,1,1],[3,5,-1,2]\}$

★ 5. Let $\mathcal{W}$ be the solution set to the matrix equation $\mathbf{AX}=\mathbf{0}_5$, where

$$\mathbf{A}=\begin{bmatrix}1&2&1&0&-1\\2&-1&0&1&3\\1&-3&-1&1&4\\2&9&4&-1&-7\end{bmatrix}.$$

   (a) Show that $\mathcal{W}$ is a subspace of $\mathbb{R}^5$.
   (b) Find a basis for $\mathcal{W}$.
   (c) Show that $\dim(\mathcal{W})+\text{rank}(\mathbf{A})=5$.
6. Let $\mathcal{W}$ be the solution set to the matrix equation $\mathbf{AX}=\mathbf{0}_4$, where

$$\mathbf{A}=\begin{bmatrix}-1&1&-2&11\\2&2&16&6\\-2&1&-7&15\\4&1&23&-9\\-1&5&10&39\end{bmatrix}.$$

   (a) Show that $\mathcal{W}$ is a subspace of $\mathbb{R}^4$.
   (b) Find a basis for $\mathcal{W}$.
   (c) Show that $\dim(\mathcal{W})+\text{rank}(\mathbf{A})=4$.
7. Let $\mathbf{f}$ be a polynomial of degree $n$. Show that the set $\{\mathbf{f},\mathbf{f}^{(1)},\mathbf{f}^{(2)},\ldots,\mathbf{f}^{(n)}\}$ is a basis for $\mathcal{P}_n$ (where $\mathbf{f}^{(i)}$ denotes the $i$th derivative of $\mathbf{f}$). (Hint: See part (b) of Exercise 20 in Section 4.4.)
8. This exercise involves a particular linear combination of powers of a matrix.
   (a) Let $\mathbf{A}$ be a $2\times 2$ matrix. Prove that there are real numbers $a_0,a_1,\ldots,a_4$, not all zero, such that $a_4\mathbf{A}^4+a_3\mathbf{A}^3+a_2\mathbf{A}^2+a_1\mathbf{A}+a_0\mathbf{I}_2=\mathbf{O}_2$. (Hint: You can assume that $\mathbf{A},\mathbf{A}^2,\mathbf{A}^3,\mathbf{A}^4$, and $\mathbf{I}_2$ are all distinct because if they are not, opposite nonzero coefficients can be chosen for any identical pair to demonstrate that the given statement holds.)
   (b) Suppose $\mathbf{B}$ is an $n\times n$ matrix. Show that there must be a nonzero polynomial $\mathbf{p}\in\mathcal{P}_{n^2}$ such that $\mathbf{p}(\mathbf{B})=\mathbf{O}_n$.
9. This exercise explores bases for special subspaces of $\mathcal{P}_5$.
   (a) Show that $B=\{(x-2),x(x-2),x^2(x-2),x^3(x-2),x^4(x-2)\}$ is a basis for $\mathcal{V}=\{\mathbf{p}\in\mathcal{P}_5\,|\,\mathbf{p}(2)=0\}$.
   ★ (b) What is $\dim(\mathcal{V})$?
   ★ (c) Find a basis for $\mathcal{W}=\{\mathbf{p}\in\mathcal{P}_5\mid\mathbf{p}(2)=\mathbf{p}(3)=0\}$.
   ★ (d) Calculate $\dim(\mathcal{W})$.

**10.** This exercise concerns linear independence within a given subset of $\mathbb{R}^4$.

**(a)** Show that $B = \{[2,3,0,-1],[-1,1,1,-1]\}$ is a linearly independent subset of $S = \{[1,4,1,-2], [-1,1,1,-1],[3,2,-1,0],[2,3,0,-1]\}$, and that no larger subset of $S$ containing $B$ is linearly independent.

★ **(b)** Show that $B$ is a basis for span($S$), and calculate dim(span($S$)).

★ **(c)** Does span$(S) = \mathbb{R}^4$? Why or why not?

**11.** This exercise concerns linear independence within a given subset of $\mathcal{P}_3$.

**(a)** Show that $B = \{2x^3 - 3x^2 + 5,\ x^3 + 4x - 7,\ 2x^3 - x^2 - 3x - 6\}$ is a linearly independent subset of $S = \{2x^3 - 3x^2 + 5,\ 4x^3 - 6x^2 - 25x + 15,\ x^3 + 4x - 7,\ 5x - 1,\ 2x^3 - x^2 - 3x - 6\}$, and that no larger subset of $S$ containing $B$ is linearly independent.

**(b)** Show that $B$ is a basis for span($S$), and calculate dim(span($S$)).

**(c)** Does span$(S) = \mathcal{P}_3$? Why or why not?

★ **12.** Let $\mathcal{V}$ be a nontrivial finite dimensional vector space.

**(a)** Let $S$ be a subset of $\mathcal{V}$ with $\dim(\mathcal{V}) \le |S|$. Find an example to show that $S$ need not span $\mathcal{V}$.

**(b)** Let $T$ be a subset of $\mathcal{V}$ with $|T| \le \dim(\mathcal{V})$. Find an example to show that $T$ need not be linearly independent.

**13.** Let $S$ be a subset of a finite dimensional vector space $\mathcal{V}$ such that $|S| = \dim(\mathcal{V})$. If $S$ is not a basis for $\mathcal{V}$, prove that $S$ neither spans $\mathcal{V}$ nor is linearly independent.

**14.** Let $\mathcal{V}$ be an $n$-dimensional vector space, and let $S$ be a subset of $\mathcal{V}$ containing exactly $n$ elements. Prove that $S$ spans $\mathcal{V}$ if and only if $S$ is linearly independent.

**15.** Let $\mathbf{A}$ be a nonsingular $n \times n$ matrix, and let $B$ be a basis for $\mathbb{R}^n$.

**(a)** Show that $B_1 = \{\mathbf{Av} \mid \mathbf{v} \in B\}$ is also a basis for $\mathbb{R}^n$. (Treat the vectors in $B$ as column vectors.)

**(b)** Show that $B_2 = \{\mathbf{vA} \mid \mathbf{v} \in B\}$ is also a basis for $\mathbb{R}^n$. (Treat the vectors in $B$ as row vectors.)

**(c)** Letting $B$ be the standard basis for $\mathbb{R}^n$, use the result of part (a) to show that the columns of $\mathbf{A}$ form a basis for $\mathbb{R}^n$.

**(d)** Prove that the rows of $\mathbf{A}$ form a basis for $\mathbb{R}^n$.

**16.** Prove that every proper nontrivial subspace of $\mathbb{R}^3$ can be thought of, from a geometric point of view, as either a line through the origin or a plane through the origin.

**17.** Prove that $\mathcal{P}$ is infinite dimensional by showing that no finite subset $S$ of $\mathcal{P}$ can span $\mathcal{P}$, as follows:

**(a)** Let $S$ be a finite subset of $\mathcal{P}$. Show that $S \subseteq \mathcal{P}_n$, for some $n$.

**(b)** Use part (a) to prove that span$(S) \subseteq \mathcal{P}_n$.

**(c)** Conclude that $S$ cannot span $\mathcal{P}$.

**18.** This exercise involves infinite dimensional vector spaces.

**(a)** Prove that if a vector space $\mathcal{V}$ has an infinite linearly independent subset, then $\mathcal{V}$ is not finite dimensional.

**(b)** Use part (a) to prove that any vector space having $\mathcal{P}$ as a subspace is not finite dimensional.

**19.** Let $B$ be a basis for a vector space $\mathcal{V}$. Prove that no subset of $\mathcal{V}$ containing $B$ (other than $B$ itself) is linearly independent. (Note: You may *not* use $\dim(\mathcal{V})$ in your proof, since $\mathcal{V}$ could be infinite dimensional.)

**20.** Let $B$ be a basis for a vector space $\mathcal{V}$. Prove that no subset of $B$ (other than $B$ itself) is a spanning set for $\mathcal{V}$. (Note: You may *not* use $\dim(\mathcal{V})$ in your proof, since $\mathcal{V}$ could be infinite dimensional.)

► **21.** The purpose of this exercise is to prove Theorem 4.13. Let $\mathcal{V}$ and $\mathcal{W}$ be as given in the theorem. Consider the set $A$ of nonnegative integers defined by $A = \{k \mid \text{a set } T \text{ exists with } T \subseteq \mathcal{W}, |T| = k, \text{ and } T \text{ linearly independent}\}$.

**(a)** Prove that $0 \in A$. (Hence, $A$ is nonempty.)

**(b)** Prove that $k \in A$ implies $k \le \dim(\mathcal{V})$. (Hint: Use Theorem 4.12.) (Hence, $A$ is finite.)

**(c)** Let $n$ be the largest integer in $A$. Let $T = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a linearly independent subset of $\mathcal{W}$ such that $|T| = n$. Explain why such a set $T$ exists and prove that span$(T) \subseteq \mathcal{W}$.

**(d)** Suppose $\mathbf{w} \in \mathcal{W}$ with $\mathbf{w} \notin$ span$(T)$. Use the definition of $T$ to show that $T \cup \{\mathbf{w}\}$ would then be a linearly dependent subset of $\mathcal{W}$.

**(e)** Use part (d) to express $\mathbf{w}$ as a linear combination of vectors in $T$.

**(f)** Explain why part (e) contradicts the assumption that $\mathbf{w} \notin$ span$(T)$. Conclude that $\mathcal{W} \subseteq$ span$(T)$.

**(g)** Use the conclusions of parts (c) and (f) to explain why $\mathcal{W}$ is finite dimensional. Then use part (b) to show $\dim(\mathcal{W}) \le \dim(\mathcal{V})$.

**(h)** Prove that if $\dim(\mathcal{W}) = \dim(\mathcal{V})$, then $\mathcal{W} = \mathcal{V}$. (Hint: Let $B$ be a basis for $\mathcal{W}$ and use part (2) of Theorem 4.12 to show that $B$ is also a basis for $\mathcal{V}$.)

**(i)** Prove the converse of part (h).

**22.** Let $\mathcal{V}$ be a vector space and let $S$ be a finite spanning set for $\mathcal{V}$. Prove that $\mathcal{V}$ is finite dimensional. (Hint: Exercise 12 in Section 4.4 asserts that span($S$) is unchanged if a redundant vector is removed from $S$. Repeatedly remove redundant vectors from $S$ until the remaining set of vectors is linearly independent.)

**23.** Let $\mathcal{V}$ be a subspace of $\mathbb{R}^n$ with $\dim(\mathcal{V}) = n-1$. (Such a subspace is called a **hyperplane** in $\mathbb{R}^n$.) Prove that there is a nonzero $\mathbf{x} \in \mathbb{R}^n$ such that $\mathcal{V} = \{\mathbf{v} \in \mathbb{R}^n \mid \mathbf{x} \cdot \mathbf{v} = 0\}$. (Hint: Set up a homogeneous system of equations whose coefficient matrix has a basis for $\mathcal{V}$ as its rows. Then notice that this $(n-1) \times n$ system has at least one nontrivial solution, say $\mathbf{x}$.)

★ **24.** True or False:

**(a)** A set $B$ of vectors in a vector space $\mathcal{V}$ is a basis for $\mathcal{V}$ if $B$ spans $\mathcal{V}$ and $B$ is linearly independent.

**(b)** All bases for $\mathcal{P}_4$ have 4 elements.

**(c)** $\dim(\mathcal{M}_{43}) = 7$.

**(d)** If $S$ is a spanning set for $\mathcal{W}$ and $\dim(\mathcal{W}) = n$, then $|S| \le n$.

**(e)** If $T$ is a linearly independent set in $\mathcal{W}$ and $\dim(\mathcal{W}) = n$, then $|T| = n$.

**(f)** If $T$ is a linearly independent set in a finite dimensional vector space $\mathcal{W}$ and $S$ is a finite spanning set for $\mathcal{W}$, then $|T| \le |S|$.

**(g)** If $\mathcal{W}$ is a subspace of a finite dimensional vector space $\mathcal{V}$, then $\dim(\mathcal{W}) < \dim(\mathcal{V})$.

**(h)** Every subspace of an infinite dimensional vector space is infinite dimensional.

**(i)** If $\mathbf{A}$ is a nonsingular $4 \times 4$ matrix, then the rows of $\mathbf{A}$ are a basis for $\mathbb{R}^4$.

## 4.6 Constructing Special Bases

In this section, we present additional methods for finding a basis for a given finite dimensional vector space, starting with either a spanning set or a linearly independent subset.

### Using the Simplified Span Method to Construct a Basis

For a given subset $S$ of $\mathbb{R}^n$, the Simplified Span Method from Section 4.3 allows us to simplify the form of span($S$) by first creating a matrix $\mathbf{A}$ whose rows are the vectors in $S$. We then row reduce $\mathbf{A}$ to obtain a reduced row echelon form matrix $\mathbf{C}$. The simplified form of span($S$) is given by the set of all linear combinations of the nonzero rows of $\mathbf{C}$. Now, each nonzero row of the matrix $\mathbf{C}$ has a (pivot) 1 in a column in which all other rows have zeroes, so the nonzero rows of $\mathbf{C}$ must be linearly independent. Thus, the nonzero rows of $\mathbf{C}$ not only span $S$ but are linearly independent as well, and so they form a basis for span($S$). In other words, whenever we use the Simplified Span Method on a subset $S$ of $\mathbb{R}^n$, we are actually creating a basis for span($S$).

**Example 1**

Let $S = \{[2,-2,3,5,5], [-1,1,4,14,-8], [4,-4,-2,-14,18], [3,-3,-1,-9,13]\}$, a subset of $\mathbb{R}^5$. We can use the Simplified Span Method to find a basis $B$ for $\mathcal{V} = \text{span}(S)$. We form the matrix $\mathbf{A}$ whose rows are the vectors in $S$, and then row reduce

$$\mathbf{A} = \begin{bmatrix} 2 & -2 & 3 & 5 & 5 \\ -1 & 1 & 4 & 14 & -8 \\ 4 & -4 & -2 & -14 & 18 \\ 3 & -3 & -1 & -9 & 13 \end{bmatrix} \quad \text{to obtain} \quad \mathbf{C} = \begin{bmatrix} 1 & -1 & 0 & -2 & 4 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, the desired basis for $\mathcal{V}$ is the set $B = \{[1,-1,0,-2,4], [0,0,1,3,-1]\}$ of nonzero rows of $\mathbf{C}$, and $\dim(\mathcal{V}) = 2$.

In general, the Simplified Span Method creates a basis of vectors with a simpler form than the original vectors. This is because a reduced row echelon form matrix has the simplest form of all matrices that are row equivalent to it.

This method can also be adapted to vector spaces other than $\mathbb{R}^n$, as in the next example.

**Example 2**

Consider the subset $S = \{x^2+3x-1,\ 3x^3+6x+15,\ 6x^3-x^2+9x+31,\ 2x^5-7x^3-22x-47\}$ of $\mathcal{P}_5$. We use the Simplified Span Method to find a basis for $\mathcal{W} = \text{span}(S)$.

Since $S$ is a subset of $\mathcal{P}_5$ instead of $\mathbb{R}^n$, we must alter our method slightly. We cannot use the polynomials in $S$ themselves as rows of a matrix, so we "peel off" their coefficients to create four 6-vectors, which we use as the rows of the following matrix:

$$\mathbf{A} = \begin{matrix} x^5 & x^4 & x^3 & x^2 & x & 1 \end{matrix}$$
$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 3 & 0 & 6 & 15 \\ 0 & 0 & 6 & -1 & 9 & 31 \\ 2 & 0 & -7 & 0 & -22 & -47 \end{bmatrix}.$$

Row reducing this matrix produces

$$\mathbf{C} = \begin{matrix} x^5 & x^4 & x^3 & x^2 & x & 1 \end{matrix}$$
$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & -4 & -6 \\ 0 & 0 & 1 & 0 & 2 & 5 \\ 0 & 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The nonzero rows of $\mathbf{C}$ yield the following three-element basis for $\mathcal{W}$:

$$D = \left\{x^5 - 4x - 6,\ x^3 + 2x + 5,\ x^2 + 3x - 1\right\}.$$

Hence, $\dim(\mathcal{W}) = 3$.

## Using the Independence Test Method to Shrink a Spanning Set to a Basis

The next theorem asserts that whenever we have a (finite or infinite) spanning set $S$ for a finite dimensional vector space $\mathcal{V}$, there is a subset of $S$ that forms a basis for $\mathcal{V}$.

**Theorem 4.14** *If $S$ is a spanning set for a finite dimensional vector space $\mathcal{V}$, then there is a set $B \subseteq S$ that is a basis for $\mathcal{V}$.*

The proof[6] of Theorem 4.14 is very similar to the first few parts of the proof of Theorem 4.13 (as outlined in Exercise 21 of Section 4.5) and is left as Exercise 16 below.

Suppose $S$ is a finite spanning set for a vector space $\mathcal{V}$. In practice, to find a subset $B$ of $S$ that is a basis for $\mathcal{V}$ (as predicted by Theorem 4.14), we eliminate certain vectors using the Independence Test Method. This process is illustrated in the following example.

### Example 3

Let $S = \{[1, 3, -2], [2, 1, 4], [0, 5, -8], [1, -7, 14]\}$, and let $\mathcal{V} = \text{span}(S)$. Theorem 4.14 indicates that some subset of $S$ is a basis for $\mathcal{V}$. We form the matrix $\mathbf{A}$ whose *columns* are the vectors in $S$, and apply the Independence Test Method to

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 3 & 1 & 5 & -7 \\ -2 & 4 & -8 & 14 \end{bmatrix} \quad \text{to obtain} \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 2 & -3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This shows that the third and fourth vectors of $S$ are linear combinations of the first two vectors, as follows:

$$[0, 5, -8] = 2[1, 3, -2] - [2, 1, 4] \quad \text{and}$$
$$[1, -7, 14] = -3[1, 3, -2] + 2[2, 1, 4].$$

As a result, these are redundant vectors and can be eliminated from $S$ without affecting $\text{span}(S)$. (See the remarks after Example 10 in Section 4.4.) Therefore, let $B = \{[1, 3, -2], [2, 1, 4]\}$ be the subset of $S$ consisting of the first two vectors alone. Since $B$ is linearly independent and has the same span as the vectors in $S$, $B$ is a subset of $S$ that forms a basis for $\mathcal{V} = \text{span}(S)$.

As Example 3 illustrates, we can find a subset $B$ of a finite spanning set $S$ that is a basis for $\text{span}(S)$ by creating the matrix whose *columns* are the vectors in $S$ and applying the Independence Test Method. We then remove the vectors corresponding

[6] Theorem 4.14 is also true for infinite dimensional vector spaces, but the proof requires advanced topics in set theory that are beyond the scope of this book.

to nonpivot columns. Because these vectors are linear combinations of the vectors corresponding to the pivot columns, they are redundant vectors that can be removed without affecting span($S$). Now, the remaining vectors (those corresponding to the pivot columns) form a linearly independent subset $B$ of $S$, because if we row reduce the matrix containing just these columns, every column becomes a pivot column. Therefore, since $B$ spans $S$ and is linearly independent, $B$ is a basis for span($S$). This procedure is illustrated in the next two examples.

**Example 4**

Consider the subset $S = \{[1,2,-1],[3,6,-3],[4,1,2],[0,0,0],[-1,5,-5]\}$ of $\mathbb{R}^3$. We use the Independence Test Method to find a subset $B$ of $S$ that is a basis for $\mathcal{V} = \text{span}(S)$. We form the matrix $\mathbf{A}$ whose columns are the vectors in $S$, and then row reduce

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 4 & 0 & -1 \\ 2 & 6 & 1 & 0 & 5 \\ -1 & -3 & 2 & 0 & -5 \end{bmatrix} \quad \text{to obtain} \quad \mathbf{C} = \begin{bmatrix} 1 & 3 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since there are pivots in the first and third columns of $\mathbf{C}$, we choose $B = \{[1,2,-1],[4,1,2]\}$, the first and third vectors in $S$. Then $B$ forms a basis for $\mathcal{V} = \text{span}(S)$. Since $|B| = 2$, $\dim(\mathcal{V}) = 2$. (Hence, $S$ does not span all of $\mathbb{R}^3$.)

This method can also be adapted to vector spaces other than $\mathbb{R}^n$.

**Example 5**

Let $S = \{x^3 - 3x^2 + 1, 2x^2 + x, 2x^3 + 3x + 2, 4x - 5\} \subseteq \mathcal{P}_3$. We use the Independence Test Method to find a subset $B$ of $S$ that is a basis for $\mathcal{V} = \text{span}(S)$. Let $\mathbf{A}$ be the matrix whose columns are the analogous vectors in $\mathbb{R}^4$ for the given vectors in $S$. Then

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ -3 & 2 & 0 & 0 \\ 0 & 1 & 3 & 4 \\ 1 & 0 & 2 & -5 \end{bmatrix}, \quad \text{which reduces to} \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Because we have pivots in the first, second, and fourth columns of $\mathbf{C}$, we choose $B = \{x^3 - 3x^2 + 1, 2x^2 + x, 4x - 5\}$, the first, second, and fourth vectors in $S$. Then $B$ is the desired basis for $\mathcal{V}$.

Notice in Example 5 that the third vector in $S$ is a redundant vector that can be eliminated since it is a linear combination of the previous vectors in $S$. Since the first two vectors in $S$ represent pivot columns, the values 2 and 3 in the third column of $\mathbf{C}$ are the coefficients of those two vectors in the linear combination; that is, $2x^3 + 3x + 2 = 2(x^3 - 3x^2 + 1) + 3(2x^2 + x)$.

It is sometimes possible to find a basis by applying the Independence Test Method to a very large or infinite spanning set. A strategy for doing so is outlined in Exercise 7, with further practice in Exercises 8 and 9.

The Simplified Span Method and the Independence Test Method for finding a basis are similar enough to cause confusion, so we contrast their various features in Table 4.2.

**TABLE 4.2 Contrasting the Simplified Span Method and Independence Test Method for finding a basis from a given spanning set $S$**

| Simplified Span Method | Independence Test Method |
|---|---|
| The vectors in $S$ become the *rows* of a matrix. | The vectors in $S$ become the *columns* of a matrix. |
| The basis created is *not* a subset of the spanning set $S$ but contains vectors with a simpler form. | The basis created *is* a subset of the spanning set $S$. |
| The nonzero rows of the reduced row echelon form matrix are used as the basis vectors. | The pivot columns of the reduced row echelon form matrix are used to determine which vectors to select from $S$. |

## Enlarging a Linearly Independent Set to a Basis

Suppose that $T = \{\mathbf{t}_1, \ldots, \mathbf{t}_k\}$ is a linearly independent set of vectors in a finite dimensional vector space $\mathcal{V}$. Because $\mathcal{V}$ is finite dimensional, it has a finite basis, say, $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$, which is a spanning set for $\mathcal{V}$. Consider the set $T \cup A$. Now, $T \cup A$ certainly spans $\mathcal{V}$ (since $A$ alone spans $\mathcal{V}$). We can therefore apply the Independence Test Method to $T \cup A$

to produce a basis $B$ for $\mathcal{V}$. If we order the vectors in $T \cup A$ so that all the vectors in $T$ are listed first, then none of these vectors will be eliminated, since no vector in $T$ is a linear combination of vectors listed earlier in $T$. In this manner we construct a basis $B$ for $\mathcal{V}$ that contains $T$. We have just proved[7] the following:

**Theorem 4.15** *Let $T$ be a linearly independent subset of a finite dimensional vector space $\mathcal{V}$. Then $\mathcal{V}$ has a basis $B$ with $T \subseteq B$.*

Compare this result with Theorem 4.14.

We now formally outline the method given just before Theorem 4.15 to enlarge a given linearly independent subset $T$ in a finite dimensional vector space $\mathcal{V}$ to a basis for $\mathcal{V}$.

**Method for Finding a Basis by Enlarging a Linearly Independent Subset (Enlarging Method)**

Suppose that $T = \{\mathbf{t}_1, \ldots, \mathbf{t}_k\}$ is a linearly independent subset of a finite dimensional vector space $\mathcal{V}$.

**Step 1:** Find a finite spanning set $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ for $\mathcal{V}$.
**Step 2:** Form the ordered spanning set $S = \{\mathbf{t}_1, \ldots, \mathbf{t}_k, \mathbf{a}_1, \ldots, \mathbf{a}_n\}$ for $\mathcal{V}$.
**Step 3:** Use the Independence Test Method on $S$ to produce a subset $B$ of $S$.

Then $B$ is a basis for $\mathcal{V}$ containing $T$.

In general, we can use the Enlarging Method only when we already know a finite spanning set $A$ for the given vector space. The basis produced by this method is easier to work with if the additional vectors in the set $A$ have a simple form. Ideally, we choose $A$ to be the standard basis for $\mathcal{V}$.

**Example 6**

Consider the linearly independent subset $T = \{[2, 0, 4, -12], [0, -1, -3, 9]\}$ of $\mathcal{V} = \mathbb{R}^4$. We use the Enlarging Method to find a basis for $\mathbb{R}^4$ that contains $T$.

**Step 1**: We choose $A$ to be the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ for $\mathbb{R}^4$.

**Step 2**: We create

$$S = \{[2, 0, 4, -12], [0, -1, -3, 9], [1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0], [0, 0, 0, 1]\}.$$

**Step 3**: The matrix

$$\begin{bmatrix} 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 4 & -3 & 0 & 0 & 1 & 0 \\ -12 & 9 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{reduces to} \quad \begin{bmatrix} 1 & 0 & 0 & -\frac{3}{4} & 0 & -\frac{1}{12} \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & \frac{3}{2} & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \end{bmatrix}.$$

Since columns 1, 2, 3, and 5 have pivots, the Independence Test Method indicates that the set $B = \{[2, 0, 4, -12], [0, -1, -3, 9], [1, 0, 0, 0], [0, 0, 1, 0]\}$ is a basis for $\mathbb{R}^4$ containing $T$.

## New Vocabulary

Enlarging Method

## Highlights

- The Simplified Span Method determines a basis $B$ *in simplified form* for the span of a given set $S$ of vectors (by row reducing the matrix whose *rows* are the vectors in $S$).
- Every spanning set $S$ of a finite dimensional vector space $\mathcal{V}$ contains a subset $B$ of $S$ that is a basis for $\mathcal{V}$.

[7] Theorem 4.15 is also true for infinite dimensional vector spaces, but the proof requires advanced topics in set theory that are beyond the scope of this book.

- The Independence Test Method determines a *subset* $B$ of a given set $S$ of vectors that is a basis for span$(S)$ (by row reducing the matrix whose *columns* are the vectors in $S$).
- Every linearly independent set $T$ of a finite dimensional vector space $\mathcal{V}$ can be enlarged to a basis for $\mathcal{V}$ containing $T$.
- The Enlarging Method determines a basis $B$ for a finite dimensional vector space $\mathcal{V}$ such that $B$ contains a given linearly independent set $T$ (by combining $T$ with a spanning set $A$ for $\mathcal{V}$ and then applying the Independence Test Method).

## Exercises for Section 4.6

**1.** For each of the given subsets $S$ of $\mathbb{R}^5$, find a basis for $\mathcal{V} = \text{span}(S)$ using the Simplified Span Method:

★ **(a)** $S = \{[1,2,3,-1,0], [3,6,8,-2,0], [-1,-1,-3,1,1], [-2,-3,-5,1,1]\}$

**(b)** $S = \{[1,3,4,-4,5], [3,4,7,2,-1], [7,4,9,1,2], [16,7,17,-4,13], [6,5,-1,2,1], [-3,4,8,2,5]\}$

**(c)** $S = \{[-1,2,4,4,1], [-3,5,9,0,45], [-3,2,0,-2,35], [7,12,50,0,37], [3,-4,-6,3,-51], [1,-2,-4,1,-21], [-1,-1,-5,-2,7]\}$

★ **(d)** $S = \{[1,1,1,1,1], [1,2,3,4,5], [0,1,2,3,4], [0,0,4,0,-1]\}$

★ **2.** Adapt the Simplified Span Method to find a basis for the subspace of $\mathcal{P}_3$ spanned by $S = \{x^3 - 3x^2 + 2, 2x^3 - 7x^2 + x - 3, 4x^3 - 13x^2 + x + 5\}$.

★ **3.** Adapt the Simplified Span Method to find a basis for the subspace of $\mathcal{M}_{32}$ spanned by

$$S = \left\{ \begin{bmatrix} 1 & 4 \\ 0 & -1 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 5 \\ 1 & -1 \\ 4 & 9 \end{bmatrix}, \begin{bmatrix} 1 & 7 \\ -1 & -2 \\ 2 & -3 \end{bmatrix}, \begin{bmatrix} 3 & 6 \\ 2 & -1 \\ 6 & 12 \end{bmatrix} \right\}.$$

**4.** For each given subset $S$ of $\mathbb{R}^3$, find a subset $B$ of $S$ that is a basis for $\mathcal{V} = \text{span}(S)$.

★ **(a)** $S = \{[1,3,-2], [2,1,4], [3,-6,18], [0,1,-1], [-2,1,-6]\}$

**(b)** $S = \{[10,-2,1], [6,-3,-12], [0,0,0], [-2,2,11], [-4,1,1]\}$

★ **(c)** $S = \{[3,-2,2], [1,2,-1], [3,-2,7], [-1,-10,6]\}$

**(d)** $S = \{[2,4,-5], [3,5,-1], [1,3,-9], [8,10,19], [4,2,15], [7,-3,9]\}$

**5.** For each given subset $S$ of $\mathcal{P}_3$, find a subset $B$ of $S$ that is a basis for $\mathcal{V} = \text{span}(S)$.

★ **(a)** $S = \{x^3 - 8x^2 + 1, 3x^3 - 2x^2 + x, 4x^3 + 2x - 10, x^3 - 20x^2 - x + 12, x^3 + 24x^2 + 2x - 13, x^3 + 14x^2 - 7x + 18\}$

**(b)** $S = \{3x^2 - 5x + 2, 3x^3 - 5x^2 + 2x, 6x^3 - x^2 - 11x + 6, 9x^3 - 19x + 10\}$

**6.** Frequently, we can find a subset of a set $S$ that is a basis for span$(S)$ by inspection. To do this, choose vectors from $S$, one by one, each of which is outside of the span of the previously chosen vectors. Stop when it is clear that all remaining vectors are in the span of the vectors you have chosen up to that point. The set of chosen vectors will then be a basis for span$(S)$. This method is useful only when it is obvious at each step whether or not the remaining vectors in $S$ are within the span of the vectors already chosen. In each part, use this method to find a subset of the given set $S$ that is a basis for span$(S)$.

★ **(a)** $S = \{[3,1,-2], [0,0,0], [6,2,-3]\}$

**(b)** $S = \{[4,-7,8], [0,1,0], [8,7,16], [-4,8,-9]\}$

**(c)** $S =$ the set of all 3-vectors whose second coordinate is zero

★ **(d)** $S =$ the set of all 3-vectors whose second coordinate is $-3$ times its first coordinate plus its third coordinate

**7.** For each given subset $S$ of $\mathcal{P}_3$, find a subset $B$ of $S$ that is a basis for $\mathcal{V} = \text{span}(S)$. (Hint: When the number of vectors in $S$ is large, first guess some smaller subset $S_1$ of $S$ which you think has the same span as $S$. Then use the Independence Test Method on $S_1$ to get a potential basis $B$. Finally, check that every vector in $S$ is in span$(B)$ in order to verify that span$(B) =$ span$(S)$. However, if you discover a vector in $S$ that is not in span$(B)$, then enlarge the set $S_1$ by adding this vector to it, and repeat the above process again.)

★ **(a)** $S =$ the set of all polynomials in $\mathcal{P}_3$ with a zero constant term

**(b)** $S = \mathcal{P}_2$

★ **(c)** $S =$ the set of all polynomials in $\mathcal{P}_3$ with the coefficient of the $x^2$ term equal to the coefficient of the $x^3$ term

**(d)** $S =$ the set of all polynomials in $\mathcal{P}_3$ with the coefficient of the $x$ term equal to 4

**8.** For each given subset $S$ of $\mathcal{M}_{33}$, find a subset $B$ of $S$ that is a basis for $\mathcal{V} = \text{span}(S)$. (Hint: Use the strategy described in Exercise 7.)

★ **(a)** $S = \{\mathbf{A} \in \mathcal{M}_{33} \mid a_{ij} = 0 \text{ or } 1\}$ (For example, $\mathbf{I}_3 \in S$.)

**(b)** $S = \{\mathbf{A} \in \mathcal{M}_{33} \mid a_{ij} = 1 \text{ or } -1\}$ (For example, the matrix $\mathbf{A}$ such that $a_{ij} = (-1)^{i+j}$ is in $S$.)

★ **(c)** $S =$ the set of all symmetric $3 \times 3$ matrices

**(d)** $S =$ the set of all skew-symmetric $3 \times 3$ matrices

**(e)** $S =$ the set of all nonsingular $3 \times 3$ matrices

**9.** Let $\mathcal{V}$ be the subspace of $\mathcal{M}_{22}$ consisting of all symmetric $2 \times 2$ matrices. Let $S$ be the set of nonsingular matrices in $\mathcal{V}$. Find a subset of $S$ that is a basis for $\text{span}(S)$. (Hint: Use the strategy described in Exercise 7.)

**10.** Enlarge each of the following linearly independent subsets $T$ of $\mathbb{R}^5$ to a basis $B$ for $\mathbb{R}^5$ containing $T$:

★ **(a)** $T = \{[1, -3, 0, 1, 4], [2, 2, 1, -3, 1]\}$

**(b)** $T = \{[1, 2, 4, 8, 16], [0, 1, 2, 4, 8], [0, 0, 1, 2, 4]\}$

★ **(c)** $T = \{[1, 0, -1, 0, 0], [0, 1, -1, 1, 0], [2, 3, -8, -1, 0]\}$

**11.** Enlarge each of the following linearly independent subsets $T$ of $\mathcal{P}_4$ to a basis $B$ for $\mathcal{P}_4$ that contains $T$:

★ **(a)** $T = \{x^3 - x^2, x^4 - 3x^3 + 5x^2 - x\}$

**(b)** $T = \{6x + 3, x^3 - 4x - 2\}$

★ **(c)** $T = \{x^4 - x^3 + x^2 - x + 1, x^3 - x^2 + x - 1, x^2 - x + 1\}$

**12.** Enlarge each of the following linearly independent subsets $T$ of $\mathcal{M}_{32}$ to a basis $B$ for $\mathcal{M}_{32}$ that contains $T$:

★ **(a)** $T = \left\{ \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} \right\}$

**(b)** $T = \left\{ \begin{bmatrix} 2 & 4 \\ -5 & 3 \\ 4 & -7 \end{bmatrix}, \begin{bmatrix} 2 & -6 \\ 3 & 4 \\ -3 & 5 \end{bmatrix}, \begin{bmatrix} 5 & -2 \\ -2 & 7 \\ 1 & -2 \end{bmatrix} \right\}$

★ **(c)** $T = \left\{ \begin{bmatrix} 3 & -1 \\ 2 & -6 \\ 5 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ -4 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 6 & 2 \\ -2 & -9 \\ 10 & 2 \end{bmatrix}, \begin{bmatrix} 3 & -4 \\ 8 & -9 \\ 5 & 1 \end{bmatrix} \right\}$

**13.** In each case, find the dimension of $\mathcal{V}$ by using an appropriate method to create a basis.

**(a)** $\mathcal{V} = \text{span}(\{[1, -4, 6, 2, -1], [1, -11, 17, 5, -3], [2, -8, 12, 4, -2], [0, 0, 0, 0, 0], [0, -7, 11, 3, -2], [-3, -16, 26, 6, -5]\})$, a subspace of $\mathbb{R}^5$

★ **(b)** $\mathcal{V} = \{\mathbf{A} \in \mathcal{M}_{33} \mid \text{trace}(\mathbf{A}) = 0\}$, a subspace of $\mathcal{M}_{33}$ (Recall that the trace of a matrix is the sum of the terms on the main diagonal.)

**(c)** $\mathcal{V} = \text{span}(\{3x^4 - 2x^3 + x^2 + 4x + 4, x^4 + 3x^3 - 2x^2 + 2x + 20, 2x^4 - x^3 + 3, 5x^4 - 3x^3 + x^2 + 4x + 7, 4x^3 - 3x^2 - 2x + 19\})$

★ **(d)** $\mathcal{V} = \{\mathbf{p} \in \mathcal{P}_6 \mid \mathbf{p} = ax^6 - bx^5 + ax^4 - cx^3 + (a + b + c)x^2 - (a - c)x + (3a - 2b + 16c)$, for real numbers $a, b$, and $c\}$

**14.** This exercise involves the dimensions of special subspaces of $\mathcal{M}_{nn}$.

**(a)** Show that each of these subspaces of $\mathcal{M}_{nn}$ has dimension $(n^2 + n)/2$.

**(i)** The set of upper triangular $n \times n$ matrices

**(ii)** The set of lower triangular $n \times n$ matrices

**(iii)** The set of symmetric $n \times n$ matrices

★ **(b)** What is the dimension of the set of skew-symmetric $n \times n$ matrices?

**15.** Let $\mathbf{A}$ be an $m \times n$ matrix. In Exercise 14 of Section 4.2 we showed that $\mathcal{V} = \{\mathbf{X} \in \mathbb{R}^n \mid \mathbf{AX} = \mathbf{0}\}$ is a subspace of $\mathbb{R}^n$. Prove that $\dim(\mathcal{V}) + \text{rank}(\mathbf{A}) = n$. (Hint: Count the nonpivot columns and the pivot columns in the reduced row echelon form matrix for $\mathbf{A}$.)

▶ **16.** Prove Theorem 4.14. This proof should be similar to the beginning of the proof of Theorem 4.13 as outlined in parts (a), (b), and (c) of Exercise 21 in Section 4.5. However, change the definition of the set $A$ in that exercise so that each set $T$ is a subset of $S$ rather than of $\mathcal{W}$.

**17.** Let $\mathcal{W}$ be a subspace of a finite dimensional vector space $\mathcal{V}$.

**(a)** Show that $\mathcal{V}$ has some basis $B$ with a subset $B'$ that is a basis for $\mathcal{W}$. (Hint: Start with a basis $B'$ for $\mathcal{W}$ and construct a corresponding basis $B$ for $\mathcal{V}$.)

★ **(b)** If $B$ is any given basis for $\mathcal{V}$, must some subset $B'$ of $B$ be a basis for $\mathcal{W}$? Prove that your answer is correct.

★ **(c)** If $B$ is any given basis for $\mathcal{V}$ and $B' \subseteq B$, is there necessarily a subspace $\mathcal{Y}$ of $\mathcal{V}$ such that $B'$ is a basis for $\mathcal{Y}$? Why or why not?

**18.** Let $\mathcal{V}$ be a finite dimensional vector space, and let $\mathcal{W}$ be a subspace of $\mathcal{V}$.

- **(a)** Prove that $\mathcal{V}$ has a subspace $\mathcal{W}'$ such that every vector in $\mathcal{V}$ can be uniquely expressed as a sum of a vector in $\mathcal{W}$ and a vector in $\mathcal{W}'$. (In other words, show that there is a subspace $\mathcal{W}'$ so that, for every $\mathbf{v}$ in $\mathcal{V}$, there are unique vectors $\mathbf{w} \in \mathcal{W}$ and $\mathbf{w}' \in \mathcal{W}'$ such that $\mathbf{v} = \mathbf{w} + \mathbf{w}'$.)
- ★ **(b)** Give an example of a subspace $\mathcal{W}$ of some finite dimensional vector space $\mathcal{V}$ for which the subspace $\mathcal{W}'$ from part (a) is not unique.

**19.** This exercise gives conditions relating a given subset of $\mathbb{R}^n$ to a basis for $\mathbb{R}^n$.

- **(a)** Let $S$ be a finite subset of $\mathbb{R}^n$. Prove that the Simplified Span Method applied to $S$ produces the standard basis for $\mathbb{R}^n$ if and only if $\text{span}(S) = \mathbb{R}^n$.
- **(b)** Let $B \subseteq \mathbb{R}^n$ with $|B| = n$, and let $\mathbf{A}$ be the $n \times n$ matrix whose rows are the vectors in $B$. Prove that $B$ is a basis for $\mathbb{R}^n$ if and only if $|\mathbf{A}| \neq 0$.

**20.** Let $\mathbf{A}$ be an $m \times n$ matrix and let $S$ be the set of vectors consisting of the rows of $\mathbf{A}$.

- **(a)** Use the Simplified Span Method to show that $\dim(\text{span}(S)) = \text{rank}(\mathbf{A})$.
- **(b)** Use the Independence Test Method to prove that $\dim(\text{span}(S)) = \text{rank}(\mathbf{A}^T)$.
- **(c)** Use parts (a) and (b) to prove that $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}^T)$. (We will state this formally as Corollary 5.11 in Section 5.3.)

**21.** Let $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ be any real numbers, with $n > 2$. Consider the $n \times n$ matrix $\mathbf{A}$ whose $(i, j)$ term is $a_{ij} = \sin(\alpha_i + \beta_j)$. Prove that $|\mathbf{A}| = 0$. (Hint: Consider $\mathbf{x}_1 = [\sin\beta_1, \sin\beta_2, \ldots, \sin\beta_n]$, $\mathbf{x}_2 = [\cos\beta_1, \cos\beta_2, \ldots, \cos\beta_n]$. Show that the row space of $\mathbf{A} \subseteq \text{span}(\{\mathbf{x}_1, \mathbf{x}_2\})$, and hence, dim(row space of $\mathbf{A}$) $< n$.)

★ **22.** True or False:

- **(a)** Given any spanning set $S$ for a finite dimensional vector space $\mathcal{V}$, there is some $B \subseteq S$ that is a basis for $\mathcal{V}$.
- **(b)** Given any linearly independent set $T$ in a finite dimensional vector space $\mathcal{V}$, there is a basis $B$ for $\mathcal{V}$ containing $T$.
- **(c)** If $S$ is a finite spanning set for $\mathbb{R}^n$, then the Simplified Span Method must produce a subset of $S$ that is a basis for $\mathbb{R}^n$.
- **(d)** If $S$ is a finite spanning set for $\mathbb{R}^n$, then the Independence Test Method produces a subset of $S$ that is a basis for $\mathbb{R}^n$.
- **(e)** If $T$ is a linearly independent set in $\mathbb{R}^n$, then the Enlarging Method must produce a subset of $T$ that is a basis for $\mathbb{R}^n$.
- **(f)** The Enlarging Method finds a basis for a vector space by adding a known spanning set to a given linearly independent set and then using the Simplified Span Method.
- **(g)** Before row reduction, the Simplified Span Method places the vectors of a given spanning set $S$ as columns in a matrix, while the Independence Test Method places the vectors of $S$ as rows.

## 4.7 Coordinatization

If $B$ is a basis for a vector space $\mathcal{V}$, then we know every vector in $\mathcal{V}$ has a unique expression as a linear combination of the vectors in $B$. In this section, we develop a process, called coordinatization, for representing any vector in a finite dimensional vector space in terms of its coefficients with respect to a given basis. We also determine how the coordinatization of a vector changes whenever we switch bases.

### Coordinates With Respect to an Ordered Basis

**Definition** An **ordered basis** for a vector space $\mathcal{V}$ is an ordered $n$-tuple of vectors $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ such that the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for $\mathcal{V}$.

In an ordered basis, the elements are written in a specific order. Thus, $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ and $(\mathbf{j}, \mathbf{i}, \mathbf{k})$ are different ordered bases for $\mathbb{R}^3$.

By Theorem 4.9, if $B = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ is an ordered basis for $\mathcal{V}$, then for every vector $\mathbf{w} \in \mathcal{V}$, there are unique scalars $a_1, a_2, \ldots, a_n$ such that $\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$. We use these scalars $a_1, a_2, \ldots, a_n$ to **coordinatize** the vector $\mathbf{w}$ as follows:

**Definition** Let $B = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ be an ordered basis for a vector space $\mathcal{V}$. Suppose that $\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n \in \mathcal{V}$. Then $[\mathbf{w}]_B$, the **coordinatization of w with respect to** $B$, is the $n$-vector $[a_1, a_2, \ldots, a_n]$.

The vector $[\mathbf{w}]_B = [a_1, a_2, \ldots, a_n]$ is frequently referred to as "**w expressed in** $B$**-coordinates**." When useful, we will express $[\mathbf{w}]_B$ as a column vector.

### Example 1

Consider the ordered basis $B = ([1, 2, 1], [2, 3, 1], [-1, 2, -3])$ for $\mathbb{R}^3$. (This is the same basis from Example 1 of Section 4.5, except that it is now "ordered.") You can easily check that the vector $[15, 7, 22]$ can be expressed as a linear combination of the vectors in $B$ as follows:

$$[15, 7, 22] = 4[1, 2, 1] + 3[2, 3, 1] + (-5)[-1, 2, -3].$$

(In fact, by Theorem 4.9, this is the *only* linear combination of these vectors that produces $[15, 7, 22]$.) Therefore, $[15, 7, 22]_B = [4, 3, -5]$.

### Example 2

It is straightforward to show that

$$C = \left( \begin{bmatrix} 2 & 1 \\ 0 & 7 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 8 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 3 \end{bmatrix} \right)$$

is an ordered basis for $\mathcal{M}_{22}$. It is also easy to verify that

$$\begin{bmatrix} -9 & -2 \\ 7 & 0 \end{bmatrix} = (-2)\begin{bmatrix} 2 & 1 \\ 0 & 7 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 \\ -1 & 8 \end{bmatrix} + (-1)\begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix} + (-11)\begin{bmatrix} 1 & 0 \\ -1 & 3 \end{bmatrix}.$$

Therefore, $\begin{bmatrix} -9 & -2 \\ 7 & 0 \end{bmatrix}_C = [-2, 6, -1, -11]$.

We next consider a particular example to illustrate geometrically how the process of coordinatization works in $\mathbb{R}^2$.

### Example 3

Let $B = ([4, 2], [1, 3])$ be an ordered basis for $\mathbb{R}^2$. Notice that $[4, 2] = 1[4, 2] + 0[1, 3]$, so $[4, 2]_B = [1, 0]$. Similarly, $[1, 3]_B = [0, 1]$. From a geometric viewpoint, converting to $B$-coordinates in $\mathbb{R}^2$ results in a new coordinate system in $\mathbb{R}^2$ with $[4, 2]$ and $[1, 3]$ as its "unit" vectors. The new coordinate grid consists of parallelograms whose sides are the vectors in $B$, as shown in Fig. 4.6. For example, $[11, 13]$ equals $[2, 3]$ when expressed in $B$-coordinates because $[11, 13] = 2[4, 2] + 3[1, 3]$. In other words, $[11, 13]_B = [2, 3]$.

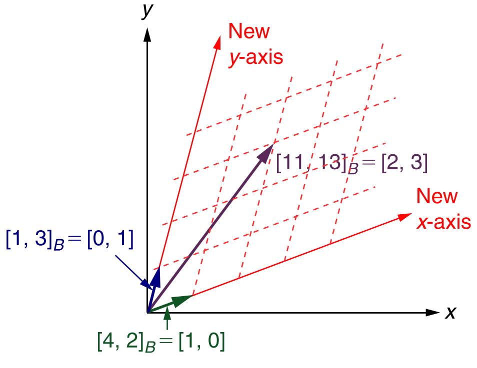


**FIGURE 4.6** A $B$-coordinate grid in $\mathbb{R}^2$: picturing $[11, 13]$ in $B$-coordinates

The first part of the next example shows that coordinatization is easy when we are working with a standard basis.

**Example 4**

Let $B = (x^3, x^2, x, 1)$, the ordered standard basis for $\mathcal{P}_3$. Finding the coordinatization of vectors with respect to $B$ is very easy because of the simple form of the vectors in $B$. For example, $[6x^3 - 2x + 18]_B = [6, 0, -2, 18]$, and $[4 - 3x + 9x^2 - 7x^3]_B = [-7, 9, -3, 4]$. Notice also that $[x^3]_B = [1, 0, 0, 0]$, $[x^2]_B = [0, 1, 0, 0]$, $[x]_B = [0, 0, 1, 0]$, and $[1]_B = [0, 0, 0, 1]$.

Similarly, the general vector $[a_1, \ldots, a_n]$ in $\mathbb{R}^n$ is written as a linear combination of the ordered standard basis $S = (\mathbf{e}_1, \ldots, \mathbf{e}_n)$ for $\mathbb{R}^n$ in a natural and unique way as $a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n$. So, for example, in $\mathbb{R}^5$, $[5, -2, 7, -6, 9]_S = [5, -2, 7, -6, 9]$. Coordinatizing with the ordered standard basis in $\mathbb{R}^n$ is easy because the coefficients in the linear combination are simply the same as the entries of the vector itself.

Also, both the first part of Example 3 and the last part of Example 4 illustrate the general principle that if $B = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$, then $[\mathbf{v}_i]_B = \mathbf{e}_i$. You are asked to prove this in Exercise 6.

## Using Row Reduction to Coordinatize a Vector

**Example 5**

Consider the subspace $\mathcal{V}$ of $\mathbb{R}^5$ spanned by the ordered basis

$$C = ([-4, 5, -1, 0, -1], [1, -3, 2, 2, 5], [1, -2, 1, 1, 3]).$$

Notice that the vectors in $\mathcal{V}$ can be put into $C$-coordinates by solving an appropriate system. For example, to find $[-23, 30, -7, -1, -7]_C$, we solve the equation

$$[-23, 30, -7, -1, -7] = a[-4, 5, -1, 0, -1] + b[1, -3, 2, 2, 5] + c[1, -2, 1, 1, 3].$$

The equivalent system is

$$\begin{cases} -4a + b + c = -23 \\ 5a - 3b - 2c = 30 \\ -a + 2b + c = -7 \\ 2b + c = -1 \\ -a + 5b + 3c = -7 \end{cases}.$$

To solve this system, we row reduce

$$\left[\begin{array}{ccc|c} -4 & 1 & 1 & -23 \\ 5 & -3 & -2 & 30 \\ -1 & 2 & 1 & -7 \\ 0 & 2 & 1 & -1 \\ -1 & 5 & 3 & -7 \end{array}\right] \quad \text{to obtain} \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Hence, the (unique) solution for the system is $a = 6$, $b = -2$, $c = 3$, and we see that $[-23, 30, -7, -1, -7]_C = [6, -2, 3]$.

On the other hand, vectors in $\mathbb{R}^5$ that are not in span($C$) cannot be expressed in $C$-coordinates. For example, the vector $[1, 2, 3, 4, 5]$ is not in $\mathcal{V} = \text{span}(C)$. To see this, consider the system

$$\begin{cases} -4a + b + c = 1 \\ 5a - 3b - 2c = 2 \\ -a + 2b + c = 3 \\ 2b + c = 4 \\ -a + 5b + 3c = 5 \end{cases}.$$

We solve this system by row reducing

$$\left[\begin{array}{ccc|c} -4 & 1 & 1 & 1 \\ 5 & -3 & -2 & 2 \\ -1 & 2 & 1 & 3 \\ 0 & 2 & 1 & 4 \\ -1 & 5 & 3 & 5 \end{array}\right] \quad \text{to obtain} \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

This result tells us that the system has no solutions, implying that the vector $[1, 2, 3, 4, 5]$ is not in span($S$).

Notice in Example 5 that the coordinatized vector $[6, -2, 3]$ is more "compact" than the original vector $[-23, 30, -7, -1, -7]$ but still contains the same essential information.

As we saw in Example 5, finding the coordinates of a vector with respect to an ordered basis typically amounts to solving a system of linear equations using row reduction. The computations we did in Example 5 motivate the following general method. Although it applies to subspaces of $\mathbb{R}^n$, we can adapt it to other finite dimensional vector spaces, such as $\mathcal{P}_n$ and $\mathcal{M}_{mn}$, as with other techniques we have examined. We handle these other vector spaces "informally" in this chapter, but we will treat them more formally in Section 5.5.

**Method for Coordinatizing a Vector With Respect to a Finite Ordered Basis (Coordinatization Method)**

Let $\mathcal{V}$ be a nontrivial subspace of $\mathbb{R}^n$, let $B = (\mathbf{v}_1, \ldots, \mathbf{v}_k)$ be an ordered basis for $\mathcal{V}$, and let $\mathbf{v} \in \mathbb{R}^n$. To calculate $[\mathbf{v}]_B$, if it exists, perform the following steps:

**Step 1:** Form an augmented matrix $[\mathbf{A} \mid \mathbf{v}]$ by using the vectors in $B$ as the *columns* of $\mathbf{A}$, in *order*, and using $\mathbf{v}$ as a column on the right.

**Step 2:** Row reduce $[\mathbf{A} \mid \mathbf{v}]$ to obtain the reduced row echelon form $[\mathbf{C} \mid \mathbf{w}]$.

**Step 3:** If there is a row of $[\mathbf{C} \mid \mathbf{w}]$ that contains all zeroes on the left and has a nonzero entry on the right, then $\mathbf{v} \notin \text{span}(B) = \mathcal{V}$, and coordinatization is not possible. Stop.

**Step 4:** Otherwise, $\mathbf{v} \in \text{span}(B) = \mathcal{V}$. Eliminate all rows consisting entirely of zeroes in $[\mathbf{C} \mid \mathbf{w}]$ to obtain $[\mathbf{I}_k \mid \mathbf{y}]$. Then, $[\mathbf{v}]_B = \mathbf{y}$, the last column of $[\mathbf{I}_k \mid \mathbf{y}]$.

**Example 6**

Let $\mathcal{V}$ be the subspace of $\mathbb{R}^3$ spanned by the ordered basis

$$B = ([2, -1, 3], [3, 2, 1]).$$

We use the Coordinatization Method to find $[\mathbf{v}]_B$, where $\mathbf{v} = [5, -6, 11]$. To do this, we set up the augmented matrix

$$\left[\begin{array}{cc|c} 2 & 3 & 5 \\ -1 & 2 & -6 \\ 3 & 1 & 11 \end{array}\right], \quad \text{which row reduces to} \quad \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right].$$

Ignoring the bottom row of zeroes, we discover $[\mathbf{v}]_B = [4, -1]$.

Similarly, applying the Coordinatization Method to the vector $[1, 2, 3]$, we see that

$$\left[\begin{array}{cc|c} 2 & 3 & 1 \\ -1 & 2 & 2 \\ 3 & 1 & 3 \end{array}\right] \quad \text{row reduces to} \quad \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right].$$

Hence, because of the third row, Step 3 of the Coordinatization Method implies that the coordinatization of $[1, 2, 3]$ with respect to $B$ is not possible.

## Fundamental Properties of Coordinatization

The following theorem shows that the coordinatization of a vector behaves in a manner similar to the original vector with respect to addition and scalar multiplication:

**Theorem 4.16** *Let $B = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ be an ordered basis for a nontrivial vector space $\mathcal{V}$. Suppose $\mathbf{w}_1, \ldots, \mathbf{w}_k \in \mathcal{V}$ and $a_1, \ldots, a_k$ are scalars. Then*

**(1)** $[\mathbf{w}_1 + \mathbf{w}_2]_B = [\mathbf{w}_1]_B + [\mathbf{w}_2]_B$

**(2)** $[a_1\mathbf{w}_1]_B = a_1[\mathbf{w}_1]_B$

**(3)** $[a_1\mathbf{w}_1 + a_2\mathbf{w}_2 + \cdots + a_k\mathbf{w}_k]_B = a_1[\mathbf{w}_1]_B + a_2[\mathbf{w}_2]_B + \cdots + a_k[\mathbf{w}_k]_B$

Fig. 4.7 illustrates part (1) of this theorem. Moving along either path from the upper left to the lower right in the diagram produces the same answer. (Such a picture is called a **commutative diagram**.)

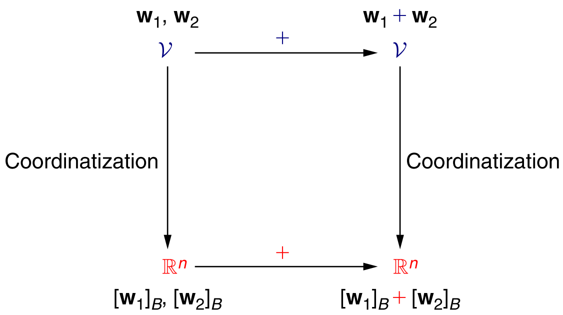


**FIGURE 4.7** Commutative diagram involving addition and coordinatization of vectors

Part (3) asserts that to put a linear combination of vectors in $\mathcal{V}$ into $B$-coordinates, we can first find the $B$-coordinates of each vector individually and then calculate the analogous linear combination in $\mathbb{R}^n$. The proof of Theorem 4.16 is left for you to do in Exercise 13.

---

### Example 7

Recall the subspace $\mathcal{V}$ of $\mathbb{R}^5$ from Example 5 spanned by the ordered basis

$$C = ([-4, 5, -1, 0, -1], [1, -3, 2, 2, 5], [1, -2, 1, 1, 3]).$$

Applying the Simplified Span Method to the vectors in $C$ produces the following vectors in $\mathcal{V} = \text{span}(C)$: $\mathbf{x} = [1, 0, -1, 0, 4]$, $\mathbf{y} = [0, 1, -1, 0, 3]$, $\mathbf{z} = [0, 0, 0, 1, 5]$. (These vectors give a more simplified basis for $\mathcal{V}$.) Applying the Coordinatization Method to $\mathbf{x}$, we find that the augmented matrix

$$\left[\begin{array}{rrr|r} -4 & 1 & 1 & 1 \\ 5 & -3 & -2 & 0 \\ -1 & 2 & 1 & -1 \\ 0 & 2 & 1 & 0 \\ -1 & 5 & 3 & 4 \end{array}\right] \quad \text{row reduces to} \quad \left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Ignoring the last two rows of zeroes, we obtain $[\mathbf{x}]_C = [1, -5, 10]$. In a similar manner we can calculate $[\mathbf{y}]_C = [1, -4, 8]$, and $[\mathbf{z}]_C = [1, -3, 7]$.

Using Theorem 4.16, it is now a simple matter to find the coordinatization of any linear combination of $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$. For example, consider the vector $2\mathbf{x} - 7\mathbf{y} + 3\mathbf{z}$, which is easily computed to be $[2, -7, 5, 3, 2]$. Theorem 4.16 asserts that

$$\begin{aligned} [2\mathbf{x} - 7\mathbf{y} + 3\mathbf{z}]_C &= 2[\mathbf{x}]_C - 7[\mathbf{y}]_C + 3[\mathbf{z}]_C \\ &= 2[1, -5, 10] - 7[1, -4, 8] + 3[1, -3, 7] = [-2, 9, -15]. \end{aligned}$$

This result is easily checked by noting that $-2[-4, 5, -1, 0, -1] + 9[1, -3, 2, 2, 5] - 15[1, -2, 1, 1, 3]$ really does equal $[2, -7, 5, 3, 2]$.

---

## The Transition Matrix for Change of Coordinates

Our next goal is to determine how the coordinates of a vector change when we convert from one ordered basis to another. We first introduce a special matrix that will assist with this conversion.

**Definition** Suppose that $\mathcal{V}$ is a nontrivial $n$-dimensional vector space with ordered bases $B$ and $C$. Let $\mathbf{P}$ be the $n \times n$ matrix whose $i$th column, for $1 \leq i \leq n$, equals $[\mathbf{b}_i]_C$, where $\mathbf{b}_i$ is the $i$th basis vector in $B$. Then $\mathbf{P}$ is called the **transition matrix from $B$-coordinates to $C$-coordinates**.

We often refer to the matrix $\mathbf{P}$ in this definition as the "**transition matrix from $B$ to $C$**."

---

### Example 8

Recall from Example 7 the subspace $\mathcal{V}$ of $\mathbb{R}^5$ that is spanned by the ordered basis

$$C = ([-4, 5, -1, 0, -1], [1, -3, 2, 2, 5], [1, -2, 1, 1, 3]).$$

Let $B = (\mathbf{x}, \mathbf{y}, \mathbf{z})$ be the other ordered basis for $\mathcal{V}$ obtained in Example 7; that is,

$$B = ([1, 0, -1, 0, 4], [0, 1, -1, 0, 3], [0, 0, 0, 1, 5]).$$

To find the transition matrix from $B$ to $C$ we must solve for the $C$-coordinates of each vector in $B$. This was already done in Example 7, and we found $[\mathbf{x}]_C = [1, -5, 10]$, $[\mathbf{y}]_C = [1, -4, 8]$, and $[\mathbf{z}]_C = [1, -3, 7]$. These vectors form the columns of the transition matrix from $B$ to $C$, namely,

$$\mathbf{P} = \begin{bmatrix} 1 & 1 & 1 \\ -5 & -4 & -3 \\ 10 & 8 & 7 \end{bmatrix}.$$

Notice that we could have obtained the same coordinatizations in Example 8 more efficiently by applying the Coordinatization Method to $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ simultaneously—that is, by row reducing the augmented matrix

$$\left[\begin{array}{rrr|rrr} -4 & 1 & 1 & 1 & 0 & 0 \\ 5 & -3 & -2 & 0 & 1 & 0 \\ -1 & 2 & 1 & -1 & -1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \\ -1 & 5 & 3 & 4 & 3 & 5 \end{array}\right] \quad \text{to obtain} \quad \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & -5 & -4 & -3 \\ 0 & 0 & 1 & 10 & 8 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

After deleting the rows of all zeroes, the columns after the augmentation bar give us the entries of the desired transition matrix $\mathbf{P}$.

Generalizing this process, we have the following:

**Method for Calculating a Transition Matrix (Transition Matrix Method)**

To find the transition matrix $\mathbf{P}$ from $B$ to $C$ where $B$ and $C$ are ordered bases for a nontrivial $k$-dimensional subspace of $\mathbb{R}^n$, use row reduction on

$$\left[\begin{array}{cccc|cccc} \text{1st} & \text{2nd} & & k\text{th} & \text{1st} & \text{2nd} & & k\text{th} \\ \text{vector} & \text{vector} & \cdots & \text{vector} & \text{vector} & \text{vector} & \cdots & \text{vector} \\ \text{in} & \text{in} & & \text{in} & \text{in} & \text{in} & & \text{in} \\ C & C & & C & B & B & & B \end{array}\right] \text{ to produce } \left[\begin{array}{c|c} \mathbf{I}_k & \mathbf{P} \\ \hline \text{rows of} & \text{zeroes} \end{array}\right].$$

In Exercise 8 you are asked to show that, in the special cases where either $B$ or $C$ is the standard basis in $\mathbb{R}^n$, there are simple expressions for the transition matrix from $B$ to $C$.

**Example 9**

Consider the following ordered bases for $\mathcal{U}_2$:

$$B = \left(\begin{bmatrix} 7 & 3 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\right) \text{ and } C = \left(\begin{bmatrix} 22 & 7 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 12 & 4 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 33 & 12 \\ 0 & 2 \end{bmatrix}\right).$$

Expressing the matrices in $B$ and $C$ as column vectors, we use the Transition Matrix Method to find the transition matrix from $B$ to $C$ by row reducing

$$\left[\begin{array}{rrr|rrr} 22 & 12 & 33 & 7 & 1 & 1 \\ 7 & 4 & 12 & 3 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 2 & 0 & -1 & 1 \end{array}\right] \quad \text{to obtain} \quad \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & -4 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Ignoring the final row of zeroes, we see that the transition matrix from $B$ to $C$ is given by

$$\mathbf{P} = \begin{bmatrix} 1 & -2 & 1 \\ -4 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

## Change of Coordinates Using the Transition Matrix

Suppose that $B$ and $C$ are ordered bases for the same vector space $\mathcal{V}$, and $\mathbf{v}$ is a vector in $\mathcal{V}$. The next theorem shows that $[\mathbf{v}]_B$ and $[\mathbf{v}]_C$ are related by the transition matrix from $B$ to $C$.

**Theorem 4.17** *Suppose that $B$ and $C$ are ordered bases for a nontrivial $n$-dimensional vector space $\mathcal{V}$, and let $\mathbf{P}$ be an $n \times n$ matrix. Then $\mathbf{P}$ is the transition matrix from $B$ to $C$ if and only if for every $\mathbf{v} \in \mathcal{V}$, $\mathbf{P}[\mathbf{v}]_B = [\mathbf{v}]_C$.*

*Proof.* Let $B$ and $C$ be ordered bases for a vector space $\mathcal{V}$, with $B = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$.

First, suppose $\mathbf{P}$ is the transition matrix from $B$ to $C$. Let $\mathbf{v} \in \mathcal{V}$. We want to show $\mathbf{P}[\mathbf{v}]_B = [\mathbf{v}]_C$. Suppose $[\mathbf{v}]_B = [a_1, \ldots, a_n]$. Then $\mathbf{v} = a_1\mathbf{b}_1 + \cdots + a_n\mathbf{b}_n$. Hence,

$$\mathbf{P}[\mathbf{v}]_B = \begin{bmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & \ddots & \vdots \\ p_{n1} & \cdots & p_{nn} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

$$= a_1 \begin{bmatrix} p_{11} \\ p_{21} \\ \vdots \\ p_{n1} \end{bmatrix} + a_2 \begin{bmatrix} p_{12} \\ p_{22} \\ \vdots \\ p_{n2} \end{bmatrix} + \cdots + a_n \begin{bmatrix} p_{1n} \\ p_{2n} \\ \vdots \\ p_{nn} \end{bmatrix}.$$

However, $\mathbf{P}$ is the transition matrix from $B$ to $C$, so the $i$th column of $\mathbf{P}$ equals $[\mathbf{b}_i]_C$. Therefore,

$$\begin{aligned} \mathbf{P}[\mathbf{v}]_B &= a_1[\mathbf{b}_1]_C + a_2[\mathbf{b}_2]_C + \cdots + a_n[\mathbf{b}_n]_C \\ &= [a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + \cdots + a_n\mathbf{b}_n]_C \qquad \text{by Theorem 4.16} \\ &= [\mathbf{v}]_C. \end{aligned}$$

Conversely, suppose that $\mathbf{P}$ is an $n \times n$ matrix and that $\mathbf{P}[\mathbf{v}]_B = [\mathbf{v}]_C$ for every $\mathbf{v} \in \mathcal{V}$. We show that $\mathbf{P}$ is the transition matrix from $B$ to $C$. By definition, it is enough to show that the $i$th column of $\mathbf{P}$ is equal to $[\mathbf{b}_i]_C$. Since $\mathbf{P}[\mathbf{v}]_B = [\mathbf{v}]_C$, for all $\mathbf{v} \in \mathcal{V}$, let $\mathbf{v} = \mathbf{b}_i$. Then since $[\mathbf{v}]_B = \mathbf{e}_i$, we have $\mathbf{P}[\mathbf{v}]_B = \mathbf{P}\mathbf{e}_i = [\mathbf{b}_i]_C$. But $\mathbf{P}\mathbf{e}_i = i$th column of $\mathbf{P}$, which completes the proof. □

### Example 10

Recall the ordered bases for $\mathcal{U}_2$ from Example 9:

$$B = \left( \begin{bmatrix} 7 & 3 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \right) \quad \text{and} \quad C = \left( \begin{bmatrix} 22 & 7 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 12 & 4 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 33 & 12 \\ 0 & 2 \end{bmatrix} \right).$$

In that example, we found that the transition matrix $\mathbf{P}$ from $B$ to $C$ is

$$\mathbf{P} = \begin{bmatrix} 1 & -2 & 1 \\ -4 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

We can use $\mathbf{P}$ to quickly change the coordinatization of any vector in $\mathcal{U}_2$ from $B$-coordinates to $C$-coordinates. For example, let $\mathbf{v} = \begin{bmatrix} 25 & 24 \\ 0 & -9 \end{bmatrix}$. Since

$$\begin{bmatrix} 25 & 24 \\ 0 & -9 \end{bmatrix} = 4\begin{bmatrix} 7 & 3 \\ 0 & 0 \end{bmatrix} + 3\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} - 6\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix},$$

we know that

$$[\mathbf{v}]_B = \begin{bmatrix} 4 \\ 3 \\ -6 \end{bmatrix}. \quad \text{But then,} \quad \mathbf{P}[\mathbf{v}]_B = \begin{bmatrix} -8 \\ -19 \\ 13 \end{bmatrix},$$

and so $[\mathbf{v}]_C = [-8, -19, 13]$ by Theorem 4.17. We can easily verify this by checking that

$$\begin{bmatrix} 25 & 24 \\ 0 & -9 \end{bmatrix} = -8\begin{bmatrix} 22 & 7 \\ 0 & 2 \end{bmatrix} - 19\begin{bmatrix} 12 & 4 \\ 0 & 1 \end{bmatrix} + 13\begin{bmatrix} 33 & 12 \\ 0 & 2 \end{bmatrix}.$$

## Algebra of the Transition Matrix

The next theorem shows that the cumulative effect of two transitions between bases is represented by the product of the transition matrices in *reverse* order.

**Theorem 4.18** *Suppose that $B$, $C$, and $D$ are ordered bases for a nontrivial finite dimensional vector space $\mathcal{V}$. Let $\mathbf{P}$ be the transition matrix from $B$ to $C$, and let $\mathbf{Q}$ be the transition matrix from $C$ to $D$. Then $\mathbf{QP}$ is the transition matrix from $B$ to $D$.*

The proof of this theorem is left as Exercise 14.

### Example 11

Consider the ordered bases $B$, $C$, and $D$ for $\mathcal{P}_2$ given by

$$\begin{aligned} B &= (-x^2 + 4x + 2,\ 2x^2 - x - 1,\ -x^2 + 2x + 1), \\ C &= (x^2 - 2x - 3,\ 2x^2 - 1,\ x^2 + x + 1), \\ \text{and}\quad D &= (-5x^2 - 15x - 18,\ -9x^2 - 20x - 23,\ 4x^2 + 8x + 9). \end{aligned}$$

Now, row reducing

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 1 & -1 & 2 & -1 \\ -2 & 0 & 1 & 4 & -1 & 2 \\ -3 & -1 & 1 & 2 & -1 & 1 \end{array}\right] \text{ to obtain } \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -9 & 3 & -5 \\ 0 & 1 & 0 & 11 & -3 & 6 \\ 0 & 0 & 1 & -14 & 5 & -8 \end{array}\right],$$

we see that the transition matrix from $B$ to $C$ is

$$\mathbf{P} = \begin{bmatrix} -9 & 3 & -5 \\ 11 & -3 & 6 \\ -14 & 5 & -8 \end{bmatrix}.$$

Similarly, row reducing

$$\left[\begin{array}{rrr|rrr} -5 & -9 & 4 & 1 & 2 & 1 \\ -15 & -20 & 8 & -2 & 0 & 1 \\ -18 & -23 & 9 & -3 & -1 & 1 \end{array}\right] \text{ to obtain } \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & -3 & 2 & -2 \\ 0 & 0 & 1 & -4 & 5 & -3 \end{array}\right],$$

we see that the transition matrix from $C$ to $D$ is

$$\mathbf{Q} = \begin{bmatrix} 2 & 0 & 1 \\ -3 & 2 & -2 \\ -4 & 5 & -3 \end{bmatrix}.$$

Then, by Theorem 4.18, the product

$$\mathbf{QP} = \begin{bmatrix} 2 & 0 & 1 \\ -3 & 2 & -2 \\ -4 & 5 & -3 \end{bmatrix}\begin{bmatrix} -9 & 3 & -5 \\ 11 & -3 & 6 \\ -14 & 5 & -8 \end{bmatrix} = \begin{bmatrix} -32 & 11 & -18 \\ 77 & -25 & 43 \\ 133 & -42 & 74 \end{bmatrix}$$

is the transition matrix from $B$ to $D$.

We will verify this result in a particular case. Consider $\mathbf{v} = -7x^2 - 2x$. You can easily check that $[\mathbf{v}]_B = [-5, -2, 8]$. Then, by Theorem 4.18,

$$[\mathbf{v}]_D = \mathbf{QP}[\mathbf{v}]_B = \begin{bmatrix} -32 & 11 & -18 \\ 77 & -25 & 43 \\ 133 & -42 & 74 \end{bmatrix} \begin{bmatrix} -5 \\ -2 \\ 8 \end{bmatrix} = \begin{bmatrix} -6 \\ 9 \\ 11 \end{bmatrix}.$$

Notice that $(-6)(-5x^2 - 15x - 18) + 9(-9x^2 - 20x - 23) + 11(4x^2 + 8x + 9)$ does, in fact, equal $\mathbf{v} = -7x^2 - 2x$.

The next theorem shows how to reverse a transition from one basis to another. The proof of this theorem is left as Exercise 15.

**Theorem 4.19** *Let $B$ and $C$ be ordered bases for a nontrivial finite dimensional vector space $\mathcal{V}$, and let $\mathbf{P}$ be the transition matrix from $B$ to $C$. Then $\mathbf{P}$ is nonsingular, and $\mathbf{P}^{-1}$ is the transition matrix from $C$ to $B$.*

Let us return to the situation in Example 11 and use the inverses of the transition matrices to calculate the $B$-coordinates of a polynomial in $\mathcal{P}_2$.

### Example 12

Consider again the bases $B$, $C$, and $D$ in Example 11 and the transition matrices $\mathbf{P}$ from $B$ to $C$ and $\mathbf{Q}$ from $C$ to $D$. From Theorem 4.19, the transition matrices from $C$ to $B$ and from $D$ to $C$, respectively, are

$$\mathbf{P}^{-1} = \begin{bmatrix} -6 & -1 & 3 \\ 4 & 2 & -1 \\ 13 & 3 & -6 \end{bmatrix} \quad \text{and} \quad \mathbf{Q}^{-1} = \begin{bmatrix} 4 & 5 & -2 \\ -1 & -2 & 1 \\ -7 & -10 & 4 \end{bmatrix}.$$

Now,

$$[\mathbf{v}]_B = \mathbf{P}^{-1}[\mathbf{v}]_C = \mathbf{P}^{-1}\left(\mathbf{Q}^{-1}[\mathbf{v}]_D\right) = \left(\mathbf{P}^{-1}\mathbf{Q}^{-1}\right)[\mathbf{v}]_D,$$

and so $\mathbf{P}^{-1}\mathbf{Q}^{-1}$ acts as the transition matrix from $D$ to $B$ (see Fig. 4.8). For example, if $\mathbf{v} = -7x^2 - 2x$ is the vector from Example 11, then

$$\begin{aligned} [\mathbf{v}]_B &= \left(\mathbf{P}^{-1}\mathbf{Q}^{-1}\right)[\mathbf{v}]_D \\ &= \begin{bmatrix} -6 & -1 & 3 \\ 4 & 2 & -1 \\ 13 & 3 & -6 \end{bmatrix} \begin{bmatrix} 4 & 5 & -2 \\ -1 & -2 & 1 \\ -7 & -10 & 4 \end{bmatrix} \begin{bmatrix} -6 \\ 9 \\ 11 \end{bmatrix} = \begin{bmatrix} -5 \\ -2 \\ 8 \end{bmatrix}, \end{aligned}$$

as expected.

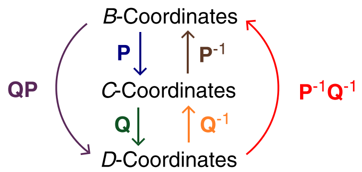


FIGURE 4.8 Transition matrices used to convert between $B$-, $C$-, and $D$-coordinates in $\mathcal{P}_2$

## Diagonalization and the Transition Matrix

The matrix $\mathbf{P}$ obtained in the process of diagonalizing an $n \times n$ matrix turns out to be a transition matrix between two different bases for $\mathbb{R}^n$, as we see in the next example.

**Example 13**

Consider

$$\mathbf{A} = \begin{bmatrix} 14 & -15 & -30 \\ 6 & -7 & -12 \\ 3 & -3 & -7 \end{bmatrix}.$$

A quick calculation produces $p_{\mathbf{A}}(x) = x^3 - 3x - 2 = (x-2)(x+1)^2$. Row reducing $(2\mathbf{I}_3 - \mathbf{A})$ yields a fundamental eigenvector $\mathbf{v}_1 = [5, 2, 1]$. The set $\{\mathbf{v}_1\}$ is a basis for the eigenspace $E_2$. Similarly, we row reduce $(-1\mathbf{I}_3 - \mathbf{A})$ to obtain fundamental eigenvectors $\mathbf{v}_2 = [1, 1, 0]$ and $\mathbf{v}_3 = [2, 0, 1]$. The set $\{\mathbf{v}_2, \mathbf{v}_3\}$ forms a basis for the eigenspace $E_{-1}$.

Let $B = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$. These vectors are linearly independent (see the remarks before Example 15 in Section 4.5), and thus $B$ is a basis for $\mathbb{R}^3$ by Theorem 4.12. Let $S$ be the standard basis. Then, the transition matrix $\mathbf{P}$ from $B$ to $S$ is simply the matrix whose columns are the vectors in $B$ (see part (b) of Exercise 8), and so

$$\mathbf{P} = \begin{bmatrix} 5 & 1 & 2 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Notice that this transition matrix is the very same matrix $\mathbf{P}$ created by the Diagonalization Method of Section 3.4!

Now, by Theorem 4.19,

$$\mathbf{P}^{-1} = \begin{bmatrix} 1 & -1 & -2 \\ -2 & 3 & 4 \\ -1 & 1 & 3 \end{bmatrix}$$

is the transition matrix from $S$ to $B$. Finally, recall from Section 3.4 that $\mathbf{P}^{-1}\mathbf{AP}$ is a diagonal matrix $\mathbf{D}$ with the eigenvalues of $\mathbf{A}$ on the main diagonal—namely,

$$\mathbf{D} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Example 13 illustrates the following general principle:

When the Diagonalization Method of Section 3.4 is successfully performed on a matrix $\mathbf{A}$, the matrix $\mathbf{P}$ obtained is the transition matrix from $B$-coordinates to standard coordinates, where $B$ is an ordered basis for $\mathbb{R}^n$ consisting of eigenvectors for $\mathbf{A}$.

We can understand the relationship between $\mathbf{A}$ and $\mathbf{D}$ in Example 13 more fully from a "change of coordinates" perspective. In fact, if $\mathbf{v}$ is any vector in $\mathbb{R}^3$ expressed in standard coordinates, we claim that $\mathbf{D}[\mathbf{v}]_B = [\mathbf{Av}]_B$. That is, multiplication by $\mathbf{D}$ when working in $B$-coordinates corresponds to first multiplying by $\mathbf{A}$ in standard coordinates, and then converting the result to $B$-coordinates (see Fig. 4.9).

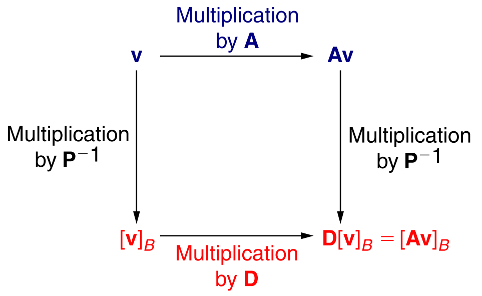


**FIGURE 4.9** Multiplication by $\mathbf{A}$ in standard coordinates corresponds to multiplication by $\mathbf{D}$ in $B$-coordinates

Why does this relationship hold? Well,

$$\mathbf{D}[\mathbf{v}]_B = (\mathbf{P}^{-1}\mathbf{AP})[\mathbf{v}]_B = (\mathbf{P}^{-1}\mathbf{A})\mathbf{P}[\mathbf{v}]_B = \mathbf{P}^{-1}\mathbf{A}[\mathbf{v}]_S = \mathbf{P}^{-1}(\mathbf{Av}) = [\mathbf{Av}]_B$$

because multiplication by $\mathbf{P}$ and $\mathbf{P}^{-1}$ perform the appropriate transitions between $B$- and $S$-coordinates. Thus, we can think of $\mathbf{D}$ as being the "$B$-coordinates version" of $\mathbf{A}$. By using a basis of eigenvectors we have converted to a new coordinate system in which multiplication by $\mathbf{A}$ has been replaced with multiplication by a diagonal matrix, which is much easier to work with because of its simpler form.

♦ **Application**: You have now covered the prerequisites for Section 8.7, "Rotation of Axes for Conic Sections."

## New Vocabulary

commutative diagram
coordinatization (of a vector with respect to an ordered basis)
Coordinatization Method
ordered basis
transition matrix (from one ordered basis to another)
Transition Matrix Method

## Highlights

- If a vector space has an ordered basis $B = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$, and if $\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$, then $\mathbf{v}$ has a unique coordinatization $[\mathbf{v}]_B = [a_1, a_2, \ldots, a_n]$ in $\mathbb{R}^n$ with respect to $B$.
- If $\mathcal{V}$ is a nontrivial subspace of $\mathbb{R}^n$ with an ordered basis $B = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$, the Coordinatization Method calculates the coordinatization with respect to $B$ of a vector $\mathbf{v} \in \mathcal{V}$ by row reducing the matrix whose first columns are $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, and whose last column is $\mathbf{v}$.
- The coordinatization with respect to an ordered basis $B$ of a linear combination of vectors, $[a_1\mathbf{w}_1 + a_2\mathbf{w}_2 + \cdots + a_k\mathbf{w}_k]_B$, is equal to the corresponding linear combination of the respective coordinatizations of the vectors: $a_1[\mathbf{w}_1]_B + a_2[\mathbf{w}_2]_B + \cdots + a_k[\mathbf{w}_k]_B$.
- The transition matrix from $B$-coordinates to $C$-coordinates is the matrix whose $i$th column is $[\mathbf{b}_i]_C$, where $\mathbf{b}_i$ is the $i$th basis vector in $B$.
- The transition matrix from $B$-coordinates to $C$-coordinates can be computed using the Transition Method: row reducing the matrix whose first columns are the vectors in $C$ and whose last columns are the vectors in $B$.
- If $B$ and $C$ are finite bases for a nontrivial vector space $\mathcal{V}$, and $\mathbf{v} \in \mathcal{V}$, then a change of coordinates from $B$ to $C$ can be obtained by multiplying by the transition matrix: that is, $[\mathbf{v}]_C = \mathbf{P}[\mathbf{v}]_B$, where $\mathbf{P}$ is the transition matrix from $B$-coordinates to $C$-coordinates.
- If $\mathbf{P}$ is the transition matrix from $B$ to $C$, and $\mathbf{Q}$ is the transition matrix from $C$ to $D$, then $\mathbf{QP}$ is the transition matrix from $B$ to $D$, and $\mathbf{P}^{-1}$ is the transition matrix from $C$ to $B$.
- When the Diagonalization Method is applied to a matrix $\mathbf{A}$ to create an (ordered) set $B$ of fundamental eigenvectors, and a diagonal matrix $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$, then the matrix $\mathbf{P}$ (whose columns are the vectors in $B$) is, in fact, the transition matrix from $B$-coordinates to standard coordinates.

## Exercises for Section 4.7

1. In each part, let $B$ represent an ordered basis for a subspace $\mathcal{V}$ of $\mathbb{R}^n$, $\mathcal{P}_n$, or $\mathcal{M}_{mn}$. Find $[\mathbf{v}]_B$, for the given $\mathbf{v} \in \mathcal{V}$.
   - ★ **(a)** $B = ([1, -4, 1], [5, -7, 2], [0, -4, 1])$; $\mathbf{v} = [2, -1, 0]$
   - **(b)** $B = ([4, 6, 0, 1], [5, 1, -1, 0], [0, 15, 1, 3], [1, 5, 0, 1])$; $\mathbf{v} = [4, -7, 2, -2]$
   - ★ **(c)** $B = ([2, 3, 1, -2, 2], [4, 3, 3, 1, -1], [1, 2, 1, -1, 1])$; $\mathbf{v} = [7, -4, 5, 13, -13]$
   - **(d)** $B = ([2, 5, 29, 2, 1], [2, 9, 49, 4, 2], [5, 2, 20, 1, 2], [2, 2, 14, 1, 1])$; $\mathbf{v} = [15, 7, 65, 3, 5]$
   - ★ **(e)** $B = (3x^2 - x + 2, x^2 + 2x - 3, 2x^2 + 3x - 1)$; $\mathbf{v} = 13x^2 - 5x + 20$
   - **(f)** $B = (2x^2 + 7x + 9, 3x^2 + 11x + 7, x^2 + 4x + 3)$; $\mathbf{v} = 2x^2 + 7x + 19$
   - ★ **(g)** $B = (2x^3 - x^2 + 3x - 1, x^3 + 2x^2 - x + 3, -3x^3 - x^2 + x + 1)$; $\mathbf{v} = 8x^3 + 11x^2 - 9x + 11$
   - ★ **(h)** $B = \left(\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix}\right)$; $\mathbf{v} = \begin{bmatrix} -3 & -2 \\ 0 & 3 \end{bmatrix}$
   - **(i)** $B = \left(\begin{bmatrix} -1 & 3 \\ 11 & 24 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -4 & -2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 11 \end{bmatrix}\right)$; $\mathbf{v} = \begin{bmatrix} -8 & -7 \\ 12 & -17 \end{bmatrix}$

★ (j) $B = \left( \begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 4 \end{bmatrix}, \begin{bmatrix} -3 & 1 & 7 \\ 1 & 2 & 5 \end{bmatrix} \right)$; $\mathbf{v} = \begin{bmatrix} 11 & 13 & -19 \\ 8 & 1 & 10 \end{bmatrix}$

2. In each part, ordered bases $B$ and $C$ are given for a subspace of $\mathbb{R}^n$, $\mathcal{P}_n$, or $\mathcal{M}_{mn}$. Find the transition matrix from $B$ to $C$.

★ (a) $B = ([1,0,0],[0,1,0],[0,0,1])$; $C = ([1,5,1],[1,6,-6],[1,3,14])$

(b) $B = ([3,1,-2],[8,1,3],[0,-2,5])$; $C = ([1,0,2],[5,2,5],[2,1,2])$

★ (c) $B = (2x^2+3x-1, 8x^2+x+1, x^2+6)$; $C = (x^2+3x+1, 3x^2+4x+1, 10x^2+17x+5)$

★ (d) $B = \left( \begin{bmatrix} 1 & 3 \\ 5 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ -4 & 1 \end{bmatrix} \right)$; $C = \left( \begin{bmatrix} -1 & 1 \\ 3 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 3 & -4 \\ -7 & 4 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix} \right)$

(e) $B = ([3,-2,12,1,-11],[4,-1,11,2,-5],[2,-1,7,1,-5])$; $C = ([2,1,1,1,5],[0,-2,6,-1,-12],[3,-4,18,-1,-25])$

★ (f) $B = (6x^4+20x^3+7x^2+19x-4, x^4+5x^3+7x^2-x+6, 5x^3+17x^2-10x+19)$; $C = (x^4+3x^3+4x-2, 2x^4+7x^3+4x^2+3x+1, 2x^4+5x^3-3x^2+8x-7)$

(g) $B = \left( \begin{bmatrix} 3 & 2 & 3 \\ 19 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 4 & 1 & 4 \\ 17 & 5 & -9 \end{bmatrix} \right)$; $C = \left( \begin{bmatrix} 4 & 6 & 4 \\ 42 & -10 & 26 \end{bmatrix}, \begin{bmatrix} 3 & 4 & 3 \\ 29 & -6 & 16 \end{bmatrix} \right)$

3. Draw the $B$-coordinate grid in $\mathbb{R}^2$ as in Example 3, where $B = ([3,2],[-2,1])$. Plot the point $(2,6)$. Convert this point to $B$-coordinates, and show that it is at the proper place on the $B$-coordinate grid.

4. In each part of this exercise, ordered bases $B$, $C$, and $D$ are given for $\mathbb{R}^n$ or $\mathcal{P}_n$. Calculate the following independently:

(i) The transition matrix $\mathbf{P}$ from $B$ to $C$

(ii) The transition matrix $\mathbf{Q}$ from $C$ to $D$

(iii) The transition matrix $\mathbf{T}$ from $B$ to $D$

Then verify Theorem 4.18 by showing that $\mathbf{T} = \mathbf{QP}$.

★ (a) $B = ([3,1],[7,2])$; $C = ([3,7],[2,5])$; $D = ([5,2],[2,1])$

(b) $B = ([3,-1,0],[2,1,2],[2,0,1])$; $C = ([2,-1,0],[-1,1,0],[0,1,1])$; $D = ([4,8,5],[5,8,3],[2,3,1])$

★ (c) $B = (x^2+2x+2, 3x^2+7x+8, 3x^2+9x+13)$; $C = (x^2+4x+1, 2x^2+x, x^2)$; $D = (7x^2-3x+2, x^2+7x-3, x^2-2x+1)$

(d) $B = (x^3+3x^2+10x+21, x^2+2x+6, 6x^3+5x^2-6x+14, x^3+x^2-x+3)$; $C = (x^2+2x+6, x^3+3x^2+10x+21, x^3+x^2-x+3, 6x^3+5x^2-6x+14)$; $D = (6x^3+5x^2-6x+14, x^3+x^2-x+3, x^3+3x^2+10x+21, x^2+2x+6)$

5. In each part of this exercise, an ordered basis $B$ is given for a subspace $\mathcal{V}$ of $\mathbb{R}^n$. Perform the following steps:

(i) Use the Simplified Span Method to find a second ordered basis $C$.

(ii) Find the transition matrix $\mathbf{P}$ from $B$ to $C$.

(iii) Use Theorem 4.19 to find the transition matrix $\mathbf{Q}$ from $C$ to $B$.

(iv) For the given vector $\mathbf{v} \in \mathcal{V}$, independently calculate $[\mathbf{v}]_B$ and $[\mathbf{v}]_C$.

(v) Check your answer to step (iv) by using $\mathbf{Q}$ and $[\mathbf{v}]_C$ to calculate $[\mathbf{v}]_B$.

★ (a) $B = ([1,-4,1,2,1],[6,-24,5,8,3],[3,-12,3,6,2])$; $\mathbf{v} = [2,-8,-2,-12,3]$

(b) $B = ([-2,5,1,1],[-5,13,4,3],[3,0,1,-4],[1,-6,-1,0])$; $\mathbf{v} = [13,-10,1,-16]$

★ (c) $B = ([3,-1,4,6],[6,7,-3,-2],[-4,-3,3,4],[-2,0,1,2])$; $\mathbf{v} = [10,14,3,12]$

(d) $B = ([-5,6,-5,45,-32],[7,-8,7,-61,44])$; $\mathbf{v} = [-1,2,-1,13,-8]$

6. Let $B = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ be an ordered basis for a vector space $\mathcal{V}$. Prove that for each $i$, $[\mathbf{v}_i]_B = \mathbf{e}_i$.

7. This exercise explores the transition matrix between two ordered bases that contain the same vectors, but in a different order.

★ (a) Let $\mathbf{u} = [-5,9,-1]$, $\mathbf{v} = [3,-9,2]$, and $\mathbf{w} = [2,-5,1]$. Find the transition matrix from the ordered basis $B = (\mathbf{u},\mathbf{v},\mathbf{w})$ to each of the following ordered bases: $C_1 = (\mathbf{v},\mathbf{w},\mathbf{u})$, $C_2 = (\mathbf{w},\mathbf{u},\mathbf{v})$, $C_3 = (\mathbf{u},\mathbf{w},\mathbf{v})$, $C_4 = (\mathbf{v},\mathbf{u},\mathbf{w})$, $C_5 = (\mathbf{w},\mathbf{v},\mathbf{u})$.

(b) Let $B$ be an ordered basis for an $n$-dimensional vector space $\mathcal{V}$. Let $C$ be another ordered basis for $\mathcal{V}$ with the same vectors as $B$ but rearranged in a different order. Prove that the transition matrix from $B$ to $C$ is obtained by rearranging rows of $\mathbf{I}_n$ in exactly the same fashion.

**8.** Let $B$ and $C$ be ordered bases for $\mathbb{R}^n$.

**(a)** Show that if $B$ is the standard basis in $\mathbb{R}^n$, then the transition matrix from $B$ to $C$ is given by

$$\begin{bmatrix} \text{1st} & \text{2nd} & & n\text{th} \\ \text{vector} & \text{vector} & \cdots & \text{vector} \\ \text{in} & \text{in} & & \text{in} \\ C & C & & C \end{bmatrix}^{-1}.$$

**(b)** Show that if $C$ is the standard basis in $\mathbb{R}^n$, then the transition matrix from $B$ to $C$ is given by

$$\begin{bmatrix} \text{1st} & \text{2nd} & & n\text{th} \\ \text{vector} & \text{vector} & \cdots & \text{vector} \\ \text{in} & \text{in} & & \text{in} \\ B & B & & B \end{bmatrix}.$$

**9.** Let $B$ and $C$ be ordered bases for $\mathbb{R}^n$. Let $\mathbf{P}$ be the matrix whose columns are the vectors in $B$ and let $\mathbf{Q}$ be the matrix whose columns are the vectors in $C$. Prove that the transition matrix from $B$ to $C$ equals $\mathbf{Q}^{-1}\mathbf{P}$. (Hint: Use Exercise 8.)

★ **10.** Consider the ordered basis $B = ([-2,1,3],[1,0,2],[-13,5,10])$ for $\mathbb{R}^3$. Suppose that $C$ is another ordered basis for $\mathbb{R}^3$ and that the transition matrix from $B$ to $C$ is given by

$$\begin{bmatrix} 1 & 9 & -1 \\ 2 & 13 & -11 \\ -1 & -8 & 3 \end{bmatrix}.$$

Find $C$. (Hint: Use Exercise 9.)

**11.** This exercise is related to Example 13.

**(a)** Verify all of the computations in Example 13, including the computation of $p_{\mathbf{A}}(x)$, the eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, the transition matrix $\mathbf{P}$, and its inverse $\mathbf{P}^{-1}$. Check that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$.

★ **(b)** Let $\mathbf{v} = [1,4,-2]$. With $B$, $\mathbf{A}$, and $\mathbf{D}$ as in Example 13, compute $\mathbf{D}[\mathbf{v}]_B$ and $[\mathbf{Av}]_B$ independently, without using multiplication by the matrices $\mathbf{P}$ or $\mathbf{P}^{-1}$ in that example. Compare your results.

**12.** Let $\mathbf{A} = \begin{bmatrix} -1 & -5 & 11 \\ 2 & 3 & -8 \\ 4 & -1 & -2 \end{bmatrix}$.

**(a)** Find all the eigenvalues for $\mathbf{A}$ and fundamental eigenvectors for each eigenvalue.

**(b)** Find a diagonal matrix $\mathbf{D}$ similar to $\mathbf{A}$.

**(c)** Let $B$ be the set of fundamental eigenvectors found in part (a). From the answer to part (a), find the transition matrix from $B$ to the standard basis without row reducing.

▶ **13.** Prove Theorem 4.16. (Hint: Use a proof by induction for part (3).)

▶ **14.** Prove Theorem 4.18. (Hint: Use Theorem 4.17.)

**15.** Prove Theorem 4.19. (Hint: Let $\mathbf{Q}$ be the transition matrix from $C$ to $B$. Prove that $\mathbf{QP} = \mathbf{I}$ by using Theorems 4.17 and 4.18.)

**16.** Suppose $\mathcal{V}$ is a nontrivial $n$-dimensional vector space, $B$ is a basis for $\mathcal{V}$, and $\mathbf{P}$ is a nonsingular $n \times n$ matrix. Prove that there is a basis $C$ for $\mathcal{V}$ such that $\mathbf{P}$ is the transition matrix from $B$ to $C$.

**17.** Let $\mathcal{V}$ be the unusual vector space of Example 7 in Section 4.1, in which the set of vectors is $\mathbb{R}^+$ (positive reals), with $\mathbf{v}_1 \oplus \mathbf{v}_2 = v_1 v_2$, and $a \odot \mathbf{v} = v^a$. Notice that all elements of $\mathcal{V}$ are scalar multiples of 2, since every positive real number $v = 2^{\log_2 v} = \log_2 v \odot 2$. Therefore, $\dim(\mathcal{V}) = 1$. Consider the ordered basis $B = (2)$ for $\mathcal{V}$.

**(a)** Compute $[4]_B$.

**(b)** Compute $[8]_B$.

**(c)** Find a general formula for $[\mathbf{v}]_B$.

**(d)** Given $[\mathbf{v}]_B$, find a formula for $\mathbf{v} \in \mathcal{V}$.

**(e)** If $C$ is the ordered basis $(3)$ for $\mathcal{V}$, find the transition matrix $\mathbf{P}$ from $B$-coordinates to $C$-coordinates.

★ **18.** True or False:

**(a)** For the ordered bases $B = (\mathbf{i},\mathbf{j},\mathbf{k})$ and $C = (\mathbf{j},\mathbf{k},\mathbf{i})$ for $\mathbb{R}^3$, we have $[\mathbf{v}]_B = [\mathbf{v}]_C$ for each $\mathbf{v} \in \mathbb{R}^3$.

(b) If $B$ is a finite ordered basis for $\mathcal{V}$ and $\mathbf{b}_i$ is the $i$th vector in $B$, then $[\mathbf{b}_i]_B = \mathbf{e}_i$.
(c) If $B = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ and $C = (\mathbf{c}_1, \ldots, \mathbf{c}_n)$ are ordered bases for a vector space $\mathcal{V}$, then the $i$th column of the transition matrix $\mathbf{Q}$ from $C$ to $B$ is $[\mathbf{c}_i]_B$.
(d) If $B$ and $C$ are ordered bases for a finite dimensional vector space $\mathcal{V}$ and $\mathbf{P}$ is the transition matrix from $B$ to $C$, then $\mathbf{P}[\mathbf{v}]_C = [\mathbf{v}]_B$ for every vector $\mathbf{v} \in \mathcal{V}$.
(e) If $B$, $C$, and $D$ are finite ordered bases for a vector space $\mathcal{V}$, $\mathbf{P}$ is the transition matrix from $B$ to $C$, and $\mathbf{Q}$ is the transition matrix from $C$ to $D$, then $\mathbf{PQ}$ is the transition matrix from $B$ to $D$.
(f) If $B$ and $C$ are ordered bases for a finite dimensional vector space $\mathcal{V}$ and if $\mathbf{P}$ is the transition matrix from $B$ to $C$, then $\mathbf{P}$ is nonsingular.
(g) If the Diagonalization Method is applied to a square matrix $\mathbf{A}$ to create a diagonal matrix $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$, then $\mathbf{P}$ is the transition matrix from standard coordinates to an ordered basis of eigenvectors for $\mathbf{A}$.

## Review Exercises for Chapter 4

1. Determine whether the subset $\{[x_1, x_2, x_3] \mid x_1 > 0,\ x_2 > 0,\ x_3 > 0\}$ of $\mathbb{R}^3$ is a vector space under the operations $[x_1, x_2, x_3] \oplus [y_1, y_2, y_3] = [x_1y_1, x_2y_2, x_3y_3]$, and $c \odot [x_1, x_2, x_3] = [(x_1)^c, (x_2)^c, (x_3)^c]$.

★ 2. Use parts (2) and (3) of Theorem 4.1 to find the zero vector $\mathbf{0}$ and the additive inverse of each vector $\mathbf{v} = [x, y]$ for the vector space $\mathbb{R}^2$ with operations $[x, y] \oplus [w, z] = [x + w + 4,\ y + z - 5]$ and $a \odot [x, y] = [ax + 4a - 4,\ ay - 5a + 5]$.

3. Which of the following subsets of the given vector spaces are subspaces? If so, prove it. If not, explain why not.
   ★ (a) $\{[3a, 2a - 1, -4a] \mid a \in \mathbb{R}\}$ in $\mathbb{R}^3$
   (b) The plane $2x + 4y - 2z = 0$ in $\mathbb{R}^3$
   ★ (c) $\left\{ \begin{bmatrix} 2a + b & -4a - 5b \\ 0 & a - 2b \end{bmatrix} \middle| \ a, b \in \mathbb{R} \right\}$ in $\mathcal{M}_{22}$
   ★ (d) All matrices that are both singular and symmetric in $\mathcal{M}_{22}$
   (e) $\{ax^3 - bx^2 + (c + 3a^2)x \mid a, b, c \in \mathbb{R}\}$ in $\mathcal{P}_3$
   ★ (f) All polynomials whose highest order nonzero term has even degree in $\mathcal{P}_4$
   (g) All functions $f$ with $y$-intercept $= 1$, in the vector space of all real-valued functions with domain $\mathbb{R}$
   (h) The line $2x - 3 = 5y + 2 = 3z + 4$ in $\mathbb{R}^3$

★ 4. For the subset $S = \{[3, 3, -2, 4], [3, 4, 0, 3], [5, 6, -1, 6], [4, 4, -3, 5]\}$ of $\mathbb{R}^4$:
   (a) Use the Simplified Span Method to find a simplified form for the vectors in span($S$). Does $S$ span $\mathbb{R}^4$?
   (b) Give a basis for span($S$). What is dim(span($S$))?

5. For the subset $S = \{2x^3 + x^2 + 11x - 4,\ x^3 + x^2 + 3x - 1,\ x^3 + 3x^2 - 7x + 3,\ 2x^3 + 3x^2 + x\}$ of $\mathcal{P}_3$:
   (a) Use the Simplified Span Method to find a simplified form for the vectors in span($S$). Does $S$ span $\mathcal{P}_3$?
   (b) Give a basis for span($S$). What is dim(span($S$))?

6. For the subset $S = \left\{ \begin{bmatrix} 16 & 4 & 25 \\ 9 & 3 & 1 \end{bmatrix}, \begin{bmatrix} 8 & -2 & 11 \\ 7 & -2 & 2 \end{bmatrix}, \begin{bmatrix} 1 & -2 & 1 \\ 2 & -2 & 1 \end{bmatrix}, \begin{bmatrix} 7 & -7 & 5 \\ 9 & -7 & 4 \end{bmatrix}, \begin{bmatrix} 9 & 4 & 6 \\ 3 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 9 & 4 & 4 \\ 2 & 5 & -1 \end{bmatrix}, \begin{bmatrix} 4 & 4 & 1 \\ 3 & -7 & 0 \end{bmatrix} \right\}$ of $\mathcal{M}_{23}$:
   (a) Use the Simplified Span Method to find a simplified form for the vectors in span($S$). Does $S$ span $\mathcal{M}_{23}$?
   (b) Give a basis for span($S$). What is dim(span($S$))?

★ 7. For the subset $S = \{[3, 5, -3], [-2, -4, 3], [1, 2, -1]\}$ of $\mathbb{R}^3$:
   (a) Use the Independence Test Method to determine whether $S$ is linearly independent. If $S$ is linearly dependent, show how to express one vector in the set as a linear combination of the others.
   (b) Find a subset of $S$ that is a basis for span($S$). Does $S$ span $\mathbb{R}^3$?
   (c) The vector $\mathbf{v} = [11, 20, -12] = 2[3, 5, -3] - 1[-2, -4, 3] + 3[1, 2, -1]$ is in span($S$). Is there a different linear combination of the vectors in $S$ that produces $\mathbf{v}$?

★ 8. For the subset $S = \{-5x^3 + 2x^2 + 5x - 2,\ 2x^3 - x^2 - 2x + 1,\ x^3 - 2x^2 - x + 2,\ -2x^3 + 2x^2 + 3x - 5\}$ of $\mathcal{P}_3$:
   (a) Use the Independence Test Method to determine whether $S$ is linearly independent. If $S$ is linearly dependent, show how to express one vector in the set as a linear combination of the others.
   (b) Find a subset of $S$ that is a basis for span($S$). Does $S$ span $\mathcal{P}_3$?
   (c) The vector $\mathbf{v} = 18x^3 - 9x^2 - 19x + 12 = -2(-5x^3 + 2x^2 + 5x - 2) + 3(2x^3 - x^2 - 2x + 1) - 1(-2x^3 + 2x^2 + 3x - 5)$ is in span($S$). Is there a different linear combination of the vectors in $S$ that produces $\mathbf{v}$?

**9.** For the subset $S = \left\{ \begin{bmatrix} 1 & 2 \\ -1 & 1 \\ -1 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 4 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & -8 \\ 11 & 9 \\ 8 & -12 \end{bmatrix}, \begin{bmatrix} 10 & 2 \\ 5 & 9 \\ 4 & 4 \end{bmatrix}, \begin{bmatrix} 8 & 4 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 25 & 2 \\ 20 & 9 \\ 9 & -21 \end{bmatrix} \right\}$ of $\mathcal{M}_{32}$:

- **(a)** Use the Independence Test Method to determine whether $S$ is linearly independent. If $S$ is linearly dependent, show how to express one vector in the set as a linear combination of the others.
- **(b)** Find a subset of $S$ that is a basis for span($S$). Does $S$ span $\mathcal{M}_{32}$?
- **(c)** The vector $\mathbf{v} = \begin{bmatrix} 0 & -4 \\ 3 & -1 \\ 3 & -11 \end{bmatrix} = 2\begin{bmatrix} 1 & 2 \\ -1 & 1 \\ -1 & 5 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 2 & 4 \\ 1 & 3 \end{bmatrix} + 1\begin{bmatrix} 1 & -8 \\ 11 & 9 \\ 8 & -12 \end{bmatrix}$ is in span($S$). Is there a different linear combination of the vectors in $S$ that produces $\mathbf{v}$?

**10.** If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a finite subset of a vector space $\mathcal{V}$, and $\mathbf{v} \in \text{span}(S)$, with $\mathbf{v} \notin S$, prove that some vector in $T = S \cup \{\mathbf{v}\}$ can be expressed in more than one way as a linear combination of vectors in $T$.

**11.** Show that $\{x, x^3 + x, x^5 + x^3, x^7 + x^5, \ldots\}$ is a linearly independent subset of $\mathcal{P}$.

**12.** Prove:

- ★ **(a)** $\{[-2, 3, -1, 4], [3, -3, 2, -4], [-2, 2, -1, 3], [3, -5, 0, -7]\}$ is a basis for $\mathbb{R}^4$.
- **(b)** $\{2x^2 - 2x + 11,\ -3x^2 - x - 11,\ x^2 + x + 3\}$ is a basis for $\mathcal{P}_2$.
- **(c)** $\left\{ \begin{bmatrix} 4 & 1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 4 \\ 2 & -1 \end{bmatrix}, \begin{bmatrix} -3 & 3 \\ 2 & -1 \end{bmatrix}, \begin{bmatrix} 3 & 17 \\ 3 & 2 \end{bmatrix} \right\}$ is a basis for $\mathcal{M}_{22}$.

★ **13.** Let $\mathcal{W}$ be the solution set to $\mathbf{AX} = \mathbf{0}_4$, where $\mathbf{A} = \begin{bmatrix} 5 & -15 & 2 & 8 \\ -3 & 9 & -1 & -5 \\ 2 & -6 & 1 & 3 \end{bmatrix}$.

- **(a)** Show that $\mathcal{W}$ is a subspace of $\mathbb{R}^4$.
- **(b)** Find a basis for $\mathcal{W}$.
- **(c)** Show that $\dim(\mathcal{W}) + \text{rank}(\mathbf{A}) = 4$.

★ **14.** Consider the subset $B = \{x^3 - 3x, x^2 - 2x, 1\}$ of $\mathcal{V} = \{\mathbf{p} \in \mathcal{P}_3 \mid \mathbf{p}'(1) = 0\}$.

- **(a)** Show that $B$ is a basis for $\mathcal{V}$. What is $\dim(\mathcal{V})$?
- **(b)** Find a basis for $\mathcal{W} = \{\mathbf{p} \in \mathcal{P}_3 \mid \mathbf{p}'(1) = \mathbf{p}''(1) = 0\}$. What is $\dim(\mathcal{W})$?

★ **15.** Consider the subset $S = \{[2, -3, 0, 1], [-6, 9, 0, -3], [4, 3, 0, 4], [8, -3, 0, 6], [1, 0, 2, 1]\}$ of $\mathbb{R}^4$. Let $\mathcal{V} = \text{span}(S)$. Use the "inspection" method from Exercise 6 of Section 4.6 to find a subset $T$ of $S$ that is a basis for $\mathcal{V}$.

**16.** Consider the subset $S = \{x^2 - x, x^3 + 2x, x^3 + x^2 + x, 3x^3 - x^2 + 4x + 5, 6x^3 - 2x^2 + 8x + 10\}$ of $\mathcal{P}_3$. Let $\mathcal{V} = \text{span}(S)$. Use the "inspection" method from Exercise 6 of Section 4.6 to find a subset $T$ of $S$ that is a basis for $\mathcal{V}$.

★ **17.** Use the Enlarging Method to enlarge the linearly independent set $T = \{[2, 1, -1, 2], [1, -2, 2, -4]\}$ to a basis for $\mathbb{R}^4$.

**18.** Use the Enlarging Method to enlarge the linearly independent set $T = \left\{ \begin{bmatrix} 3 & -1 \\ 0 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ 0 & -1 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ 0 & 1 \\ 0 & 4 \end{bmatrix} \right\}$ to a basis for $\mathcal{M}_{32}$.

**19.** Consider the set $S$ of all polynomials in $\mathcal{P}_4$ of the form $\{\mathbf{p} \in \mathcal{P}_4 \mid \mathbf{p} = ax^4 + bx^3 + (3a - 2b)x^2 + (5a - 3b)x + (-2b)\}$. Find a subset of $S$ that is a basis for span($S$).

**20.** In each case, let $B$ represent an ordered basis $B$ for a subspace $\mathcal{V}$ of $\mathbb{R}^n$, $\mathcal{P}_n$, or $\mathcal{M}_{mn}$. For the given vector $\mathbf{v}$, find $[\mathbf{v}]_B$.

- ★ **(a)** $B = ([2, 1, 2], [5, 0, 1], [-6, 2, 1])$; $\mathbf{v} = [1, -7, -9]$
- **(b)** $B = \{5x^3 - x^2 + 3x + 1, -9x^3 + 3x^2 - 3x - 2, 6x^3 - x^2 + 4x + 1\}$; $\mathbf{v} = 9x^3 + 2x^2 + 13x + 1$
- ★ **(c)** $B = \left\{ \begin{bmatrix} -3 & 3 & 11 \\ 5 & -2 & 2 \end{bmatrix}, \begin{bmatrix} -10 & 3 & 28 \\ 4 & -6 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 11 & 10 \\ 14 & -16 & 3 \end{bmatrix} \right\}$; $\mathbf{v} = \begin{bmatrix} -43 & -5 & 97 \\ -9 & -8 & -9 \end{bmatrix}$

**21.** For the given ordered bases $B$, $C$ (for a subspace $\mathcal{V}$ of $\mathbb{R}^n$, $\mathcal{P}_n$, or $\mathcal{M}_{mn}$), find $[\mathbf{v}]_B$, and the transition matrix $\mathbf{P}$ from $B$ to $C$. Then use $\mathbf{P}$ and $[\mathbf{v}]_B$ to find $[\mathbf{v}]_C$.

- ★ **(a)** $B = ([26, -47, -10], [9, -16, -1], [-3, 10, 37])$; $C = ([2, -3, 4], [-3, 5, -1], [5, -10, -9])$; $\mathbf{v} = [126, -217, 14]$

★ **(b)** $B = (x^2+3x+1, 3x^2+11x+5, -2x^2+4x+4)$; $C = (-7x^2+7x+9, 13x^2-7x-13, -16x^2+18x+22)$; $\mathbf{v} = -13x^2-11x+3$

**(c)** $B = \left(\begin{bmatrix}3 & 1\\2 & 2\end{bmatrix}, \begin{bmatrix}-1 & 3\\-1 & 7\end{bmatrix}, \begin{bmatrix}2 & -1\\3 & -1\end{bmatrix}, \begin{bmatrix}2 & 1\\2 & 4\end{bmatrix}\right)$; $C = \left(\begin{bmatrix}-3 & 19\\-12 & 30\end{bmatrix}, \begin{bmatrix}12 & 3\\11 & 11\end{bmatrix}, \begin{bmatrix}3 & 6\\0 & 12\end{bmatrix}, \begin{bmatrix}0 & -3\\1 & -5\end{bmatrix}\right)$; $\mathbf{v} = \begin{bmatrix}10 & 3\\9 & 7\end{bmatrix}$

**22.** Consider the ordered bases $B = ([3,-1,2,-1], [2,6,1,2], [3,-1,3,1], [2,1,-2,1])$; $C = ([10,5,4,3], [4,-3,7,-1], [15,10,8,6], [18,9,10,5])$; $D = ([5,5,4,3], [6,-2,5,0], [4,7,-1,3], [8,4,6,2])$.

**(a)** Find the transition matrix $\mathbf{P}$ from $B$ to $C$.

**(b)** Find the transition matrix $\mathbf{Q}$ from $C$ to $D$.

**(c)** Verify that the transition matrix $\mathbf{R}$ from $B$ to $D$ is equal to $\mathbf{QP}$.

**(d)** Use the answer to part (c) to find the transition matrix from $D$ to $B$.

**23.** Let $\mathbf{A} = \begin{bmatrix}-30 & -48 & 24\\-32 & -46 & 24\\-104 & -156 & 80\end{bmatrix}$.

**(a)** Find all the eigenvalues for $\mathbf{A}$ and fundamental eigenvectors for each eigenvalue.

**(b)** Find a diagonal matrix $\mathbf{D}$ similar to $\mathbf{A}$.

★ **(c)** Let $B$ be the set of fundamental eigenvectors found in part (a). From the answer to part (a), find the transition matrix from $B$ to the standard basis without row reducing.

**24.** Consider the ordered basis $B = ([1,2,-1,5], [3,7,0,19], [-2,-2,9,-4])$ for a subspace $\mathcal{V}$ of $\mathbb{R}^4$.

**(a)** Use the Simplified Span Method to find a second ordered basis $C$.

**(b)** Find the transition matrix $\mathbf{P}$ from $B$ to $C$.

**(c)** Suppose that $[\mathbf{v}]_C = [-21,-29,68]$ for some vector $\mathbf{v} \in \mathcal{V}$. Use the answer to part (b) to calculate $[\mathbf{v}]_B$.

**(d)** For the vector $\mathbf{v}$ in part (c), what is $\mathbf{v}$ expressed in standard coordinates?

**25.** Let $B$, $C$ be ordered bases for $\mathbb{R}^n$, and let $\mathbf{P}$ be the transition matrix from $B$ to $C$. If $\mathbf{C}$ is the matrix whose columns are the vectors of $C$, show that $\mathbf{CP}$ is the matrix whose columns are the respective vectors of $B$.

★ **26.** True or False:

**(a)** To prove that some set with given operations is not a vector space, we only need to find a single counterexample for one of the ten vector space properties.

**(b)** If $\mathbf{A}$ is an $m \times n$ matrix and $\mathcal{V} = \{\mathbf{X} \in \mathbb{R}^n \mid \mathbf{AX} = \mathbf{0}\}$, then $\mathcal{V}$ is a vector space using the usual operations in $\mathbb{R}^n$.

**(c)** The set of integers is a subspace of $\mathbb{R}$.

**(d)** Every subspace of a vector space contains the zero vector from the vector space.

**(e)** The union of two subspaces of the same vector space is also a subspace of the vector space.

**(f)** If $S$ is a subset of a vector space $\mathcal{V}$, and $S$ contains at least one nonzero vector, then span($S$) is a subspace of $\mathcal{V}$ containing an infinite number of vectors.

**(g)** If $S$ is a complete set of fundamental eigenvectors found for an eigenvalue $\lambda$ using the Diagonalization Method, then $S$ spans $E_\lambda$.

**(h)** If $S_1$ and $S_2$ are two nonempty subsets of a vector space having no vectors in common, then $\text{span}(S_1) \neq \text{span}(S_2)$.

**(i)** Performing the Simplified Span Method on a subset $S$ of $\mathbb{R}^n$ that is already a basis for $\mathbb{R}^n$ will yield the same set $S$.

**(j)** Performing the Independence Test Method on a subset $T$ of $\mathbb{R}^n$ that is already a basis for $\mathbb{R}^n$ will yield the same set $T$.

**(k)** The set $\{1\}$ is a linearly independent subset of the vector space $\mathcal{V} = \mathbb{R}^+$ under the operations $\mathbf{v}_1 \oplus \mathbf{v}_2 = v_1 \cdot v_2$ and $a \odot \mathbf{v} = v^a$ discussed in Example 7 in Section 4.1.

**(l)** Every set of distinct eigenvectors of an $n \times n$ matrix corresponding to the same eigenvalue is linearly independent.

**(m)** The rows of a nonsingular matrix form a linearly independent set of vectors.

**(n)** If $T$ is a linearly independent subset of a vector space $\mathcal{V}$, and $\mathbf{v} \in \mathcal{V}$ with $\mathbf{v} \notin \text{span}(T)$, then $T \cup \{\mathbf{v}\}$ is linearly independent.

**(o)** If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a subset of a vector space such that $a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = \mathbf{0}$, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent.

**(p)** If $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a linearly dependent subset of a vector space, then there is a scalar $c$ such that $\mathbf{v}_2 = c\mathbf{v}_1$.
**(q)** If $T$ is a linearly independent subset of a vector space $\mathcal{V}$, then $T$ is a basis for $\text{span}(T)$.
**(r)** The dimension of the trivial vector space is 1.
**(s)** If $S$ and $T$ are subsets of a finite dimensional vector space $\mathcal{V}$ such that $S$ spans $\mathcal{V}$, $T$ is linearly independent, and $|S| = |T|$, then both $S$ and $T$ are bases for $\mathcal{V}$.
**(t)** If a vector space $\mathcal{V}$ has an infinite dimensional subspace $\mathcal{W}$, then $\mathcal{V}$ is infinite dimensional.
**(u)** $\dim(\mathcal{U}_n) = \frac{n(n+1)}{2}$.
**(v)** If $\mathcal{W}$ is a subspace of a finite dimensional vector space $\mathcal{V}$, and if $B$ is a basis for $\mathcal{W}$, then there is a basis for $\mathcal{V}$ that contains $B$.
**(w)** If $B$ and $C$ are ordered bases for a finite dimensional vector space $\mathcal{V}$ and if $\mathbf{P}$ is the transition matrix from $B$ to $C$, then $\mathbf{P}^T$ is the transition matrix from $C$ to $B$.
**(x)** If $B$ and $C$ are ordered bases for a finite dimensional vector space $\mathcal{V}$ and if $\mathbf{P}$ is the transition matrix from $B$ to $C$, then $\mathbf{P}$ is a square matrix.
**(y)** If $B$ is an ordered basis for $\mathbb{R}^n$, and $S$ is the standard basis for $\mathbb{R}^n$, then the transition matrix from $B$ to $S$ is the matrix whose columns are the vectors in $B$.
**(z)** After a row reduction using the Transition Matrix Method, the desired transition matrix is the matrix to the right of the augmentation bar.

# Chapter 5

# Linear Transformations

### Transforming Space

Consider a particular figure displayed on a computer screen. The edges of the figure can be thought of as vectors. Suppose we want to move or alter the figure in some way, such as translating it to a new position on the screen. We need a means of calculating the new position for each of the original vectors. This suggests that we need another "tool" in our arsenal: functions that move a given set of vectors in a prescribed "linear" manner. Such functions that map the vectors in one vector space "linearly" to those in another are called linear transformations. Just as general vector spaces are abstract generalizations of $\mathbb{R}^n$, we will find that linear transformations are abstract generalizations of matrix multiplication.

In this chapter, we investigate linear transformations and their properties. We show that the effect of any linear transformation between nontrivial finite dimensional vector spaces is equivalent to multiplication by a corresponding matrix. We also introduce the kernel and range of a linear transformation, as well as special types of linear transformations: one-to-one, onto, and isomorphism. Finally, we revisit eigenvalues and eigenvectors in the context of linear transformations.

## 5.1 Introduction to Linear Transformations

In this section, we introduce linear transformations and examine their elementary properties.

### Functions

If you are not familiar with the terms *domain*, *codomain*, *range*, *image*, and *pre-image* in the context of functions, read Appendix B before proceeding. The following example illustrates some of these terms:

**Example 1**

Let $f\colon \mathcal{M}_{23} \to \mathcal{M}_{22}$ be given by

$$f\left(\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\right) = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}.$$

Then $f$ is a function that maps one vector space to another. The domain of $f$ is $\mathcal{M}_{23}$, the codomain of $f$ is $\mathcal{M}_{22}$, and the range of $f$ is the set of all $2 \times 2$ matrices with second row entries equal to zero. The image of $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ under $f$ is $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$. The matrix $\begin{bmatrix} 1 & 2 & 10 \\ 11 & 12 & 13 \end{bmatrix}$ is one of the pre-images of $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ under $f$. Also, the image under $f$ of the set $S$ of all matrices of the form $\begin{bmatrix} 7 & * & * \\ * & * & * \end{bmatrix}$ (where "$*$" represents any real number) is the set $f(S)$ containing all matrices of the form $\begin{bmatrix} 7 & * \\ 0 & 0 \end{bmatrix}$. Finally, the pre-image under $f$ of the set $T$ of all matrices of the form $\begin{bmatrix} a & a+2 \\ 0 & 0 \end{bmatrix}$ is the set $f^{-1}(T)$ consisting of all matrices of the form $\begin{bmatrix} a & a+2 & * \\ * & * & * \end{bmatrix}$.

### Linear Transformations

**Definition** Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces, and let $f\colon \mathcal{V} \to \mathcal{W}$ be a function from $\mathcal{V}$ to $\mathcal{W}$. (That is, for each vector $\mathbf{v} \in \mathcal{V}$, $f(\mathbf{v})$ denotes exactly one vector of $\mathcal{W}$.) Then $f$ is a **linear transformation** if and only if both of the following are true:

**(1)** $f(\mathbf{v}_1 + \mathbf{v}_2) = f(\mathbf{v}_1) + f(\mathbf{v}_2)$, for all $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}$

**(2)** $f(c\mathbf{v}) = cf(\mathbf{v})$, for all $c \in \mathbb{R}$ and all $\mathbf{v} \in \mathcal{V}$.

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Properties (1) and (2) insist that the operations of addition and scalar multiplication give the same result on vectors whether the operations are performed before $f$ is applied (in $\mathcal{V}$) or after $f$ is applied (in $\mathcal{W}$). Thus, a linear transformation is a function between vector spaces that "preserves" the operations that give structure to the spaces. The commutative diagram in Fig. 5.1 illustrates Property (1).

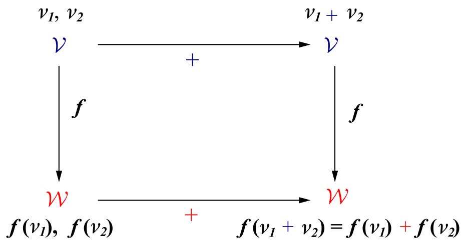


**FIGURE 5.1** $f(\mathbf{v}_1 + \mathbf{v}_2) = f(\mathbf{v}_1) + f(\mathbf{v}_2)$

To determine whether a given function $f$ from a vector space $\mathcal{V}$ to a vector space $\mathcal{W}$ is a linear transformation, we need only verify properties (1) and (2) in the definition, as in the next three examples.

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**Example 2**

Consider the mapping $f\colon \mathcal{M}_{mn} \to \mathcal{M}_{nm}$, given by $f(\mathbf{A}) = \mathbf{A}^T$ for any $m \times n$ matrix $\mathbf{A}$. We will show that $f$ is a linear transformation.

**(1)** We must show that $f(\mathbf{A}_1 + \mathbf{A}_2) = f(\mathbf{A}_1) + f(\mathbf{A}_2)$, for matrices $\mathbf{A}_1, \mathbf{A}_2 \in \mathcal{M}_{mn}$. However, $f(\mathbf{A}_1 + \mathbf{A}_2) = (\mathbf{A}_1 + \mathbf{A}_2)^T = \mathbf{A}_1^T + \mathbf{A}_2^T$ (by part (2) of Theorem 1.13) $= f(\mathbf{A}_1) + f(\mathbf{A}_2)$.

**(2)** We must show that $f(c\mathbf{A}) = cf(\mathbf{A})$, for all $c \in \mathbb{R}$ and for all $\mathbf{A} \in \mathcal{M}_{mn}$. However, $f(c\mathbf{A}) = (c\mathbf{A})^T = c\left(\mathbf{A}^T\right)$ (by part (3) of Theorem 1.13) $= cf(\mathbf{A})$.

Hence, $f$ is a linear transformation.

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**Example 3**

Consider the function $g\colon \mathcal{P}_n \to \mathcal{P}_{n-1}$ given by $g(\mathbf{p}) = \mathbf{p}'$, the derivative of $\mathbf{p}$. We will show that $g$ is a linear transformation.

**(1)** We must show that $g(\mathbf{p}_1 + \mathbf{p}_2) = g(\mathbf{p}_1) + g(\mathbf{p}_2)$, for all $\mathbf{p}_1, \mathbf{p}_2 \in \mathcal{P}_n$. Now, $g(\mathbf{p}_1 + \mathbf{p}_2) = (\mathbf{p}_1 + \mathbf{p}_2)'$. From calculus we know that the derivative of a sum is the sum of the derivatives, so $(\mathbf{p}_1 + \mathbf{p}_2)' = \mathbf{p}_1' + \mathbf{p}_2' = g(\mathbf{p}_1) + g(\mathbf{p}_2)$.

**(2)** We must show that $g(c\mathbf{p}) = cg(\mathbf{p})$, for all $c \in \mathbb{R}$ and $\mathbf{p} \in \mathcal{P}_n$. Now, $g(c\mathbf{p}) = (c\mathbf{p})'$. Again, from calculus we know that the derivative of a constant times a function is equal to the constant times the derivative of the function, so $(c\mathbf{p})' = c(\mathbf{p}') = cg(\mathbf{p})$.

Hence, $g$ is a linear transformation.

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**Example 4**

Let $\mathcal{V}$ be an $n$-dimensional vector space, and let $B$ be an ordered basis for $\mathcal{V}$. Then every element $\mathbf{v} \in \mathcal{V}$ has its coordinatization $[\mathbf{v}]_B$ with respect to $B$. Consider the mapping $f\colon \mathcal{V} \to \mathbb{R}^n$ given by $f(\mathbf{v}) = [\mathbf{v}]_B$. We will show that $f$ is a linear transformation.

Let $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}$. By Theorem 4.16, $[\mathbf{v}_1 + \mathbf{v}_2]_B = [\mathbf{v}_1]_B + [\mathbf{v}_2]_B$. Hence,

$$f(\mathbf{v}_1 + \mathbf{v}_2) = [\mathbf{v}_1 + \mathbf{v}_2]_B = [\mathbf{v}_1]_B + [\mathbf{v}_2]_B = f(\mathbf{v}_1) + f(\mathbf{v}_2).$$

Next, let $c \in \mathbb{R}$ and $\mathbf{v} \in \mathcal{V}$. Again by Theorem 4.16, $[c\mathbf{v}]_B = c[\mathbf{v}]_B$. Hence,

$$f(c\mathbf{v}) = [c\mathbf{v}]_B = c[\mathbf{v}]_B = cf(\mathbf{v}).$$

Thus, $f$ is a linear transformation from $\mathcal{V}$ to $\mathbb{R}^n$.

---

Not every function between vector spaces is a linear transformation. For example, consider the function $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $h([x, y]) = [x+1, y-2] = [x, y] + [1, -2]$. In this case, $h$ merely adds $[1, -2]$ to each vector $[x, y]$ (see Fig. 5.2). This type of mapping is called a **translation**. However, $h$ is not a linear transformation. To show that it is not, we have to produce a counterexample to verify that either property (1) or property (2) of the definition fails. Property (1) fails, since $h([1, 2] + [3, 4]) = h([4, 6]) = [5, 4]$, while $h([1, 2]) + h([3, 4]) = [2, 0] + [4, 2] = [6, 2]$.

In general, when given a function $f$ between vector spaces, we do not always know right away whether $f$ is a linear transformation. If we suspect that either property (1) or (2) does not hold for $f$, then we look for a counterexample.

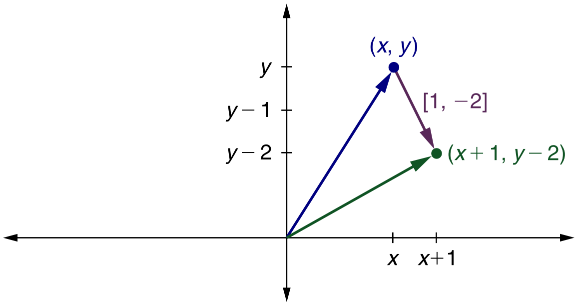


FIGURE 5.2 A translation in $\mathbb{R}^2$

## Linear Operators and Some Geometric Examples

An important type of linear transformation is one that maps a vector space to itself.

**Definition** Let $\mathcal{V}$ be a vector space. A **linear operator** on $\mathcal{V}$ is a linear transformation whose domain and codomain are both $\mathcal{V}$.

### Example 5

If $\mathcal{V}$ is any vector space, then the mapping $i\colon \mathcal{V} \to \mathcal{V}$ given by $i(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v} \in \mathcal{V}$ is a linear operator, known as the **identity linear operator**. Also, the constant mapping $z\colon \mathcal{V} \to \mathcal{V}$ given by $z(\mathbf{v}) = \mathbf{0}_{\mathcal{V}}$, is a linear operator known as the **zero linear operator** (see Exercise 2).

The next few examples exhibit important geometric operators. In these examples, assume that all vectors begin at the origin.

### Example 6

**Reflections:** Consider the mapping $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $f([a_1, a_2, a_3]) = [a_1, a_2, -a_3]$. This mapping "reflects" the vector $[a_1, a_2, a_3]$ through the $xy$-plane, which acts like a "mirror" (see Fig. 5.3). Now, since

$$\begin{aligned} f([a_1, a_2, a_3] + [b_1, b_2, b_3]) &= f([a_1 + b_1, a_2 + b_2, a_3 + b_3]) \\ &= [a_1 + b_1, a_2 + b_2, -(a_3 + b_3)] \\ &= [a_1, a_2, -a_3] + [b_1, b_2, -b_3] \\ &= f([a_1, a_2, a_3]) + f([b_1, b_2, b_3]), \qquad \text{and} \end{aligned}$$

$$f(c[a_1, a_2, a_3]) = f([ca_1, ca_2, ca_3]) = [ca_1, ca_2, -ca_3] = c[a_1, a_2, -a_3] = cf([a_1, a_2, a_3]),$$

we see that $f$ is a linear operator. Similarly, reflection through the $xz$-plane or the $yz$-plane is also a linear operator on $\mathbb{R}^3$ (see Exercise 4).

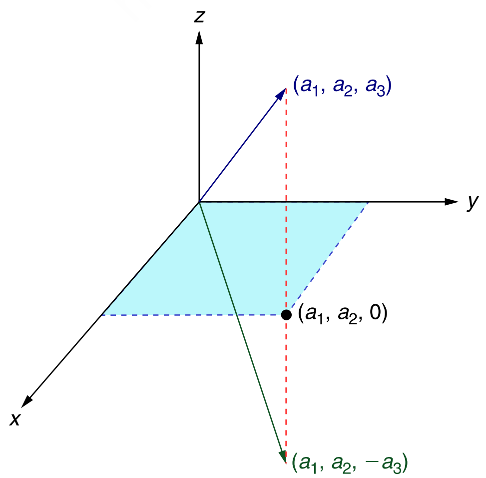


FIGURE 5.3 Reflection in $\mathbb{R}^3$ through the $xy$-plane

## Example 7

**Contractions and Dilations:** Consider the mapping $g: \mathbb{R}^n \to \mathbb{R}^n$ given by scalar multiplication by $k$, where $k \in \mathbb{R}$; that is, $g(\mathbf{v}) = k\mathbf{v}$, for $\mathbf{v} \in \mathbb{R}^n$. The function $g$ is a linear operator (see Exercise 3). If $|k| > 1$, $g$ represents a **dilation** (lengthening) of the vectors in $\mathbb{R}^n$; if $|k| < 1$, $g$ represents a **contraction** (shrinking).

## Example 8

**Projections:** Consider the mapping $h: \mathbb{R}^3 \to \mathbb{R}^3$ given by $h([a_1, a_2, a_3]) = [a_1, a_2, 0]$. This mapping takes each vector in $\mathbb{R}^3$ to a corresponding vector in the $xy$-plane (see Fig. 5.4). Similarly, consider the mapping $j: \mathbb{R}^4 \to \mathbb{R}^4$ given by $j([a_1, a_2, a_3, a_4]) = [0, a_2, 0, a_4]$. This mapping takes each vector in $\mathbb{R}^4$ to a corresponding vector whose first and third coordinates are zero. The functions $h$ and $j$ are both linear operators (see Exercise 5). Such mappings, where at least one of the coordinates is "zeroed out," are examples of **projection mappings**. You can verify that all such mappings are linear operators. (Other types of projection mappings are illustrated in Exercises 6 and 7.)

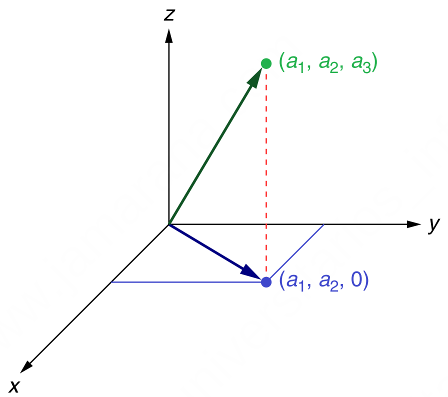


FIGURE 5.4 Projection of $[a_1, a_2, a_3]$ to the $xy$-plane

## Example 9

**Rotations:** Let $\theta$ be a fixed angle in $\mathbb{R}^2$, and let $l: \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$l\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}.$$

In Exercise 9 you are asked to show that $l$ rotates $[x, y]$ counterclockwise through the angle $\theta$ (see Fig. 5.5).

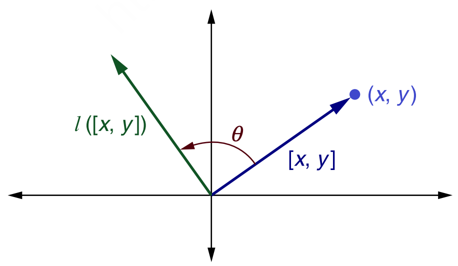


FIGURE 5.5 Counterclockwise rotation of $[x, y]$ through an angle $\theta$ in $\mathbb{R}^2$

Now, let $\mathbf{v}_1 = [x_1, y_1]$ and $\mathbf{v}_2 = [x_2, y_2]$ be two vectors in $\mathbb{R}^2$. Then,

$$l(\mathbf{v}_1 + \mathbf{v}_2) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} (\mathbf{v}_1 + \mathbf{v}_2)$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \mathbf{v}_1 + \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \mathbf{v}_2$$
$$= l(\mathbf{v}_1) + l(\mathbf{v}_2).$$

Similarly, $l(c\mathbf{v}) = cl(\mathbf{v})$, for any $c \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^2$. Hence, $l$ is a linear operator.

Beware! Not all geometric operations are linear operators. Recall that the translation function is not a linear operator!

## Multiplication Transformation

The linear operator in Example 9 is actually a special case of the next example, which shows that multiplication by an $m \times n$ matrix is always a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

**Example 10**

Let $\mathbf{A}$ be a given $m \times n$ matrix. We show that the function $f: \mathbb{R}^n \to \mathbb{R}^m$ defined by $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$, for all $\mathbf{x} \in \mathbb{R}^n$, is a linear transformation. Let $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$. Then $f(\mathbf{x}_1 + \mathbf{x}_2) = \mathbf{A}(\mathbf{x}_1 + \mathbf{x}_2) = \mathbf{A}\mathbf{x}_1 + \mathbf{A}\mathbf{x}_2 = f(\mathbf{x}_1) + f(\mathbf{x}_2)$. Also, let $\mathbf{x} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then, $f(c\mathbf{x}) = \mathbf{A}(c\mathbf{x}) = c(\mathbf{A}\mathbf{x}) = cf(\mathbf{x})$.

For a specific example of the multiplication transformation, consider the matrix $\mathbf{A} = \begin{bmatrix} -1 & 4 & 2 \\ 5 & 6 & -3 \end{bmatrix}$. The mapping given by

$$f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -1 & 4 & 2 \\ 5 & 6 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_1 + 4x_2 + 2x_3 \\ 5x_1 + 6x_2 - 3x_3 \end{bmatrix}$$

is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$. In the next section, we will show that the converse of the result in Example 10 also holds; every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is equivalent to multiplication by an appropriate $m \times n$ matrix.

## Elementary Properties of Linear Transformations

We now prove some basic properties of linear transformations. From here on, we usually use italicized capital letters, such as "$L$," to represent linear transformations.

**Theorem 5.1** *Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces, and let $L: \mathcal{V} \to \mathcal{W}$ be a linear transformation. Let $\mathbf{0}_\mathcal{V}$ be the zero vector in $\mathcal{V}$ and $\mathbf{0}_\mathcal{W}$ be the zero vector in $\mathcal{W}$. Then*

**(1)** $L(\mathbf{0}_\mathcal{V}) = \mathbf{0}_\mathcal{W}$

**(2)** $L(-\mathbf{v}) = -L(\mathbf{v})$, *for all* $\mathbf{v} \in \mathcal{V}$

**(3)** $L(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n) = a_1L(\mathbf{v}_1) + a_2L(\mathbf{v}_2) + \cdots + a_nL(\mathbf{v}_n)$, *for all* $a_1, \ldots, a_n \in \mathbb{R}$, *and* $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathcal{V}$, *for* $n \geq 2$.

*Proof.*
**Part (1):**

| | |
|---|---|
| $L(\mathbf{0}_\mathcal{V}) = L(0\mathbf{0}_\mathcal{V})$ | part (2) of Theorem 4.1, in $\mathcal{V}$ |
| $= 0L(\mathbf{0}_\mathcal{V})$ | property (2) of linear transformation |
| $= \mathbf{0}_\mathcal{W}$ | part (2) of Theorem 4.1, in $\mathcal{W}$ |

**Part (2):**

| | |
|---|---|
| $L(-\mathbf{v}) = L(-1\mathbf{v})$ | part (3) of Theorem 4.1, in $\mathcal{V}$ |
| $= -1(L(\mathbf{v}))$ | property (2) of linear transformation |
| $= -L(\mathbf{v})$ | part (3) of Theorem 4.1, in $\mathcal{W}$ |

**Part (3):** (Abridged) This part is proved by induction. We prove the Base Step ($n=2$) here and leave the Inductive Step as Exercise 29. For the Base Step, we must show that $L(a_1\mathbf{v}_1+a_2\mathbf{v}_2)=a_1L(\mathbf{v}_1)+a_2L(\mathbf{v}_2)$. But,

$$\begin{aligned} L(a_1\mathbf{v}_1+a_2\mathbf{v}_2) &= L(a_1\mathbf{v}_1)+L(a_2\mathbf{v}_2) && \text{property (1) of linear transformation}\\ &= a_1L(\mathbf{v}_1)+a_2L(\mathbf{v}_2) && \text{property (2) of linear transformation.}\end{aligned}$$

□

The next theorem asserts that the composition $L_2\circ L_1$ of linear transformations $L_1$ and $L_2$ is again a linear transformation (see Appendix B for a review of composition of functions).

**Theorem 5.2** *Let $\mathcal{V}_1$, $\mathcal{V}_2$, and $\mathcal{V}_3$ be vector spaces. Let $L_1\colon\mathcal{V}_1\to\mathcal{V}_2$ and $L_2\colon\mathcal{V}_2\to\mathcal{V}_3$ be linear transformations. Then $L_2\circ L_1\colon\mathcal{V}_1\to\mathcal{V}_3$ given by $(L_2\circ L_1)(\mathbf{v})=L_2(L_1(\mathbf{v}))$, for all $\mathbf{v}\in\mathcal{V}_1$, is a linear transformation.*

*Proof.* (Abridged) To show that $L_2\circ L_1$ is a linear transformation, we must show that for all $c\in\mathbb{R}$ and $\mathbf{v},\mathbf{v}_1,\mathbf{v}_2\in\mathcal{V}$,

$$(L_2\circ L_1)(\mathbf{v}_1+\mathbf{v}_2)=(L_2\circ L_1)(\mathbf{v}_1)+(L_2\circ L_1)(\mathbf{v}_2)$$
$$\text{and}\qquad (L_2\circ L_1)(c\mathbf{v})=c(L_2\circ L_1)(\mathbf{v}).$$

The first property holds since

$$\begin{aligned}(L_2\circ L_1)(\mathbf{v}_1+\mathbf{v}_2) &= L_2(L_1(\mathbf{v}_1+\mathbf{v}_2))\\ &= L_2(L_1(\mathbf{v}_1)+L_1(\mathbf{v}_2)) && \text{because } L_1 \text{ is a linear transformation}\\ &= L_2(L_1(\mathbf{v}_1))+L_2(L_1(\mathbf{v}_2)) && \text{because } L_2 \text{ is a linear transformation}\\ &= (L_2\circ L_1)(\mathbf{v}_1)+(L_2\circ L_1)(\mathbf{v}_2).\end{aligned}$$

We leave the proof of the second property as Exercise 31. □

### Example 11

Let $L_1$ represent the rotation of vectors in $\mathbb{R}^2$ through a fixed angle $\theta$ (as in Example 9), and let $L_2$ represent the reflection of vectors in $\mathbb{R}^2$ through the $x$-axis. That is, if $\mathbf{v}=[v_1,v_2]$, then

$$L_1(\mathbf{v})=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}\quad\text{and}\quad L_2(\mathbf{v})=\begin{bmatrix}v_1\\ -v_2\end{bmatrix}.$$

Because $L_1$ and $L_2$ are both linear transformations, Theorem 5.2 asserts that

$$L_2(L_1(\mathbf{v}))=L_2\left(\begin{bmatrix}v_1\cos\theta-v_2\sin\theta\\ v_1\sin\theta+v_2\cos\theta\end{bmatrix}\right)=\begin{bmatrix}v_1\cos\theta-v_2\sin\theta\\ -v_1\sin\theta-v_2\cos\theta\end{bmatrix}$$

is also a linear transformation. $L_2\circ L_1$ represents a rotation of $\mathbf{v}$ through $\theta$ followed by a reflection through the $x$-axis.

Theorem 5.2 generalizes naturally to more than two linear transformations. That is, if $L_1, L_2, \ldots, L_k$ are linear transformations and the composition $L_k\circ\cdots\circ L_2\circ L_1$ makes sense, then $L_k\circ\cdots\circ L_2\circ L_1$ is also a linear transformation.

## Linear Transformations and Subspaces

The final theorem of this section assures us that, under a linear transformation $L\colon\mathcal{V}\to\mathcal{W}$, subspaces of $\mathcal{V}$ "correspond" to subspaces of $\mathcal{W}$ and vice versa.

**Theorem 5.3** *Let $L\colon\mathcal{V}\to\mathcal{W}$ be a linear transformation.*

**(1)** *If $\mathcal{V}'$ is a subspace of $\mathcal{V}$, then $L(\mathcal{V}')=\{L(\mathbf{v})\mid\mathbf{v}\in\mathcal{V}'\}$, the image of $\mathcal{V}'$ in $\mathcal{W}$, is a subspace of $\mathcal{W}$. In particular, the range of $L$ is a subspace of $\mathcal{W}$.*

**(2)** *If $\mathcal{W}'$ is a subspace of $\mathcal{W}$, then $L^{-1}(\mathcal{W}')=\{\mathbf{v}\mid L(\mathbf{v})\in\mathcal{W}'\}$, the pre-image of $\mathcal{W}'$ in $\mathcal{V}$, is a subspace of $\mathcal{V}$.*

We prove part (1) and leave part (2) as Exercise 33.

*Proof.* **Part (1)**: Suppose that $L\colon \mathcal{V} \to \mathcal{W}$ is a linear transformation and that $\mathcal{V}'$ is a subspace of $\mathcal{V}$. Now, $L\left(\mathcal{V}'\right)$, the image of $\mathcal{V}'$ in $\mathcal{W}$ (see Fig. 5.6), is certainly nonempty (why?). Hence, to show that $L\left(\mathcal{V}'\right)$ is a subspace of $\mathcal{W}$, we must prove that $L\left(\mathcal{V}'\right)$ is closed under addition and scalar multiplication.

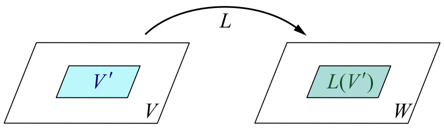


**FIGURE 5.6** Under a linear transformation $L\colon \mathcal{V} \to \mathcal{W}$, the image $L(\mathcal{V}')$ of a subspace $\mathcal{V}'$ of $\mathcal{V}$ is a subspace of $\mathcal{W}$

First, suppose that $\mathbf{w}_1, \mathbf{w}_2 \in L\left(\mathcal{V}'\right)$. Then, by definition of $L\left(\mathcal{V}'\right)$, we have $\mathbf{w}_1 = L(\mathbf{v}_1)$ and $\mathbf{w}_2 = L(\mathbf{v}_2)$, for some $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}'$. Then, $\mathbf{w}_1 + \mathbf{w}_2 = L(\mathbf{v}_1) + L(\mathbf{v}_2) = L(\mathbf{v}_1 + \mathbf{v}_2)$ because $L$ is a linear transformation. However, since $\mathcal{V}'$ is a subspace of $\mathcal{V}$, $(\mathbf{v}_1 + \mathbf{v}_2) \in \mathcal{V}'$. Thus, $(\mathbf{w}_1 + \mathbf{w}_2)$ is the image of $(\mathbf{v}_1 + \mathbf{v}_2) \in \mathcal{V}'$, and so $(\mathbf{w}_1 + \mathbf{w}_2) \in L\left(\mathcal{V}'\right)$. Hence, $L\left(\mathcal{V}'\right)$ is closed under addition.

Next, suppose that $c \in \mathbb{R}$ and $\mathbf{w} \in L\left(\mathcal{V}'\right)$. By definition of $L\left(\mathcal{V}'\right)$, $\mathbf{w} = L(\mathbf{v})$, for some $\mathbf{v} \in \mathcal{V}'$. Then, $c\mathbf{w} = cL(\mathbf{v}) = L(c\mathbf{v})$ since $L$ is a linear transformation. Now, $c\mathbf{v} \in \mathcal{V}'$, because $\mathcal{V}'$ is a subspace of $\mathcal{V}$. Thus, $c\mathbf{w}$ is the image of $c\mathbf{v} \in \mathcal{V}'$, and so $c\mathbf{w} \in L\left(\mathcal{V}'\right)$. Hence, $L\left(\mathcal{V}'\right)$ is closed under scalar multiplication. □

**Example 12**

Let $L\colon \mathcal{M}_{22} \to \mathbb{R}^3$, where $L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = [b, 0, c]$. $L$ is a linear transformation (verify!). By Theorem 5.3, the range of any linear transformation is a subspace of the codomain. Hence, the range of $L = \{[b, 0, c] \mid b, c \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$.

Also, consider the subspace $\mathcal{U}_2 = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \middle| \, a, b, d \in \mathbb{R} \right\}$ of $\mathcal{M}_{22}$. Then the image of $\mathcal{U}_2$ under $L$ is $\{[b, 0, 0] \mid b \in \mathbb{R}\}$. This image is a subspace of $\mathbb{R}^3$, as Theorem 5.3 asserts. Finally, consider the subspace $\mathcal{W} = \{[b, e, 2b] \mid b, e \in \mathbb{R}\}$ of $\mathbb{R}^3$. The pre-image of $\mathcal{W}$ consists of all matrices in $\mathcal{M}_{22}$ of the form $\begin{bmatrix} a & b \\ 2b & d \end{bmatrix}$. Notice that this pre-image is a subspace of $\mathcal{M}_{22}$, as claimed by Theorem 5.3.

## New Vocabulary

codomain (of a linear transformation)
composition of linear transformations
contraction (mapping)
dilation (mapping)
domain (of a linear transformation)
identity linear operator
image (of a vector in the domain)
linear operator
linear transformation
pre-image (of a vector in the codomain)
projection (mapping)
range (of a linear transformation)
reflection (mapping)
rotation (mapping)
shear (mapping) (see Exercise 11)
translation (mapping)
zero linear operator

## Highlights

- If $\mathcal{V}$ and $\mathcal{W}$ are vector spaces, a linear transformation $L\colon \mathcal{V} \to \mathcal{W}$ is a function from $\mathcal{V}$ to $\mathcal{W}$ that preserves the operations of addition and scalar multiplication. That is, $L(\mathbf{v}_1 + \mathbf{v}_2) = L(\mathbf{v}_1) + L(\mathbf{v}_2)$, for all $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}$, and $L(c\mathbf{v}) = cL(\mathbf{v})$, for all $\mathbf{v} \in \mathcal{V}$.
- A linear transformation $L\colon \mathcal{V} \to \mathcal{W}$ always maps $\mathbf{0}_{\mathcal{V}}$, the zero vector of the domain, to $\mathbf{0}_{\mathcal{W}}$, the zero vector of the codomain.
- If $L$ is a linear transformation, then $L(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n) = a_1L(\mathbf{v}_1) + a_2L(\mathbf{v}_2) + \cdots + a_nL(\mathbf{v}_n)$, for all $a_1, \ldots, a_n \in \mathbb{R}$, and $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathcal{V}$. That is, the image of a linear combination of vectors is the linear combination of the images of the vectors using the same coefficients.
- A linear operator is a linear transformation from a vector space to itself.

- A nontrivial translation of the plane ($\mathbb{R}^2$) or of space ($\mathbb{R}^3$) is never a linear operator, but all of the following are linear operators: contraction (of $\mathbb{R}^n$), dilation (of $\mathbb{R}^n$), reflection of space through the $xy$-plane (or $xz$-plane or $yz$-plane), rotation of the plane about the origin through a given angle $\theta$, projection (of $\mathbb{R}^n$) in which one or more of the coordinates are zeroed out.
- For a given $m \times n$ matrix $\mathbf{A}$, the mapping $L: \mathbb{R}^n \to \mathbb{R}^m$ given by $L(\mathbf{v}) = \mathbf{Av}$ (multiplication of vectors in $\mathbb{R}^n$ on the left by $\mathbf{A}$) is a linear transformation.
- Multiplying a vector $\mathbf{v}$ in $\mathbb{R}^2$ on the left by the matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ rotates $\mathbf{v}$ counterclockwise about the origin through the angle $\theta$.
- If $\mathcal{V}_1$, $\mathcal{V}_2$, and $\mathcal{V}_3$ are vector spaces, and $L_1: \mathcal{V}_1 \to \mathcal{V}_2$ and $L_2: \mathcal{V}_2 \to \mathcal{V}_3$ are linear transformations, then the composition $L_2 \circ L_1: \mathcal{V}_1 \to \mathcal{V}_3$ is a linear transformation.
- Under a linear transformation $L: \mathcal{V} \to \mathcal{W}$, the image of any subspace of $\mathcal{V}$ is a subspace of $\mathcal{W}$, and the pre-image of any subspace of $\mathcal{W}$ is a subspace of $\mathcal{V}$.

## Exercises for Section 5.1

1. Determine which of the following functions are linear transformations. Prove that your answers are correct. Which are linear operators?
   ★ (a) $f: \mathbb{R}^2 \to \mathbb{R}^2$ given by $f([x, y]) = [3x - 4y, -x + 2y]$
   ★ (b) $h: \mathbb{R}^4 \to \mathbb{R}^4$ given by $h([x_1, x_2, x_3, x_4]) = [x_1 + 2, x_2 - 1, x_3, -3]$
   (c) $k: \mathbb{R}^3 \to \mathbb{R}^3$ given by $k([x_1, x_2, x_3]) = [x_2, |x_3|, x_1]$
   ★ (d) $l: \mathcal{M}_{22} \to \mathcal{M}_{22}$ given by $l\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a - 2c + d & 3b - c \\ -4a & b + c - 3d \end{bmatrix}$
   (e) $z: \mathcal{M}_{22} \to \mathcal{M}_{11}$ given by $z(\mathbf{A}) = [3 \ \ 4]\,\mathbf{A}\begin{bmatrix} 5 \\ 2 \end{bmatrix}$
   ★ (f) $r: \mathcal{P}_3 \to \mathcal{P}_2$ given by $r(ax^3 + bx^2 + cx + d) = (\sqrt[3]{a})x^2 - b^2x + c$
   (g) $s: \mathbb{R}^3 \to \mathbb{R}^2$ given by $s([x_1, x_2, x_3]) = [\cos(x_1 + x_3), \sin(x_2 - x_3)]$
   ★ (h) $t: \mathcal{P}_3 \to \mathbb{R}$ given by $t(a_3x^3 + a_2x^2 + a_1x + a_0) = a_3 + a_2 + a_1 + a_0$
   (i) $u: \mathbb{R}^4 \to \mathbb{R}^4$ given by $u([x_1, x_2, x_3, x_4]) = [x_2, x_1, x_4, x_3]$
   ★ (j) $v: \mathcal{P}_2 \to \mathbb{R}$ given by $v\left(ax^2 + bx + c\right) = abc$
   ★ (k) $g: \mathcal{M}_{32} \to \mathcal{P}_4$ given by $g\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}\right) = a_{11}x^4 - a_{21}x^2 + a_{31}$
   ★ (l) $e: \mathbb{R}^2 \to \mathbb{R}$ given by $e([x, y]) = \sqrt{x^2 + y^2}$
   (m) $w: \mathbb{R}^2 \to \mathbb{R}^2$ given by $w([x_1, x_2]) = [e^{x_1} - 1, \ e^{x_2} - 1]$
2. Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces.
   (a) Show that the identity mapping $i: \mathcal{V} \to \mathcal{V}$ given by $i(\mathbf{v}) = \mathbf{v}$, for all $\mathbf{v} \in \mathcal{V}$, is a linear operator.
   (b) Show that the zero mapping $z: \mathcal{V} \to \mathcal{W}$ given by $z(\mathbf{v}) = \mathbf{0}_{\mathcal{W}}$, for all $\mathbf{v} \in \mathcal{V}$, is a linear transformation. (Hence, if $\mathcal{V} = \mathcal{W}$, the zero mapping is a linear operator.)
3. Let $k$ be a fixed scalar in $\mathbb{R}$. Show that the mapping $f: \mathbb{R}^n \to \mathbb{R}^n$ given by $f(\mathbf{v}) = k\mathbf{v}$ is a linear operator.
4. This exercise involves particular types of reflections in $\mathbb{R}^3$ and $\mathbb{R}^2$.
   (a) Show that $f: \mathbb{R}^3 \to \mathbb{R}^3$ given by $f([x, y, z]) = [-x, y, z]$ (reflection of a vector through the $yz$-plane) is a linear operator.
   (b) What mapping from $\mathbb{R}^3$ to $\mathbb{R}^3$ would reflect a vector through the $xz$-plane? Is it a linear operator? Why or why not?
   (c) What mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$ would reflect a vector through the $x$-axis? through the $y$-axis? Are these linear operators? Why or why not?
   (d) What mapping from $\mathbb{R}^3$ to $\mathbb{R}^3$ would reflect a vector through the $x$-axis? through the $y$-axis? through the $z$-axis? Are these linear operators? Why or why not?
5. Show that the projection mappings $h: \mathbb{R}^3 \to \mathbb{R}^3$ given by $h([a_1, a_2, a_3]) = [a_1, a_2, 0]$ and $j: \mathbb{R}^4 \to \mathbb{R}^4$ given by $j([a_1, a_2, a_3, a_4]) = [0, a_2, 0, a_4]$ are linear operators.

**6.** The mapping $f: \mathbb{R}^n \to \mathbb{R}$ given by $f([x_1, x_2, \ldots, x_i, \ldots, x_n]) = x_i$ is another type of projection mapping. Show that $f$ is a linear transformation.

**7.** Let $\mathbf{x}$ be a fixed nonzero vector in $\mathbb{R}^3$. Show that the mapping $g: \mathbb{R}^3 \to \mathbb{R}^3$ given by $g(\mathbf{y}) = \mathbf{proj_x y}$ is a linear operator.

**8.** Let $\mathbf{x}$ be a fixed vector in $\mathbb{R}^n$. Prove that $L: \mathbb{R}^n \to \mathbb{R}$ given by $L(\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ is a linear transformation.

**9.** Let $\theta$ be a fixed angle in the $xy$-plane. Show that the linear operator $L: \mathbb{R}^2 \to \mathbb{R}^2$ given by $L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ rotates the vector $[x, y]$ counterclockwise through the angle $\theta$ in the plane. (Hint: Consider the vector $[x', y']$, obtained by rotating $[x, y]$ counterclockwise through the angle $\theta$. Let $r = \sqrt{x^2 + y^2}$. Then $x = r\cos\alpha$ and $y = r\sin\alpha$, where $\alpha$ is the angle shown in Fig. 5.7. Notice that $x' = r(\cos(\theta + \alpha))$ and $y' = r(\sin(\theta + \alpha))$. Then show that $L([x, y]) = [x', y']$.)

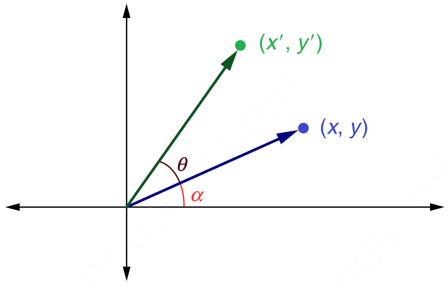


**FIGURE 5.7** The vectors $[x, y]$ and $[x', y']$

**10.** This exercise involves particular types of rotations in $\mathbb{R}^3$.

**(a)** Explain why the mapping $L: \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

is a linear operator.

**(b)** Show that the mapping $L$ in part (a) rotates every vector in $\mathbb{R}^3$ about the $z$-axis through an angle of $\theta$ (as measured relative to the $xy$-plane).

★ **(c)** What matrix should be multiplied times $[x, y, z]$ to create the linear operator that rotates $\mathbb{R}^3$ about the $y$-axis through an angle $\theta$ (relative to the $xz$-plane)? (Hint: When looking down from the positive $y$-axis toward the $xz$-plane in a right-handed system, the positive $z$-axis rotates 90° counterclockwise into the positive $x$-axis.)

**11.** **Shears**: Let $f_1, f_2: \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$f_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + ky \\ y \end{bmatrix}$$

and

$$f_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ kx + y \end{bmatrix}.$$

The mapping $f_1$ is called a **shear in the $x$-direction with factor** $k$; $f_2$ is called a **shear in the $y$-direction with factor** $k$. The effect of these functions (for $k > 1$) on the vector $[1, 1]$ is shown in Fig. 5.8. Show that $f_1$ and $f_2$ are linear operators directly, without using Example 10.

**12.** Let $f: \mathcal{M}_{nn} \to \mathbb{R}$ be given by $f(\mathbf{A}) = \text{trace}(\mathbf{A})$. (The trace is defined in Exercise 13 of Section 1.4.) Prove that $f$ is a linear transformation.

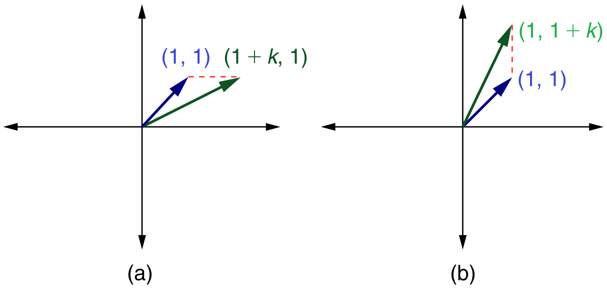


**FIGURE 5.8** (a) Shear in the $x$-direction; (b) shear in the $y$-direction (both for $k > 0$)

**13.** This exercise explores two related functions from $\mathcal{M}_{nn}$ to $\mathcal{M}_{nn}$.

**(a)** Show that the mappings $g, h\colon \mathcal{M}_{nn} \to \mathcal{M}_{nn}$ given by $g(\mathbf{A}) = \mathbf{A} + \mathbf{A}^T$ and $h(\mathbf{A}) = \mathbf{A} - \mathbf{A}^T$ are linear operators on $\mathcal{M}_{nn}$.

**(b)** Show that the range of the linear operator $g$ from part (a) is the set of all symmetric $n \times n$ matrices.

**(c)** Show that the range of the linear operator $h$ from part (a) is the set of all skew-symmetric $n \times n$ matrices.

**14.** This exercise concerns linear transformations involving integration.

**(a)** Show that if $\mathbf{p} \in \mathcal{P}_n$, then the (indefinite integral) function $f\colon \mathcal{P}_n \to \mathcal{P}_{n+1}$, where $f(\mathbf{p})$ is the vector $\int \mathbf{p}(x)\,dx$ with zero constant term, is a linear transformation.

**(b)** Show that if $\mathbf{p} \in \mathcal{P}_n$, then the (definite integral) function $g\colon \mathcal{P}_n \to \mathbb{R}$ given by $g(\mathbf{p}) = \int_a^b \mathbf{p}\,dx$ is a linear transformation, for any fixed $a, b \in \mathbb{R}$.

**15.** Let $\mathcal{V}$ be the vector space of all functions $f$ from $\mathbb{R}$ to $\mathbb{R}$ that are infinitely differentiable (that is, for which $f^{(n)}$, the $n$th derivative of $f$, exists for every integer $n \geq 1$). Use induction and Theorem 5.2 to show that for any given integer $k \geq 1$, $L\colon \mathcal{V} \to \mathcal{V}$ given by $L(f) = f^{(k)}$ is a linear operator.

**16.** This exercise explores several linear transformations with domain $\mathcal{M}_{mn}$ that involve multiplication by a fixed matrix.

**(a)** Consider the function $f\colon \mathcal{M}_{nn} \to \mathcal{M}_{nn}$ given by $f(\mathbf{A}) = \mathbf{BA}$, where $\mathbf{B}$ is some fixed $n \times n$ matrix. Show that $f$ is a linear operator.

**(b)** Consider the function $g\colon \mathcal{M}_{nn} \to \mathcal{M}_{nn}$ given by $g(\mathbf{A}) = \mathbf{AC}$, where $\mathbf{C}$ is some fixed $n \times n$ matrix. Show that $g$ is a linear operator.

**(c)** Consider the function $h\colon \mathcal{M}_{mn} \to \mathcal{M}_{pk}$ given by $h(\mathbf{A}) = \mathbf{DAE}$, where $\mathbf{D}$ is some fixed $p \times m$ matrix, and $\mathbf{E}$ is some fixed $n \times k$ matrix. Show that $h$ is a linear transformation.[1]

**17.** Let $\mathbf{B}$ be a fixed nonsingular matrix in $\mathcal{M}_{nn}$. Show that the mapping $f\colon \mathcal{M}_{nn} \to \mathcal{M}_{nn}$ given by $f(\mathbf{A}) = \mathbf{B}^{-1}\mathbf{AB}$ is a linear operator.

**18.** Let $a$ be a fixed real number.

**(a)** Let $L\colon \mathcal{P}_n \to \mathbb{R}$ be given by $L(\mathbf{p}(x)) = \mathbf{p}(a)$. (That is, $L$ evaluates polynomials in $\mathcal{P}_n$ at $x = a$.) Show that $L$ is a linear transformation.

**(b)** Let $L\colon \mathcal{P}_n \to \mathcal{P}_n$ be given by $L(\mathbf{p}(x)) = \mathbf{p}(x + a)$. (For example, when $a$ is positive, $L$ shifts the graph of $\mathbf{p}(x)$ to the *left* by $a$ units.) Prove that $L$ is a linear operator.

**19.** Let $\mathbf{A}$ be a fixed matrix in $\mathcal{M}_{nn}$. Define $f\colon \mathcal{P}_n \to \mathcal{M}_{nn}$ by

$$f(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0) = a_n\mathbf{A}^n + a_{n-1}\mathbf{A}^{n-1} + \cdots + a_1\mathbf{A} + a_0\mathbf{I}_n.$$

Show that $f$ is a linear transformation.

**20.** Let $\mathcal{V}$ be the unusual vector space from Example 7 in Section 4.1.

**(a)** Show that $L\colon \mathcal{V} \to \mathbb{R}$ given by $L(x) = \ln(x)$ is a linear transformation.

**(b)** Show that $M\colon \mathbb{R} \to \mathcal{V}$ given by $M(x) = e^x$ is a linear transformation.

**21.** Let $\mathcal{V}$ be a vector space, and let $\mathbf{x} \neq \mathbf{0}$ be a fixed vector in $\mathcal{V}$. Prove that the translation function $f\colon \mathcal{V} \to \mathcal{V}$ given by $f(\mathbf{v}) = \mathbf{v} + \mathbf{x}$ is not a linear transformation.

**22.** Show that if $\mathbf{A}$ is a fixed matrix in $\mathcal{M}_{mn}$ and $\mathbf{y} \neq \mathbf{0}$ is a fixed vector in $\mathbb{R}^m$, then the mapping $f\colon \mathbb{R}^n \to \mathbb{R}^m$ given by $f(\mathbf{x}) = \mathbf{Ax} + \mathbf{y}$ is not a linear transformation by showing that part (1) of Theorem 5.1 fails for $f$.

[1] The subscripts for the codomain in part (c) of Exercise 16 were chosen as a special tribute to Paul Klingsberg, also known as PK, who has taught from every edition of this textbook, and has provided many useful suggestions for improvement.

**23.** Prove that, for $n > 1$, $f\colon \mathcal{M}_{nn} \to \mathbb{R}$ given by $f(\mathbf{A}) = |\mathbf{A}|$ is not a linear transformation.

**24.** Suppose $L_1\colon \mathcal{V} \to \mathcal{W}$ is a linear transformation and $L_2\colon \mathcal{V} \to \mathcal{W}$ is defined by $L_2(\mathbf{v}) = L_1(2\mathbf{v})$. Show that $L_2$ is a linear transformation.

**25.** Suppose $L\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a linear operator and $L([1,0,0]) = [4,-3,7]$, $L([0,1,0]) = [6,1,-1]$, and $L([0,0,1]) = [-3,5,1]$. Find $L([2,4,-3])$. Give a formula for $L([x,y,z])$, for any $[x,y,z] \in \mathbb{R}^3$.

★ **26.** Suppose $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear operator and $L(\mathbf{i}+\mathbf{j}) = \mathbf{i} - 3\mathbf{j}$ and $L(-2\mathbf{i}+3\mathbf{j}) = -4\mathbf{i}+2\mathbf{j}$. Express $L(\mathbf{i})$ and $L(\mathbf{j})$ as linear combinations of $\mathbf{i}$ and $\mathbf{j}$.

**27.** Let $L\colon \mathcal{V} \to \mathcal{W}$ be a linear transformation. Show that $L(\mathbf{x}-\mathbf{y}) = L(\mathbf{x}) - L(\mathbf{y})$, for all vectors $\mathbf{x}, \mathbf{y} \in \mathcal{V}$.

**28.** Part (3) of Theorem 5.1 assures us that if $L\colon \mathcal{V} \to \mathcal{W}$ is a linear transformation, then $L(a\mathbf{v}_1 + b\mathbf{v}_2) = aL(\mathbf{v}_1) + bL(\mathbf{v}_2)$, for all $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}$ and all $a, b \in \mathbb{R}$. Prove that the converse of this statement is true. (Hint: Consider two cases: first $a = b = 1$ and then $b = 0$.)

▶ **29.** Finish the proof of part (3) of Theorem 5.1 by doing the Inductive Step.

**30.** This exercise explores linear independence with linear transformations.

- **(a)** Suppose that $L\colon \mathcal{V} \to \mathcal{W}$ is a linear transformation. Show that if $\{L(\mathbf{v}_1), L(\mathbf{v}_2), \ldots, L(\mathbf{v}_n)\}$ is a linearly independent set of $n$ distinct vectors in $\mathcal{W}$, for some vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathcal{V}$, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a linearly independent set in $\mathcal{V}$.
- ★ **(b)** Find a counterexample to the converse of part (a).

▶ **31.** Finish the proof of Theorem 5.2 by proving property (2) of a linear transformation for $L_2 \circ L_1$.

**32.** Show that every linear operator $L\colon \mathbb{R} \to \mathbb{R}$ has the form $L(\mathbf{x}) = c\mathbf{x}$, for some $c \in \mathbb{R}$.

▶ **33.** Finish the proof of Theorem 5.3 by showing that if $L\colon \mathcal{V} \to \mathcal{W}$ is a linear transformation and $\mathcal{W}'$ is a subspace of $\mathcal{W}$ with pre-image $L^{-1}\left(\mathcal{W}'\right)$, then $L^{-1}\left(\mathcal{W}'\right)$ is a subspace of $\mathcal{V}$.

**34.** Let $L_1, L_2\colon \mathcal{V} \to \mathcal{W}$ be linear transformations. Define $(L_1 \oplus L_2)\colon \mathcal{V} \to \mathcal{W}$ by $(L_1 \oplus L_2)(\mathbf{v}) = L_1(\mathbf{v}) + L_2(\mathbf{v})$ (where the latter addition takes place in $\mathcal{W}$). Also define $(c \odot L_1)\colon \mathcal{V} \to \mathcal{W}$ by $(c \odot L_1)(\mathbf{v}) = c\,(L_1(\mathbf{v}))$ (where the latter scalar multiplication takes place in $\mathcal{W}$).

- **(a)** Show that $(L_1 \oplus L_2)$ and $(c \odot L_1)$ are linear transformations.
- **(b)** Use the results in part (a) above and part (b) of Exercise 2 to show that the set of all linear transformations from $\mathcal{V}$ to $\mathcal{W}$ is a vector space under the operations $\oplus$ and $\odot$.

**35.** Let $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a nonzero linear operator. Show that $L$ maps a line to either a line or a point.

**36.** Draw a commutative diagram, similar to the one in Fig. 5.1, that illustrates the second property of a linear transformation, $f(c\mathbf{v}) = cf(\mathbf{v})$.

**37.** For the vector space $\mathcal{V}$ given in Example 8 of Section 4.1, show that $L\colon \mathcal{V} \to \mathbb{R}^2$ given by $L(\mathbf{v}) = \mathbf{v} + [1,-2]$ is a linear transformation.

★ **38.** True or False:

- **(a)** If $L\colon \mathcal{V} \to \mathcal{W}$ is a function between vector spaces for which $L(c\mathbf{v}) = cL(\mathbf{v})$, then $L$ is a linear transformation.
- **(b)** If $\mathcal{V}$ is an $n$-dimensional vector space with ordered basis $B$, then $L\colon \mathcal{V} \to \mathbb{R}^n$ given by $L(\mathbf{v}) = [\mathbf{v}]_B$ is a linear transformation.
- **(c)** The function $L\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $L([x,y,z]) = [x+1,\ y-2,\ z+3]$ is a linear operator.
- **(d)** If $\mathbf{A}$ is a $4 \times 3$ matrix, then $L(\mathbf{v}) = \mathbf{A}\mathbf{v}$ is a linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^3$.
- **(e)** A linear transformation from $\mathcal{V}$ to $\mathcal{W}$ always maps $\mathbf{0}_{\mathcal{V}}$ to $\mathbf{0}_{\mathcal{W}}$.
- **(f)** If $M_1\colon \mathcal{V} \to \mathcal{W}$ and $M_2\colon \mathcal{W} \to \mathcal{X}$ are linear transformations, then $M_1 \circ M_2$ is a well-defined linear transformation.
- **(g)** If $L\colon \mathcal{V} \to \mathcal{W}$ is a linear transformation, then the image of any subspace of $\mathcal{V}$ is a subspace of $\mathcal{W}$.
- **(h)** If $L\colon \mathcal{V} \to \mathcal{W}$ is a linear transformation, then the pre-image of $\{\mathbf{0}_{\mathcal{W}}\}$ is a subspace of $\mathcal{V}$.

## 5.2 The Matrix of a Linear Transformation

In this section we show that the behavior of any linear transformation $L\colon \mathcal{V} \to \mathcal{W}$ is determined by its effect on a basis for $\mathcal{V}$. In particular, when $\mathcal{V}$ and $\mathcal{W}$ are nontrivial and finite dimensional, and ordered bases for $\mathcal{V}$ and $\mathcal{W}$ are chosen, we can obtain a matrix corresponding to $L$ that is useful in computing images under $L$. Finally, we investigate how the matrix for $L$ changes as the bases for $\mathcal{V}$ and $\mathcal{W}$ change.

## A Linear Transformation Is Determined by Its Action on a Basis

If the action of a linear transformation $L: \mathcal{V} \to \mathcal{W}$ on a basis for $\mathcal{V}$ is known, then the action of $L$ can be computed for all elements of $\mathcal{V}$, as we see in the next example.

**Example 1**

You can quickly verify that

$$B = ([0,4,0,1], [-2,5,0,2], [-3,5,1,1], [-1,2,0,1])$$

is an ordered basis for $\mathbb{R}^4$. Now suppose that $L: \mathbb{R}^4 \to \mathbb{R}^3$ is a linear transformation for which

$$L([0,4,0,1]) = [3,1,2], \qquad L([-2,5,0,2]) = [2,-1,1],$$
$$L([-3,5,1,1]) = [-4,3,0], \text{ and } L([-1,2,0,1]) = [6,1,-1].$$

We can use the values of $L$ on $B$ to compute $L$ for other vectors in $\mathbb{R}^4$. For example, let $\mathbf{v} = [-4,14,1,4]$. By using row reduction, we see that $[\mathbf{v}]_B = [2,-1,1,3]$ (verify!). So,

$$\begin{aligned} L(\mathbf{v}) &= L(2[0,4,0,1] - 1[-2,5,0,2] + 1[-3,5,1,1] + 3[-1,2,0,1]) \\ &= 2L([0,4,0,1]) - 1L([-2,5,0,2]) + 1L([-3,5,1,1]) + 3L([-1,2,0,1]) \\ &= 2[3,1,2] - [2,-1,1] + [-4,3,0] + 3[6,1,-1] \\ &= [18,9,0]. \end{aligned}$$

In general, if $\mathbf{v} \in \mathbb{R}^4$ and $[\mathbf{v}]_B = [k_1, k_2, k_3, k_4]$, then

$$\begin{aligned} L(\mathbf{v}) &= k_1[3,1,2] + k_2[2,-1,1] + k_3[-4,3,0] + k_4[6,1,-1] \\ &= [3k_1 + 2k_2 - 4k_3 + 6k_4,\ k_1 - k_2 + 3k_3 + k_4,\ 2k_1 + k_2 - k_4]. \end{aligned}$$

Thus, we have derived a general formula for $L$ from its effect on the basis $B$.

Example 1 illustrates the next theorem.

**Theorem 5.4** *Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for a vector space $\mathcal{V}$. Let $\mathcal{W}$ be a vector space, and let $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n$ be any $n$ vectors in $\mathcal{W}$. Then there is a unique linear transformation $L: \mathcal{V} \to \mathcal{W}$ such that $L(\mathbf{v}_1) = \mathbf{w}_1$, $L(\mathbf{v}_2) = \mathbf{w}_2$, ..., $L(\mathbf{v}_n) = \mathbf{w}_n$.*

*Proof.* (Abridged) Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for $\mathcal{V}$, and let $\mathbf{v} \in \mathcal{V}$. Then $\mathbf{v} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$, for some unique $a_i$'s in $\mathbb{R}$. Let $\mathbf{w}_1, \ldots, \mathbf{w}_n$ be any vectors in $\mathcal{W}$. Define $L: \mathcal{V} \to \mathcal{W}$ by $L(\mathbf{v}) = a_1\mathbf{w}_1 + a_2\mathbf{w}_2 + \cdots + a_n\mathbf{w}_n$.

To show that $L$ is a linear transformation, we must prove that $L(\mathbf{x}_1 + \mathbf{x}_2) = L(\mathbf{x}_1) + L(\mathbf{x}_2)$ and $L(c\mathbf{x}_1) = cL(\mathbf{x}_1)$, for all $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{V}$ and all $c \in \mathbb{R}$. Suppose that $\mathbf{x}_1 = d_1\mathbf{v}_1 + \cdots + d_n\mathbf{v}_n$ and $\mathbf{x}_2 = e_1\mathbf{v}_1 + \cdots + e_n\mathbf{v}_n$. Then, by definition of $L$, $L(\mathbf{x}_1) = d_1\mathbf{w}_1 + \cdots + d_n\mathbf{w}_n$ and $L(\mathbf{x}_2) = e_1\mathbf{w}_n + \cdots + e_n\mathbf{w}_n$. Now, grouping like terms,

$$L(\mathbf{x}_1) + L(\mathbf{x}_2) = (d_1 + e_1)\mathbf{w}_1 + \cdots + (d_n + e_n)\mathbf{w}_n.$$

However,

$$\mathbf{x}_1 + \mathbf{x}_2 = (d_1 + e_1)\mathbf{v}_1 + \cdots + (d_n + e_n)\mathbf{v}_n,$$
$$\text{so,} \quad L(\mathbf{x}_1 + \mathbf{x}_2) = (d_1 + e_1)\mathbf{w}_1 + \cdots + (d_n + e_n)\mathbf{w}_n,$$

again by definition of $L$. Hence, $L(\mathbf{x}_1) + L(\mathbf{x}_2) = L(\mathbf{x}_1 + \mathbf{x}_2)$.

Similarly, suppose $\mathbf{x} \in \mathcal{V}$, and $\mathbf{x} = t_1\mathbf{v}_1 + \cdots + t_n\mathbf{v}_n$. Then, $c\mathbf{x} = ct_1\mathbf{v}_1 + \cdots + ct_n\mathbf{v}_n$, and so $L(c\mathbf{x}) = ct_1\mathbf{w}_1 + \cdots + ct_n\mathbf{w}_n = cL(\mathbf{x})$. Hence, $L$ is a linear transformation.

Finally, the proof of the uniqueness assertion is straightforward and is left as Exercise 23. □

## The Matrix of a Linear Transformation

Our next goal is to show that every linear transformation between nontrivial finite dimensional vector spaces can be expressed as a matrix multiplication. This will allow us to solve problems involving linear transformations by performing matrix multiplications, which can easily be done by computer. As we will see, the matrix for a linear transformation is determined by the ordered bases $B$ and $C$ chosen for the domain and codomain, respectively. Our goal is to find a matrix that takes the $B$-coordinates of a vector in the domain to the $C$-coordinates of its image vector in the codomain.

Recall the linear transformation $L: \mathbb{R}^4 \to \mathbb{R}^3$ with the ordered basis $B$ for $\mathbb{R}^4$ from Example 1. For $\mathbf{v} \in \mathbb{R}^4$, we let $[\mathbf{v}]_B = [k_1, k_2, k_3, k_4]$, and obtained the following formula for $L$:

$$L(\mathbf{v}) = [3k_1 + 2k_2 - 4k_3 + 6k_4,\ k_1 - k_2 + 3k_3 + k_4,\ 2k_1 + k_2 - k_4].$$

Now, to keep matters simple, we select the standard basis $C = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ for the codomain $\mathbb{R}^3$, so that the $C$-coordinates of vectors in the codomain are the same as the vectors themselves. (That is, $L(\mathbf{v}) = [L(\mathbf{v})]_C$, since $C$ is the standard basis.) Then this formula for $L$ takes the $B$-coordinates of each vector in the domain to the $C$-coordinates of its image vector in the codomain. Now, notice that if

$$\mathbf{A}_{BC} = \begin{bmatrix} 3 & 2 & -4 & 6 \\ 1 & -1 & 3 & 1 \\ 2 & 1 & 0 & -1 \end{bmatrix}, \quad \text{then } \mathbf{A}_{BC} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 3k_1 + 2k_2 - 4k_3 + 6k_4 \\ k_1 - k_2 + 3k_3 + k_4 \\ 2k_1 + k_2 - k_4 \end{bmatrix}.$$

Hence, the matrix $\mathbf{A}$ contains all of the information needed for carrying out the linear transformation $L$ with respect to the chosen bases $B$ and $C$.

A similar process can be used for any linear transformation between nontrivial finite dimensional vector spaces.

**Theorem 5.5** *Let $\mathcal{V}$ and $\mathcal{W}$ be nontrivial vector spaces, with $\dim(\mathcal{V}) = n$ and $\dim(\mathcal{W}) = m$. Let $B = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ and $C = (\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m)$ be ordered bases for $\mathcal{V}$ and $\mathcal{W}$, respectively. Let $L: \mathcal{V} \to \mathcal{W}$ be a linear transformation. Then there is a unique $m \times n$ matrix $\mathbf{A}_{BC}$ such that $\mathbf{A}_{BC}[\mathbf{v}]_B = [L(\mathbf{v})]_C$, for all $\mathbf{v} \in \mathcal{V}$. (That is, $\mathbf{A}_{BC}$ times the coordinatization of $\mathbf{v}$ with respect to $B$ gives the coordinatization of $L(\mathbf{v})$ with respect to $C$.)*

*Furthermore, for $1 \le i \le n$, the $i$th column of $\mathbf{A}_{BC} = [L(\mathbf{v}_i)]_C$.*

Theorem 5.5 asserts that once ordered bases for $\mathcal{V}$ and $\mathcal{W}$ have been selected, *each linear transformation $L: \mathcal{V} \to \mathcal{W}$ is equivalent to multiplication by a unique corresponding matrix*. The matrix $\mathbf{A}_{BC}$ in this theorem is known as the **matrix of the linear transformation $L$ with respect to the ordered bases** $B$ (for $\mathcal{V}$) **and** $C$ (for $\mathcal{W}$).[2] Theorem 5.5 also says that the matrix $\mathbf{A}_{BC}$ is computed as follows: find the image of each domain basis element $\mathbf{v}_i$ in turn, and then express these images in $C$-coordinates to get the respective columns of $\mathbf{A}_{BC}$.

The subscripts $B$ and $C$ on $\mathbf{A}$ are sometimes omitted when the bases being used are clear from context. Beware! If a different ordered basis is chosen for $\mathcal{V}$ or $\mathcal{W}$, the matrix for the linear transformation will probably change.

*Proof.* Consider the $m \times n$ matrix $\mathbf{A}_{BC}$ whose $i$th column equals $[L(\mathbf{v}_i)]_C$, for $1 \le i \le n$. Let $\mathbf{v} \in \mathcal{V}$. We first prove that $\mathbf{A}_{BC}[\mathbf{v}]_B = [L(\mathbf{v})]_C$.

Suppose that $[\mathbf{v}]_B = [k_1, k_2, \ldots, k_n]$. Then $\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n$, and $L(\mathbf{v}) = k_1L(\mathbf{v}_1) + k_2L(\mathbf{v}_2) + \cdots + k_nL(\mathbf{v}_n)$, by Theorem 5.1. Hence,

$$\begin{aligned} [L(\mathbf{v})]_C &= [k_1L(\mathbf{v}_1) + k_2L(\mathbf{v}_2) + \cdots + k_nL(\mathbf{v}_n)]_C \\ &= k_1\,[L(\mathbf{v}_1)]_C + k_2\,[L(\mathbf{v}_2)]_C + \cdots + k_n\,[L(\mathbf{v}_n)]_C && \text{by Theorem 4.16} \\ &= k_1(\text{1st column of } \mathbf{A}_{BC}) + k_2(\text{2nd column of } \mathbf{A}_{BC}) + \cdots + k_n(n\text{th column of } \mathbf{A}_{BC}) \\ &= \mathbf{A}_{BC}\begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} = \mathbf{A}_{BC}[\mathbf{v}]_B. \end{aligned}$$

[2] If $L$ is a linear operator, and the same ordered basis $B$ is used for both the domain and codomain, then we call $\mathbf{A}_{BB}$ "the matrix for $L$ with respect to the (ordered) basis $B$."

To complete the proof, we need to establish the uniqueness of $\mathbf{A}_{BC}$. Suppose that $\mathbf{H}$ is an $m \times n$ matrix such that $\mathbf{H}[\mathbf{v}]_B = [L(\mathbf{v})]_C$ for all $\mathbf{v} \in \mathcal{V}$. We will show that $\mathbf{H} = \mathbf{A}_{BC}$. It is enough to show that the $i$th column of $\mathbf{H}$ equals the $i$th column of $\mathbf{A}_{BC}$, for $1 \le i \le n$. Consider the $i$th vector, $\mathbf{v}_i$, of the ordered basis $B$ for $\mathcal{V}$. Since $[\mathbf{v}_i]_B = \mathbf{e}_i$, we have: $i$th column of $\mathbf{H} = \mathbf{H}\mathbf{e}_i = \mathbf{H}[\mathbf{v}_i]_B = [L(\mathbf{v}_i)]_C$, and this is the $i$th column of $\mathbf{A}_{BC}$. □

Notice that in the special case where the codomain $\mathcal{W}$ is $\mathbb{R}^m$, and the basis $C$ for $\mathcal{W}$ is the standard basis, Theorem 5.5 asserts that the $i$th column of $\mathbf{A}_{BC}$ is simply $L(\mathbf{v}_i)$ itself (why?).

### Example 2

Table 5.1 lists the matrices corresponding to some geometric linear operators on $\mathbb{R}^3$, with respect to the standard basis. The columns of each matrix are quickly calculated using Theorem 5.5, since we simply find the images $L(\mathbf{e}_1)$, $L(\mathbf{e}_2)$, and $L(\mathbf{e}_3)$ of the domain basis elements $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$. (Each image is equal to its coordinatization in the codomain since we are using the standard basis for the codomain as well.)

Once the matrix for each transformation is calculated, we can easily find the image of any vector using matrix multiplication. For example, to find the effect of the reflection $L_1$ in Table 5.1 on the vector $[3, -4, 2]$, we simply multiply by the matrix for $L_1$ to get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ -2 \end{bmatrix}.$$

**TABLE 5.1** Matrices for several geometric linear operators on $\mathbb{R}^3$

| Transformation | Formula | Matrix |
|---|---|---|
| Reflection (through $xy$-plane) | $L_1\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) = \begin{bmatrix} a_1 \\ a_2 \\ -a_3 \end{bmatrix}$ | $\begin{matrix} L_1(\mathbf{e}_1) & L_1(\mathbf{e}_2) & L_1(\mathbf{e}_3) \end{matrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ |
| Contraction or dilation | $L_2\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) = \begin{bmatrix} ca_1 \\ ca_2 \\ ca_3 \end{bmatrix}$, for $c \in \mathbb{R}$ | $\begin{matrix} L_2(\mathbf{e}_1) & L_2(\mathbf{e}_2) & L_2(\mathbf{e}_3) \end{matrix}$ $\begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix}$ |
| Projection (onto $xy$-plane) | $L_3\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) = \begin{bmatrix} a_1 \\ a_2 \\ 0 \end{bmatrix}$ | $\begin{matrix} L_3(\mathbf{e}_1) & L_3(\mathbf{e}_2) & L_3(\mathbf{e}_3) \end{matrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ |
| Rotation (about $z$-axis through angle $\theta$ relative to the $xy$-plane) | $L_4\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) = \begin{bmatrix} a_1\cos\theta - a_2\sin\theta \\ a_1\sin\theta + a_2\cos\theta \\ a_3 \end{bmatrix}$ | $\begin{matrix} L_4(\mathbf{e}_1) & L_4(\mathbf{e}_2) & L_4(\mathbf{e}_3) \end{matrix}$ $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ |
| Shear (in the $z$-direction with factor $k$) (analog of Exercise 11 in Section 5.1) | $L_5\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) = \begin{bmatrix} a_1 + ka_3 \\ a_2 + ka_3 \\ a_3 \end{bmatrix}$ | $\begin{matrix} L_5(\mathbf{e}_1) & L_5(\mathbf{e}_2) & L_5(\mathbf{e}_3) \end{matrix}$ $\begin{bmatrix} 1 & 0 & k \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}$ |

### Example 3

We will find the matrix for the linear transformation $L: \mathcal{P}_3 \to \mathbb{R}^3$ given by

$$L(a_3x^3 + a_2x^2 + a_1x + a_0) = [a_0 + a_1, 2a_2, a_3 - a_0]$$

with respect to the standard ordered bases $B = (x^3, x^2, x, 1)$ for $\mathcal{P}_3$ and $C = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ for $\mathbb{R}^3$. We first need to find $L(\mathbf{v})$, for each $\mathbf{v} \in B$. By definition of $L$, we have

$$L(x^3) = [0, 0, 1], \quad L(x^2) = [0, 2, 0], \quad L(x) = [1, 0, 0], \text{ and } L(1) = [1, 0, -1].$$

Since we are using the standard basis $C$ for $\mathbb{R}^3$, each of these images in $\mathbb{R}^3$ is its own $C$-coordinatization. Then by Theorem 5.5, the matrix $\mathbf{A}_{BC}$ for $L$ is the matrix whose columns are these images; that is,

$$\mathbf{A}_{BC} = \begin{array}{c} \begin{matrix} L(x^3) & L(x^2) & L(x) & L(1) \end{matrix} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \end{array}.$$

We will compute $L(5x^3 - x^2 + 3x + 2)$ using this matrix. Now, $\left[5x^3 - x^2 + 3x + 2\right]_B = [5, -1, 3, 2]$. Hence, multiplication by $\mathbf{A}_{BC}$ gives

$$\left[L(5x^3 - x^2 + 3x + 2)\right]_C = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix}.$$

Since $C$ is the standard basis for $\mathbb{R}^3$, we have $L(5x^3 - x^2 + 3x + 2) = [5, -2, 3]$, which can be quickly verified to be the correct answer.

## Example 4

We will find the matrix for the same linear transformation $L: \mathcal{P}_3 \to \mathbb{R}^3$ of Example 3 with respect to the different ordered bases

$$D = (x^3 + x^2,\ x^2 + x,\ x + 1,\ 1)$$
$$\text{and} \quad E = ([-2, 1, -3], [1, -3, 0], [3, -6, 2]).$$

You should verify that $D$ and $E$ are bases for $\mathcal{P}_3$ and $\mathbb{R}^3$, respectively.

We first need to find $L(\mathbf{v})$, for each $\mathbf{v} \in D$. By definition of $L$, we have $L(x^3 + x^2) = [0, 2, 1]$, $L(x^2 + x) = [1, 2, 0]$, $L(x + 1) = [2, 0, -1]$, and $L(1) = [1, 0, -1]$. Now we must find the coordinatization of each of these images in terms of the basis $E$ for $\mathbb{R}^3$. We therefore row reduce the matrix

$$\left[\begin{array}{ccc|cccc} -2 & 1 & 3 & 0 & 1 & 2 & 1 \\ 1 & -3 & -6 & 2 & 2 & 0 & 0 \\ -3 & 0 & 2 & 1 & 0 & -1 & -1 \end{array}\right],$$

whose first 3 columns are the vectors in $E$, and whose last 4 columns are the images under $L$ listed above, to obtain

$$\left[\begin{array}{ccc|cccc} 1 & 0 & 0 & -1 & -10 & -15 & -9 \\ 0 & 1 & 0 & 1 & 26 & 41 & 25 \\ 0 & 0 & 1 & -1 & -15 & -23 & -14 \end{array}\right].$$

By Theorem 5.5, the last 4 columns give us

$$\mathbf{A}_{DE} = \begin{bmatrix} -1 & -10 & -15 & -9 \\ 1 & 26 & 41 & 25 \\ -1 & -15 & -23 & -14 \end{bmatrix}.$$

We will compute $L(5x^3 - x^2 + 3x + 2)$ using this matrix. We must first find the representation for $5x^3 - x^2 + 3x + 2$ in terms of the basis $D$. Solving $5x^3 - x^2 + 3x + 2 = a(x^3 + x^2) + b(x^2 + x) + c(x + 1) + d(1)$ for $a$, $b$, $c$, and $d$, we get the unique solution $a = 5$, $b = -6$, $c = 9$, and $d = -7$ (verify!). Hence, $\left[5x^3 - x^2 + 3x + 2\right]_D = [5, -6, 9, -7]$. Then

$$\left[L(5x^3 - x^2 + 3x + 2)\right]_E = \begin{bmatrix} -1 & -10 & -15 & -9 \\ 1 & 26 & 41 & 25 \\ -1 & -15 & -23 & -14 \end{bmatrix} \begin{bmatrix} 5 \\ -6 \\ 9 \\ -7 \end{bmatrix} = \begin{bmatrix} -17 \\ 43 \\ -24 \end{bmatrix}.$$

This answer represents a coordinate vector in terms of the basis $E$, and so

$$L(5x^3 - x^2 + 3x + 2) = -17 \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} + 43 \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} - 24 \begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix},$$

which agrees with the answer in Example 3.

## Finding the New Matrix for a Linear Transformation After a Change of Basis

The next theorem indicates precisely how the matrix for a linear transformation changes when we alter the bases for the domain and codomain.

**Theorem 5.6** *Let $\mathcal{V}$ and $\mathcal{W}$ be two nontrivial finite dimensional vector spaces with ordered bases $B$ and $C$, respectively. Let $L: \mathcal{V} \to \mathcal{W}$ be a linear transformation with matrix $\mathbf{A}_{BC}$ with respect to bases $B$ and $C$. Suppose that $D$ and $E$ are other ordered bases for $\mathcal{V}$ and $\mathcal{W}$, respectively. Let $\mathbf{P}$ be the transition matrix from $B$ to $D$, and let $\mathbf{Q}$ be the transition matrix from $C$ to $E$. Then the matrix $\mathbf{A}_{DE}$ for $L$ with respect to bases $D$ and $E$ is given by $\mathbf{A}_{DE} = \mathbf{Q}\mathbf{A}_{BC}\mathbf{P}^{-1}$.*

The situation in Theorem 5.6 is summarized in Fig. 5.9.

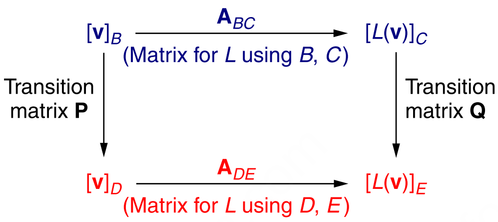


**FIGURE 5.9** Relationship between matrices $\mathbf{A}_{BC}$ and $\mathbf{A}_{DE}$ for a linear transformation under a change of basis

*Proof.* For all $\mathbf{v} \in \mathcal{V}$,

$$\begin{aligned}
& \mathbf{A}_{BC}[\mathbf{v}]_B = [L(\mathbf{v})]_C && \text{by Theorem 5.5} \\
\Rightarrow\ & \mathbf{Q}\mathbf{A}_{BC}[\mathbf{v}]_B = \mathbf{Q}[L(\mathbf{v})]_C && \\
\Rightarrow\ & \mathbf{Q}\mathbf{A}_{BC}[\mathbf{v}]_B = [L(\mathbf{v})]_E && \text{because } \mathbf{Q} \text{ is the transition matrix from } C \text{ to } E \\
\Rightarrow\ & \mathbf{Q}\mathbf{A}_{BC}\mathbf{P}^{-1}[\mathbf{v}]_D = [L(\mathbf{v})]_E && \text{because } \mathbf{P}^{-1} \text{ is the transition matrix from } D \text{ to } B
\end{aligned}$$

However, $\mathbf{A}_{DE}$ is the *unique* matrix such that $\mathbf{A}_{DE}[\mathbf{v}]_D = [L(\mathbf{v})]_E$, for all $\mathbf{v} \in \mathcal{V}$. Hence, $\mathbf{A}_{DE} = \mathbf{Q}\mathbf{A}_{BC}\mathbf{P}^{-1}$. □

Theorem 5.6 gives us an alternate method for finding the matrix of a linear transformation with respect to one pair of bases when the matrix for another pair of bases is known.

### Example 5

Recall the linear transformation $L: \mathcal{P}_3 \to \mathbb{R}^3$ from Examples 3 and 4, given by

$$L(a_3x^3 + a_2x^2 + a_1x + a_0) = [a_0 + a_1, 2a_2, a_3 - a_0].$$

Example 3 shows that the matrix for $L$ using the standard bases $B$ (for $\mathcal{P}_3$) and $C$ (for $\mathbb{R}^3$) is

$$\mathbf{A}_{BC} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}.$$

Also, in Example 4, we computed directly to find the matrix $\mathbf{A}_{DE}$ for the ordered bases $D = (x^3 + x^2,\ x^2 + x,\ x + 1,\ 1)$ for $\mathcal{P}_3$ and $E = ([-2, 1, -3], [1, -3, 0], [3, -6, 2])$ for $\mathbb{R}^3$. Instead, we now use Theorem 5.6 to calculate $\mathbf{A}_{DE}$. Now, the transition matrix $\mathbf{Q}$ from bases $C$ to $E$ is easily calculated to be

$$\mathbf{Q} = \begin{bmatrix} -6 & -2 & 3 \\ 16 & 5 & -9 \\ -9 & -3 & 5 \end{bmatrix}. \quad \text{(Verify!)}$$

Also, the transition matrix $\mathbf{P}^{-1}$ from bases $D$ to $B$ is

$$\mathbf{P}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}. \quad \text{(Verify!)}$$

Hence,

$$\mathbf{A}_{DE} = \mathbf{Q}\mathbf{A}_{BC}\mathbf{P}^{-1} = \begin{bmatrix} -6 & -2 & 3 \\ 16 & 5 & -9 \\ -9 & -3 & 5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -10 & -15 & -9 \\ 1 & 26 & 41 & 25 \\ -1 & -15 & -23 & -14 \end{bmatrix},$$

which agrees with the result obtained for $\mathbf{A}_{DE}$ in Example 4.

## Linear Operators and Similarity

Suppose $L$ is a linear operator on a nontrivial finite dimensional vector space $\mathcal{V}$. If $C$ is a basis for $\mathcal{V}$, then there is some matrix $\mathbf{A}_{CC}$ for $L$ with respect to $C$. Also, if $D$ is another basis for $\mathcal{V}$, then there is some matrix $\mathbf{A}_{DD}$ for $L$ with respect to $D$. Let $\mathbf{P}$ be the transition matrix from $D$ to $C$ (see Fig. 5.10). (Then $\mathbf{P}^{-1}$ is the transition matrix from $C$ to $D$.) Notice that by Theorem 5.6 we have $\mathbf{A}_{CC} = \mathbf{P}\mathbf{A}_{DD}\mathbf{P}^{-1}$, and so $\mathbf{A}_{DD} = \mathbf{P}^{-1}\mathbf{A}_{CC}\mathbf{P}$. Thus, by the definition of similar matrices, $\mathbf{A}_{CC}$ and $\mathbf{A}_{DD}$ are similar. This argument shows that any two matrices for the same linear operator with respect to different bases are similar.

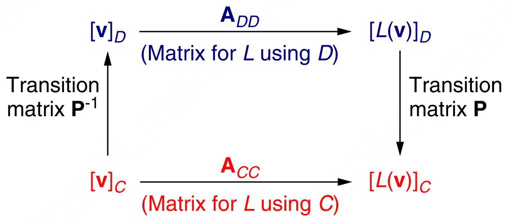


**FIGURE 5.10** Relationship between matrices $\mathbf{A}_{CC}$ and $\mathbf{A}_{DD}$ for a linear operator under a change of basis

### Example 6

Consider the linear operator $L\colon \mathbb{R}^3 \to \mathbb{R}^3$ whose matrix with respect to the standard basis $C$ for $\mathbb{R}^3$ is

$$\mathbf{A}_{CC} = \frac{1}{7}\begin{bmatrix} 6 & 3 & -2 \\ 3 & -2 & 6 \\ -2 & 6 & 3 \end{bmatrix}.$$

We will use eigenvectors to find another basis $D$ for $\mathbb{R}^3$ so that with respect to $D$, $L$ has a much simpler matrix representation. Now, $p_{\mathbf{A}_{CC}}(x) = |x\mathbf{I}_3 - \mathbf{A}_{CC}| = x^3 - x^2 - x + 1 = (x-1)^2(x+1)$ (verify!).

By row reducing $(1\mathbf{I}_3 - \mathbf{A}_{CC})$ and $(-1\mathbf{I}_3 - \mathbf{A}_{CC})$ we find the basis $\{[3,1,0], [-2,0,1]\}$ for the eigenspace $E_1$ for $\mathbf{A}_{CC}$ and the basis $\{[1,-3,2]\}$ for the eigenspace $E_{-1}$ for $\mathbf{A}_{CC}$. (Again, verify!) A quick check verifies that $D = \{[3,1,0], [-2,0,1], [1,-3,2]\}$ is a basis for $\mathbb{R}^3$ consisting of eigenvectors for $\mathbf{A}_{CC}$.

Now, recall from the remarks right before this example that $\mathbf{A}_{DD} = \mathbf{P}^{-1}\mathbf{A}_{CC}\mathbf{P}$, where $\mathbf{P}$ is the transition matrix from $D$ to $C$. But since $C$ is the standard basis, the matrix whose columns are the vectors in $D$ is the transition matrix from $D$ to $C$. Thus,

$$\mathbf{P} = \begin{bmatrix} 3 & -2 & 1 \\ 1 & 0 & -3 \\ 0 & 1 & 2 \end{bmatrix}, \quad \text{with} \quad \mathbf{P}^{-1} = \frac{1}{14}\begin{bmatrix} 3 & 5 & 6 \\ -2 & 6 & 10 \\ 1 & -3 & 2 \end{bmatrix},$$

which represents the transition matrix from $C$ to $D$. Finally,

$$\mathbf{A}_{DD} = \mathbf{P}^{-1}\mathbf{A}_{CC}\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

a diagonal matrix with the eigenvalues 1 and $-1$ on the main diagonal.

Written in this form, the operator $L$ is more comprehensible. Compare $\mathbf{A}_{DD}$ to the matrix for a reflection through the $xy$-plane given in Table 5.1. Now, because $D$ is not the standard basis for $\mathbb{R}^3$, $L$ is *not* a reflection through the $xy$-plane. But we can show that $L$ is a reflection of all vectors in $\mathbb{R}^3$ through the plane formed by the two basis vectors for $E_1$ (that is, the plane is the eigenspace $E_1$ itself). By the uniqueness assertion in Theorem 5.4, it is enough to show that $L$ acts as a reflection through the plane $E_1$ for each of the three basis vectors of $D$.

Since $[3, 1, 0]$ and $[-2, 0, 1]$ are in the plane $E_1$, we need to show that $L$ "reflects" these vectors to themselves. But this is true since $L([3, 1, 0]) = 1[3, 1, 0] = [3, 1, 0]$, and similarly for $[-2, 0, 1]$. Finally, notice that $[1, -3, 2]$ is orthogonal to the plane $E_1$ (since it is orthogonal to both $[3, 1, 0]$ and $[-2, 0, 1]$). Therefore, we need to show that $L$ "reflects" this vector to its opposite. But, $L([1, -3, 2]) = -1[1, -3, 2] = -[1, -3, 2]$, and we are done. Hence, $L$ is a reflection through the plane $E_1$.

Because the matrix $\mathbf{A}_{DD}$ in Example 6 is diagonal, it is easy to see that $p_{\mathbf{A}_{DD}}(x) = (x-1)^2(x+1)$. In Exercise 6 of Section 3.4, you were asked to prove that similar matrices have the same characteristic polynomial. Therefore, $p_{\mathbf{A}_{CC}}(x)$ also equals $(x-1)^2(x+1)$.

## Matrix for the Composition of Linear Transformations

Our final theorem for this section shows how to find the corresponding matrix for the composition of linear transformations. The proof is left as Exercise 15.

**Theorem 5.7** *Let $\mathcal{V}_1$, $\mathcal{V}_2$, and $\mathcal{V}_3$ be nontrivial finite dimensional vector spaces with ordered bases $B$, $C$, and $D$, respectively. Let $L_1: \mathcal{V}_1 \to \mathcal{V}_2$ be a linear transformation with matrix $\mathbf{A}_{BC}$ with respect to bases $B$ and $C$, and let $L_2: \mathcal{V}_2 \to \mathcal{V}_3$ be a linear transformation with matrix $\mathbf{A}_{CD}$ with respect to bases $C$ and $D$. Then the matrix $\mathbf{A}_{BD}$ for the composite linear transformation $L_2 \circ L_1: \mathcal{V}_1 \to \mathcal{V}_3$ with respect to bases $B$ and $D$ is the product $\mathbf{A}_{CD}\mathbf{A}_{BC}$.*

Theorem 5.7 can be generalized to compositions of several linear transformations, as in the next example.

### Example 7

Let $L_1, L_2, \ldots, L_5$ be the geometric linear operators on $\mathbb{R}^3$ given in Table 5.1. Let $\mathbf{A}_1, \ldots, \mathbf{A}_5$ be the matrices for these operators using the standard basis for $\mathbb{R}^3$. Then, the matrix for the composition $L_4 \circ L_5$ is

$$\mathbf{A}_4\mathbf{A}_5 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & k \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & k\cos\theta - k\sin\theta \\ \sin\theta & \cos\theta & k\sin\theta + k\cos\theta \\ 0 & 0 & 1 \end{bmatrix}.$$

Similarly, the matrix for the composition $L_2 \circ L_3 \circ L_1 \circ L_5$ is

$$\mathbf{A}_2\mathbf{A}_3\mathbf{A}_1\mathbf{A}_5 = \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & k \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c & 0 & kc \\ 0 & c & kc \\ 0 & 0 & 0 \end{bmatrix}.$$

- **Supplemental Material:** You have now covered the prerequisites for Section 7.3, "Complex Vector Spaces."
- **Application**: You have now covered the prerequisites for Section 8.8, "Computer Graphics."

## New Vocabulary

matrix for a linear transformation

## Highlights

- Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces, with $\dim(\mathcal{V}) = n \geq 1$. Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for $\mathcal{V}$, and let $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n$ be any $n$ (not necessarily distinct) vectors in $\mathcal{W}$. Then there is a unique linear transformation $L: \mathcal{V} \to \mathcal{W}$ such that $L(\mathbf{v}_i) = \mathbf{w}_i$, for $1 \leq i \leq n$ (that is, once the images of a basis for the domain are specified).
- Let $L: \mathcal{V} \to \mathcal{W}$ be a linear transformation between nontrivial finite dimensional vector spaces. Suppose $B = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ and $C = (\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m)$ are ordered bases for $\mathcal{V}$ and $\mathcal{W}$, respectively. Then there is a unique $m \times n$ matrix $\mathbf{A}_{BC}$ for $L$ with respect to $B$ and $C$ such that $\mathbf{A}_{BC}[\mathbf{v}]_B = [L(\mathbf{v})]_C$, for all $\mathbf{v} \in \mathcal{V}$. In fact, the $i$th column of $\mathbf{A}_{BC}$ $= [L(\mathbf{v}_i)]_C$, the $C$-coordinatization of the $i$th vector in $B$.
- Let $L: \mathcal{V} \to \mathcal{W}$ be a linear transformation between nontrivial finite dimensional vector spaces, having matrix $\mathbf{A}_{BC}$ with respect to ordered bases $B$ and $C$. If $D$ and $E$ are other ordered bases for $\mathcal{V}$ and $\mathcal{W}$, respectively, and $\mathbf{P}$ is the transition matrix from $B$ to $D$, and $\mathbf{Q}$ is the transition matrix from $C$ to $E$, then the matrix $\mathbf{A}_{DE}$ for $L$ in terms of bases $D$ and $E$ is $\mathbf{A}_{DE} = \mathbf{Q}\mathbf{A}_{BC}\mathbf{P}^{-1}$.
- Matrices (with respect to the standard basis) for several useful geometric operators on $\mathbb{R}^3$ are given in Table 5.1.
- Let $L: \mathcal{V} \to \mathcal{V}$ be a linear operator on a nontrivial finite dimensional vector space, having matrix $\mathbf{A}_{DD}$ with respect to an ordered basis $D$, and matrix $\mathbf{A}_{CC}$ with respect to an ordered basis $C$. If $\mathbf{P}$ is the transition matrix from $D$ to $C$, then $\mathbf{A}_{CC} = \mathbf{P}\mathbf{A}_{DD}\mathbf{P}^{-1}$ (and hence, $\mathbf{A}_{DD}$ and $\mathbf{A}_{CC}$ are similar matrices).
- If $L_1: \mathcal{V}_1 \to \mathcal{V}_2$ is a linear transformation with matrix $\mathbf{A}_{BC}$ with respect to ordered bases $B$ and $C$, and $L_2: \mathcal{V}_2 \to \mathcal{V}_3$ is a linear transformation with matrix $\mathbf{A}_{CD}$ with respect to ordered bases $C$ and $D$, then the matrix $\mathbf{A}_{BD}$ for $L_2 \circ L_1$: $\mathcal{V}_1 \to \mathcal{V}_3$ with respect to bases $B$ and $D$ is given by $\mathbf{A}_{BD} = \mathbf{A}_{CD}\mathbf{A}_{BC}$.

## Exercises for Section 5.2

**1.** Verify that the correct matrix is given for each of the geometric linear operators in Table 5.1.

**2.** For each of the following linear transformations $L: \mathcal{V} \to \mathcal{W}$, find the matrix for $L$ with respect to the standard bases for $\mathcal{V}$ and $\mathcal{W}$.

★ **(a)** $L: \mathbb{R}^3 \to \mathbb{R}^3$ given by $L([x, y, z]) = [-6x + 4y - z, -2x + 3y - 5z, 3x - y + 7z]$

**(b)** $L: \mathbb{R}^4 \to \mathbb{R}^3$ given by $L([w, x, y, z]) = [w + 4x - 3z, x + 2y + 7z, 2w - x + 5y]$

★ **(c)** $L: \mathcal{P}_3 \to \mathbb{R}^3$ given by $L(ax^3 + bx^2 + cx + d) = [4a - b + 3c + 3d, a + 3b - c + 5d, -2a - 7b + 5c - d]$

**(d)** $L: \mathcal{P}_3 \to \mathcal{M}_{22}$ given by

$$L(\mathbf{p}(x)) = \begin{bmatrix} \mathbf{p}(1) & \mathbf{p}(2) \\ \mathbf{p}(3) & \mathbf{p}(4) \end{bmatrix}$$

**(e)** $L: \mathcal{M}_{22} \to \mathcal{M}_{22}$ given by

$$L(\mathbf{A}) = \begin{bmatrix} 3 & 8 \\ 2 & 5 \end{bmatrix} \mathbf{A}$$

**3.** For each of the following linear transformations $L: \mathcal{V} \to \mathcal{W}$, find the matrix $\mathbf{A}_{BC}$ for $L$ with respect to the given bases $B$ for $\mathcal{V}$ and $C$ for $\mathcal{W}$ using the method of Theorem 5.5:

★ **(a)** $L: \mathbb{R}^3 \to \mathbb{R}^2$ given by $L([x, y, z]) = [-2x + 3z, x + 2y - z]$ with $B = ([1, -3, 2], [-4, 13, -3], [2, -3, 20])$ and $C = ([-2, -1], [5, 3])$

**(b)** $L: \mathbb{R}^2 \to \mathbb{R}^3$ given by $L([x, y]) = [13x - 9y, -x - 2y, -11x + 6y]$ with $B = ([3, -4], [-5, 6])$ and $C = ([1, 2, -3], [0, 1, -1], [-1, -3, 5])$

★ **(c)** $L: \mathbb{R}^2 \to \mathcal{P}_2$ given by $L([a, b]) = (-a + 5b)x^2 + (3a - b)x + 2b$ with $B = ([5, 3], [3, 2])$ and $C = (3x^2 - 2x, -2x^2 + 2x - 1, x^2 - x + 1)$

**(d)** $L: \mathcal{M}_{22} \to \mathbb{R}^3$ given by $L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = [a + 2c - 2d, 3a - b + d, 5c - d]$ with $B = \left(\begin{bmatrix} 6 & 5 \\ 4 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ -3 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}\right)$ and $C = ([4, 3, -3], [8, 6, -7], [-7, -5, 6])$

★ **(e)** $L: \mathcal{P}_2 \to \mathcal{M}_{23}$ given by

$$L(ax^2 + bx + c) = \begin{bmatrix} -a & 2b + c & 3a - c \\ a + b & c & -2a + b - c \end{bmatrix}$$

with $B = (-5x^2 - x - 1, -6x^2 + 3x + 1, 2x + 1)$ and $C = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix},\right.$
$\left.\begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$

**4.** In each case, find the matrix $\mathbf{A}_{DE}$ for the given linear transformation $L: \mathcal{V} \to \mathcal{W}$ with respect to the given bases $D$ and $E$ by first finding the matrix for $L$ with respect to the standard bases $B$ and $C$ for $\mathcal{V}$ and $\mathcal{W}$, respectively, and then using the method of Theorem 5.6.

★ **(a)** $L: \mathbb{R}^3 \to \mathbb{R}^3$ given by $L([a, b, c]) = [-2a + b, -b - c, a + 3c]$ with $D = ([15, -6, 4], [2, 0, 1], [3, -1, 1])$ and $E = ([1, -3, 1], [0, 3, -1], [2, -2, 1])$

★ **(b)** $L: \mathcal{M}_{22} \to \mathbb{R}^2$ given by

$$L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = [6a - b + 3c - 2d, -2a + 3b - c + 4d]$$

with

$$D = \left(\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right)$$

and $E = ([-2, 5], [-1, 2])$

**(c)** $L: \mathcal{P}_2 \to \mathcal{M}_{22}$ given by

$$L\left(ax^2 + bx + c\right) = \begin{bmatrix} a + 4b & b - 2c \\ 3a + c & a + b + c \end{bmatrix}$$

with

$$D = (2x - 1, -5x^2 + 3x - 1, x^2 - 2x + 1)$$

$$\text{and } E = \left(\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix}\right)$$

**(d)** $L: \mathcal{P}_3 \to \mathbb{R}^3$ given by $L(\mathbf{p}(x)) = [\mathbf{p}(-1), \mathbf{p}(0), \mathbf{p}(1)]$, with

$$D = \left(x^3 + x^2 + x + 1, 3x^2 + 2x + 1, 6x + 2, 6\right)$$

and $E = ([1, -1, -2], [-2, 3, 6], [-4, 6, 13])$

**5.** Verify that the same matrix is obtained for $L$ in Exercise 3(d) by first finding the matrix for $L$ with respect to the standard bases and then using the method of Theorem 5.6.

**6.** In each case, find the matrix $\mathbf{A}_{BB}$ for each of the given linear operators $L: \mathcal{V} \to \mathcal{V}$ with respect to the given basis $B$ by using the method of Theorem 5.5. Then, check your answer by calculating the matrix for $L$ using the standard basis and applying the method of Theorem 5.6.

★ **(a)** $L: \mathbb{R}^2 \to \mathbb{R}^2$ given by $L([x, y]) = [2x - y, x - 3y]$ with $B = ([4, -1], [-7, 2])$

★ **(b)** $L: \mathcal{P}_2 \to \mathcal{P}_2$ given by $L(ax^2 + bx + c) = (b - 2c)x^2 + (2a + c)x + (a - b - c)$ with $B = (2x^2 + 2x - 1, x, -3x^2 - 2x + 1)$

**(c)** $L: \mathcal{M}_{22} \to \mathcal{M}_{22}$ given by

$$L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 2a - 3b - c + d & a + 5b - 2c - d \\ 4b + 3d & 5a + 3c \end{bmatrix}$$

with

$$B = \left(\begin{bmatrix} 1 & 0 \\ -2 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -3 & 6 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ -2 & 7 \end{bmatrix}\right)$$

**(d)** $L: \mathcal{M}_{22} \to \mathcal{M}_{22}$ given by

$$L(\mathbf{A}) = \mathbf{A}\begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}$$

with

$$B = \left(\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 2 & -1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}\right)$$

**7.** This exercise involves matrices for linear transformations related to calculus.

★ **(a)** Let $L: \mathcal{P}_3 \to \mathcal{P}_2$ be given by $L(\mathbf{p}) = \mathbf{p}'$, for $\mathbf{p} \in \mathcal{P}_3$. Find the matrix for $L$ with respect to the standard bases for $\mathcal{P}_3$ and $\mathcal{P}_2$. Use this matrix to calculate $L(4x^3 - 5x^2 + 6x - 7)$ by matrix multiplication.

**(b)** Let $L: \mathcal{P}_2 \to \mathcal{P}_3$ be the indefinite integral linear transformation; that is, $L(\mathbf{p})$ is the vector $\int \mathbf{p}(x)\, dx$ with zero constant term. Find the matrix for $L$ with respect to the standard bases for $\mathcal{P}_2$ and $\mathcal{P}_3$. Use this matrix to calculate $L(2x^2 - x + 5)$ by matrix multiplication.

**(c)** Let $L: \mathcal{P}_4 \to \mathbb{R}$ be given by $L(\mathbf{p}) = \int_1^2 \mathbf{p}(x)\, dx$. Find the matrix for $L$ with respect to the standard bases for $\mathcal{P}_4$ and $\mathbb{R}$. Use this matrix to calculate $\int_1^2 (2x^4 - 5x^3 + 3x^2 - 7x + 6)\, dx$ by matrix multiplication.

**8.** Let $L: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operator that performs a counterclockwise rotation through an angle of $\frac{\pi}{6}$ radians (30°).

★ **(a)** Find the matrix for $L$ with respect to the standard basis for $\mathbb{R}^2$.

**(b)** Find the matrix for $L$ with respect to the basis $B = ([4, -5], [-2, 3])$.

**9.** Let $L: \mathcal{M}_{23} \to \mathcal{M}_{32}$ be given by $L(\mathbf{A}) = \mathbf{A}^T$.

**(a)** Find the matrix for $L$ with respect to the standard bases.

★ **(b)** Find the matrix for $L$ with respect to the bases

$$B = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \text{ for } \mathcal{M}_{23},$$

$$\text{and } C = \left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & -1 \end{bmatrix}\right) \text{ for } \mathcal{M}_{32}.$$

★ **10.** Let $B$ be a basis for $\mathcal{V}_1$, $C$ be a basis for $\mathcal{V}_2$, and $D$ be a basis for $\mathcal{V}_3$. Suppose $L_1: \mathcal{V}_1 \to \mathcal{V}_2$ and $L_2: \mathcal{V}_2 \to \mathcal{V}_3$ are represented, respectively, by the matrices

$$\mathbf{A}_{BC} = \begin{bmatrix} -2 & 3 & -1 \\ 4 & 0 & -2 \end{bmatrix} \quad \text{and} \quad \mathbf{A}_{CD} = \begin{bmatrix} 4 & -1 \\ 2 & 0 \\ -1 & -3 \end{bmatrix}.$$

Find the matrix $\mathbf{A}_{BD}$ representing the composition $L_2 \circ L_1: \mathcal{V}_1 \to \mathcal{V}_3$.

**11.** Let $L_1: \mathbb{R}^4 \to \mathbb{R}^3$ be given by $L_1([w, x, y, z]) = [2w - x + y, 3x - 2y + z, 4w + 5y - z]$, and let $L_2: \mathbb{R}^3 \to \mathbb{R}^2$ be given by $L_2([x, y, z]) = [6x + y - 3z, 5x - 4y + 7z]$.

**(a)** Find the matrices for $L_1$ and $L_2$ with respect to the standard bases in each case.

**(b)** Find the matrix for $L_2 \circ L_1$ with respect to the standard bases for $\mathbb{R}^3$ and $\mathbb{R}^2$ using Theorem 5.7.

**(c)** Check your answer to part (b) by computing $(L_2 \circ L_1)([x, y, z])$ and finding the matrix for $L_2 \circ L_1$ directly from this result.

**12.** Let $\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, the matrix representing the counterclockwise rotation of $\mathbb{R}^2$ about the origin through an angle $\theta$.

**(a)** Use Theorem 5.7 to show that

$$\mathbf{A}^2 = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}.$$

**(b)** Generalize the result of part (a) to show that for any integer $n \geq 1$,

$$\mathbf{A}^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}.$$

**13.** Let $B = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ be an ordered basis for a vector space $\mathcal{V}$. Find the matrix with respect to $B$ for each of the following linear operators $L: \mathcal{V} \to \mathcal{V}$:

★ **(a)** $L(\mathbf{v}) = \mathbf{v}$, for all $\mathbf{v} \in \mathcal{V}$ (identity linear operator)

**(b)** $L(\mathbf{v}) = \mathbf{0}$, for all $\mathbf{v} \in \mathcal{V}$ (zero linear operator)

★ **(c)** $L(\mathbf{v}) = c\mathbf{v}$, for all $\mathbf{v} \in \mathcal{V}$, and for some fixed $c \in \mathbb{R}$ (scalar linear operator)

**(d)** $L: \mathcal{V} \to \mathcal{V}$ given by $L(\mathbf{v}_1) = \mathbf{v}_2$, $L(\mathbf{v}_2) = \mathbf{v}_3$, $\ldots$, $L(\mathbf{v}_{n-1}) = \mathbf{v}_n$, $L(\mathbf{v}_n) = \mathbf{v}_1$ (forward replacement of basis vectors)

★ **(e)** $L: \mathcal{V} \to \mathcal{V}$ given by $L(\mathbf{v}_1) = \mathbf{v}_n$, $L(\mathbf{v}_2) = \mathbf{v}_1$, $\ldots$, $L(\mathbf{v}_{n-1}) = \mathbf{v}_{n-2}$, $L(\mathbf{v}_n) = \mathbf{v}_{n-1}$ (reverse replacement of basis vectors)

**14.** Let $L: \mathbb{R}^n \to \mathbb{R}$ be a linear transformation. Prove that there is a vector $\mathbf{x}$ in $\mathbb{R}^n$ such that $L(\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{y} \in \mathbb{R}^n$.

▶ **15.** Prove Theorem 5.7.

**16.** Let $L: \mathbb{R}^3 \to \mathbb{R}^3$ be given by $L([x, y, z]) = [-2x - 2y + 2z, 5x + 3y - z, 4x + 2y]$.

**(a)** What is the matrix for $L$ with respect to the standard basis for $\mathbb{R}^3$?

**(b)** What is the matrix for $L$ with respect to the basis

$$B = ([-1, 2, 1], [-2, 3, 2], [0, 1, 1])?$$

**(c)** What does your answer to part (b) tell you about the vectors in $B$? Explain.

**17.** In Example 6, verify that $p_{\mathbf{A}_{BB}}(x) = (x-1)^2(x+1)$, $\{[3, 1, 0], [-2, 0, 1]\}$ is a basis for the eigenspace $E_1$, $\{[1, -3, 2]\}$ is a basis for the eigenspace $E_{-1}$, the transition matrices $\mathbf{P}$ and $\mathbf{P}^{-1}$ are as indicated, and, finally, $\mathbf{A}_{DD} = \mathbf{P}^{-1}\mathbf{A}_{BB}\mathbf{P}$ is a diagonal matrix with entries 1, 1, and $-1$, respectively, on the main diagonal.

**18.** Let $L: \mathbb{R}^3 \to \mathbb{R}^3$ be the linear operator whose matrix with respect to the standard basis $B$ for $\mathbb{R}^3$ is

$$\mathbf{A}_{BB} = \frac{1}{9}\begin{bmatrix} 8 & 2 & 2 \\ 2 & 5 & -4 \\ 2 & -4 & 5 \end{bmatrix}.$$

★ **(a)** Calculate and factor $p_{\mathbf{A}_{BB}}(x)$. (Be sure to incorporate $\frac{1}{9}$ correctly into your calculations.)

★ **(b)** Solve for a basis for each eigenspace for $L$. Combine these to form a basis $C$ for $\mathbb{R}^3$.

★ **(c)** Find the transition matrix $\mathbf{P}$ from $C$ to $B$.

**(d)** Calculate $\mathbf{A}_{CC}$ using $\mathbf{A}_{BB}$, $\mathbf{P}$ and $\mathbf{P}^{-1}$.

**(e)** Use $\mathbf{A}_{CC}$ to give a geometric description of the operator $L$, as was done in Example 6.

**19.** Let $L$ be a linear operator on a vector space $\mathcal{V}$ with ordered basis $B = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$. Suppose that $k$ is a nonzero real number, and let $C$ be the ordered basis $(k\mathbf{v}_1, \ldots, k\mathbf{v}_n)$ for $\mathcal{V}$. Show that $\mathbf{A}_{BB} = \mathbf{A}_{CC}$.

**20.** Let $B = ([a, b], [c, d])$ be a basis for $\mathbb{R}^2$. Then $ad - bc \neq 0$ (why?). Let $L: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator such that $L([a, b]) = [c, d]$ and $L([c, d]) = [a, b]$. Show that the matrix for $L$ with respect to the standard basis for $\mathbb{R}^2$ is

$$\frac{1}{ad - bc}\begin{bmatrix} cd - ab & a^2 - c^2 \\ d^2 - b^2 & ab - cd \end{bmatrix}.$$

**21.** This exercise explores the reflection of a vector through a line.

**(a)** Let $L: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation where $L(\mathbf{v})$ is the reflection of $\mathbf{v}$ through the line $y = mx$. (Assume that the initial point of $\mathbf{v}$ is the origin.) Show that the matrix for $L$ with respect to the standard basis for $\mathbb{R}^2$ is

$$\frac{1}{1 + m^2}\begin{bmatrix} 1 - m^2 & 2m \\ 2m & m^2 - 1 \end{bmatrix}.$$

(Hint: Let $\mathbf{A}$ represent a reflection through the $x$-axis, and let $\mathbf{P}$ represent a counterclockwise rotation of the plane that moves the $x$-axis to the line $y = mx$ (as in Example 9 of Section 5.1). Consider $\mathbf{PAP}^{-1}$.

**(b)** Let $L: \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation where $L(\mathbf{v})$ is the reflection of $\mathbf{v}$ through the line through the origin parallel to the nonzero vector $\mathbf{r} = [a, b, c]$. (Assume that the initial point of $\mathbf{v}$ is the origin.) Show that the matrix for $L$ with respect to the standard basis for $\mathbb{R}^2$ is

$$\frac{1}{a^2+b^2+c^2}\begin{bmatrix} a^2-b^2-c^2 & 2ab & 2ac \\ 2ab & -a^2+b^2-c^2 & 2bc \\ 2ac & 2bc & -a^2-b^2+c^2 \end{bmatrix}.$$

(Hint: Use the formula in Exercise 21 of Section 1.2 to compute the images of the standard basis vectors.)

**22.** Let $\mathcal{V}$ and $\mathcal{W}$ be finite dimensional vector spaces, and let $\mathcal{Y}$ be a subspace of $\mathcal{V}$. Suppose that $L: \mathcal{Y} \to \mathcal{W}$ is a linear transformation. Prove that there is a linear transformation $L': \mathcal{V} \to \mathcal{W}$ such that $L'(\mathbf{y}) = L(\mathbf{y})$ for every $\mathbf{y} \in \mathcal{Y}$. ($L'$ is called an **extension** of $L$ to $\mathcal{V}$.)

▶ **23.** Prove the uniqueness assertion in Theorem 5.4. (Hint: Let $\mathbf{v}$ be any vector in $\mathcal{V}$. Show that there is only one possible answer for $L(\mathbf{v})$ by expressing $L(\mathbf{v})$ as a linear combination of the $\mathbf{w}_i$'s.)

★ **24.** True or False:

**(a)** If $L: \mathcal{V} \to \mathcal{W}$ is a linear transformation, and $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for $\mathcal{V}$, then for any $\mathbf{v} \in \mathcal{V}$, $L(\mathbf{v})$ can be computed if $L(\mathbf{v}_1), L(\mathbf{v}_2), \ldots, L(\mathbf{v}_n)$ are known.

**(b)** There is a unique linear transformation $L: \mathbb{R}^3 \to \mathcal{P}_3$ such that $L([1, 0, 0]) = x^3 - x^2$, $L([0, 1, 0]) = x^3 - x^2$, and $L([0, 0, 1]) = x^3 - x^2$.

**(c)** If $\mathcal{V}, \mathcal{W}$ are nontrivial finite dimensional vector spaces and $L: \mathcal{V} \to \mathcal{W}$ is a linear transformation, then there is a unique matrix $\mathbf{A}$ corresponding to $L$.

**(d)** If $L: \mathcal{V} \to \mathcal{W}$ is a linear transformation and $B$ is a (finite) nonempty ordered basis for $\mathcal{V}$, and $C$ is a (finite) nonempty ordered basis for $\mathcal{W}$, then $[\mathbf{v}]_B = \mathbf{A}_{BC}[L(\mathbf{v})]_C$.

**(e)** If $L: \mathcal{V} \to \mathcal{W}$ is a linear transformation and $B = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ is a nonempty ordered basis for $\mathcal{V}$, and $C$ is a (finite) nonempty ordered basis for $\mathcal{W}$, then the $i$th column of $\mathbf{A}_{BC}$ is $[L(\mathbf{v}_i)]_C$.

**(f)** The matrix for the projection of $\mathbb{R}^3$ onto the $xz$-plane (with respect to the standard basis) is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

**(g)** If $L: \mathcal{V} \to \mathcal{W}$ is a linear transformation, and $B$ and $D$ are (finite nonempty) ordered bases for $\mathcal{V}$, and $C$ and $E$ are (finite nonempty) ordered bases for $\mathcal{W}$, then $\mathbf{A}_{DE}\mathbf{P} = \mathbf{Q}\mathbf{A}_{BC}$, where $\mathbf{P}$ is the transition matrix from $B$ to $D$, and $\mathbf{Q}$ is the transition matrix from $C$ to $E$.

**(h)** If $L: \mathcal{V} \to \mathcal{V}$ is a linear operator on a nontrivial finite dimensional vector space, and $B$ and $D$ are ordered bases for $\mathcal{V}$, then $\mathbf{A}_{BB}$ is similar to $\mathbf{A}_{DD}$.

**(i)** Similar square matrices have identical characteristic polynomials.

**(j)** If $L_1, L_2: \mathbb{R}^2 \to \mathbb{R}^2$ are linear transformations with matrices $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, respectively, with respect to the standard basis, then the matrix for $L_2 \circ L_1$ with respect to the standard basis equals $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

## 5.3 The Dimension Theorem

In this section, we introduce two special subspaces associated with a linear transformation $L: \mathcal{V} \to \mathcal{W}$: the kernel of $L$ (a subspace of $\mathcal{V}$) and the range of $L$ (a subspace of $\mathcal{W}$). We illustrate techniques for calculating bases for both the kernel and range and show their dimensions are related to the rank of any matrix for the linear transformation. We then use this to show that any matrix and its transpose have the same rank.

### Kernel and Range

**Definition** Let $L: \mathcal{V} \to \mathcal{W}$ be a linear transformation. The **kernel** of $L$, denoted by $\ker(L)$, is the subset of all vectors in $\mathcal{V}$ that map to $\mathbf{0}_{\mathcal{W}}$. That is, $\ker(L) = \{\mathbf{v} \in \mathcal{V} \mid L(\mathbf{v}) = \mathbf{0}_{\mathcal{W}}\}$. The **range** of $L$, or, range$(L)$, is the subset of all vectors in $\mathcal{W}$ that are the image of some vector in $\mathcal{V}$. That is, range$(L) = \{L(\mathbf{v}) \mid \mathbf{v} \in \mathcal{V}\}$.

Remember that the kernel[3] is a subset of the *domain* and that the range is a subset of the *codomain*. Since the kernel of $L\colon \mathcal{V} \to \mathcal{W}$ is the pre-image of the subspace $\{\mathbf{0}_{\mathcal{W}}\}$ of $\mathcal{W}$, it must be a subspace of $\mathcal{V}$ by Theorem 5.3. That theorem also assures us that the range of $L$ is a subspace of $\mathcal{W}$. Hence, we have

**Theorem 5.8** *If $L\colon \mathcal{V} \to \mathcal{W}$ is a linear transformation, then the kernel of $L$ is a subspace of $\mathcal{V}$ and the range of $L$ is a subspace of $\mathcal{W}$.*

**Example 1**

**Projection:** For $n \geq 3$, consider the linear operator $L\colon \mathbb{R}^n \to \mathbb{R}^n$ given by $L([a_1, a_2, \ldots, a_n]) = [a_1, a_2, 0, \ldots, 0]$. Now, $\ker(L)$ consists of those elements of the domain that map to $[0, 0, \ldots, 0]$, the zero vector of the codomain. Hence, for vectors in the kernel, $a_1 = a_2 = 0$, but $a_3, \ldots, a_n$ can have any values. Thus,

$$\ker(L) = \{\, [0, 0, a_3, \ldots, a_n] \mid a_3, \ldots, a_n \in \mathbb{R}\} .$$

Notice that $\ker(L)$ is a subspace of the domain and that $\dim(\ker(L)) = n - 2$, because the standard basis vectors $\mathbf{e}_3, \ldots, \mathbf{e}_n$ of $\mathbb{R}^n$ span $\ker(L)$.

Also, $\text{range}(L)$ consists of those elements of the codomain $\mathbb{R}^n$ that are images of domain elements. Hence, $\text{range}(L) = \{\, [a_1, a_2, 0, \ldots, 0] \mid a_1, a_2 \in \mathbb{R}\}$. Notice that $\text{range}(L)$ is a subspace of the codomain and that $\dim(\text{range}(L)) = 2$, since the standard basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ span $\text{range}(L)$.

**Example 2**

**Differentiation:** Consider the linear transformation $L\colon \mathcal{P}_3 \to \mathcal{P}_2$ given by $L(ax^3 + bx^2 + cx + d) = 3ax^2 + 2bx + c$. Now, $\ker(L)$ consists of the polynomials in $\mathcal{P}_3$ that map to the zero polynomial in $\mathcal{P}_2$. However, if $3ax^2 + 2bx + c = 0$, we must have $a = b = c = 0$. Hence, $\ker(L) = \left\{ 0x^3 + 0x^2 + 0x + d \,\middle|\, d \in \mathbb{R} \right\}$; that is, $\ker(L)$ is just the subset of $\mathcal{P}_3$ of all constant polynomials. Notice that $\ker(L)$ is a subspace of $\mathcal{P}_3$ and that $\dim(\ker(L)) = 1$ because the single polynomial "1" spans $\ker(L)$.

Also, $\text{range}(L)$ consists of all polynomials in the codomain $\mathbb{R}^2$ of the form $3ax^2 + 2bx + c$. Since every polynomial $Ax^2 + Bx + C$ of degree 2 or less can be expressed in this form (take $a = A/3$, $b = B/2$, $c = C$), $\text{range}(L)$ is all of $\mathcal{P}_2$. Therefore, $\text{range}(L)$ is a subspace of $\mathcal{P}_2$, and $\dim(\text{range}(L)) = 3$.

**Example 3**

**Rotation:** Recall that the linear transformation $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

for some (fixed) angle $\theta$, represents the counterclockwise rotation of any vector $[x, y]$ with initial point at the origin through the angle $\theta$.

Now, $\ker(L)$ consists of all vectors in the domain $\mathbb{R}^2$ that map to $[0, 0]$ in the codomain $\mathbb{R}^2$. However, only $[0, 0]$ itself is rotated by $L$ to the zero vector. Hence, $\ker(L) = \{[0, 0]\}$. Notice that $\ker(L)$ is a subspace of $\mathbb{R}^2$, and $\dim(\ker(L)) = 0$.

Also, $\text{range}(L)$ is all of the codomain $\mathbb{R}^2$ because every nonzero vector $\mathbf{v}$ in $\mathbb{R}^2$ is the image of the vector of the same length at the angle $\theta$ *clockwise* from $\mathbf{v}$. Thus, $\text{range}(L) = \mathbb{R}^2$ and so, $\text{range}(L)$ is a subspace of $\mathbb{R}^2$ with $\dim(\text{range}(L)) = 2$.

## Finding the Kernel From the Matrix of a Linear Transformation

Consider the linear transformation $L\colon \mathbb{R}^n \to \mathbb{R}^m$ given by $L(\mathbf{X}) = \mathbf{AX}$, where $\mathbf{A}$ is a (fixed) $m \times n$ matrix and $\mathbf{X} \in \mathbb{R}^n$. Now, $\ker(L)$ is the subspace of all vectors $\mathbf{X}$ in the domain $\mathbb{R}^n$ that are solutions of the homogeneous system $\mathbf{AX} = \mathbf{0}$. If $\mathbf{B}$ is the reduced row echelon form matrix for $\mathbf{A}$, we find a basis for $\ker(L)$ by solving for fundamental solutions to the system $\mathbf{BX} = \mathbf{0}$ by systematically setting each independent variable equal to 1 in turn, while setting the others equal to 0. Thus, $\dim(\ker(L))$ equals the number of independent variables in the system $\mathbf{BX} = \mathbf{0}$.

We present an example of this technique.

[3] Some textbooks refer to the kernel of $L$ as the **nullspace** of $L$.

**Example 4**

Let $L: \mathbb{R}^5 \to \mathbb{R}^4$ be given by $L(\mathbf{X}) = \mathbf{AX}$, where

$$\mathbf{A} = \begin{bmatrix} 8 & 4 & 16 & 32 & 0 \\ 4 & 2 & 10 & 22 & -4 \\ -2 & -1 & -5 & -11 & 7 \\ 6 & 3 & 15 & 33 & -7 \end{bmatrix}.$$

To solve for $\ker(L)$, we first row reduce $\mathbf{A}$ to

$$\mathbf{B} = \begin{bmatrix} 1 & \frac{1}{2} & 0 & -2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The homogeneous system $\mathbf{BX} = \mathbf{0}$ has independent variables $x_2$ and $x_4$, and

$$\begin{cases} x_1 = -\frac{1}{2}x_2 + 2x_4 \\ x_3 = \qquad\quad - 3x_4 \\ x_5 = \qquad 0 \end{cases}.$$

The solution set of this system is

$$\ker(L) = \left\{ \left[ -\frac{1}{2}b + 2d, b, -3d, d, 0 \right] \,\middle|\, b, d \in \mathbb{R} \right\}.$$

The fundamental solutions are obtained here by first setting $x_2 = 1$ and $x_4 = 0$ to obtain $\mathbf{v}_1 = [-\frac{1}{2}, 1, 0, 0, 0]$, and then setting $x_2 = 0$ and $x_4 = 1$, yielding $\mathbf{v}_2 = [2, 0, -3, 1, 0]$. The set $\{\mathbf{v}_1, \mathbf{v}_2\}$ forms a basis for $\ker(L)$, and thus, $\dim(\ker(L)) = 2$, the number of independent variables. The entire subspace $\ker(L)$ consists of all linear combinations of these basis vectors; that is,

$$\ker(L) = \{b\mathbf{v}_1 + d\mathbf{v}_2 \,|\, b, d \in \mathbb{R}\}.$$

Finally, note that we could have eliminated fractions in this basis, just as we did with fundamental solutions in Section 2.2 by replacing $\mathbf{v}_1$ with $2\mathbf{v}_1 = [-1, 2, 0, 0, 0]$.

Example 4 illustrates the following general technique:

**Method for Finding a Basis for the Kernel of a Linear Transformation (Kernel Method)**
Let $L: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation given by $L(\mathbf{X}) = \mathbf{AX}$ for some $m \times n$ matrix $\mathbf{A}$. To find a basis for $\ker(L)$, perform the following steps:

**Step 1:** Find $\mathbf{B}$, the reduced row echelon form of $\mathbf{A}$.
**Step 2:** Calculate the fundamental solutions $\mathbf{v}_1, \ldots, \mathbf{v}_k$ for the homogeneous system $\mathbf{BX} = \mathbf{0}$.
**Step 3:** The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a basis for $\ker(L)$. (We can replace any $\mathbf{v}_i$ with $c\mathbf{v}_i$, where $c \neq 0$, to eliminate fractions.)

The method for finding a basis for $\ker(L)$ is practically identical to Step 3 of the Diagonalization Method of Section 3.4, in which we create a basis of fundamental eigenvectors for the eigenspace $E_\lambda$ for a matrix $\mathbf{A}$. This is to be expected, since $E_\lambda$ is really the kernel of the linear transformation $L$ whose matrix is $(\lambda\mathbf{I}_n - \mathbf{A})$.

## Finding the Range From the Matrix of a Linear Transformation

Next, we determine a method for finding a basis for the range of $L: \mathbb{R}^n \to \mathbb{R}^m$ given by $L(\mathbf{X}) = \mathbf{AX}$. In Section 1.5 we saw that $\mathbf{AX}$ can be expressed as a linear combination of the columns of $\mathbf{A}$. In particular, if $\mathbf{X} = [x_1, \ldots, x_n]$, then $\mathbf{AX} = x_1$ (1st column of $\mathbf{A}$) $+ \cdots + x_n$ ($n$th column of $\mathbf{A}$). Thus, range$(L)$ is spanned by the set of columns of $\mathbf{A}$; that is, range$(L)$ = span({columns of $\mathbf{A}$}). Note that $L(\mathbf{e}_i)$ equals the $i$th column of $\mathbf{A}$. Thus, we can also say that $\{L(\mathbf{e}_1), \ldots, L(\mathbf{e}_n)\}$ spans range$(L)$.

The fact that the columns of $\mathbf{A}$ span range$(L)$ combined with the Independence Test Method yields the following general technique for finding a basis for the range:

**Method for Finding a Basis for the Range of a Linear Transformation (Range Method)**
Let $L: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation given by $L(\mathbf{X}) = \mathbf{AX}$, for some $m \times n$ matrix $\mathbf{A}$. To find a basis for range$(L)$, perform the following steps:

**Step 1:** Find $\mathbf{B}$, the reduced row echelon form of $\mathbf{A}$.
**Step 2:** Form the set of those columns of $\mathbf{A}$ whose corresponding columns in $\mathbf{B}$ have pivots. This set is a basis for range$(L)$.

**Example 5**

Consider the linear transformation $L: \mathbb{R}^5 \to \mathbb{R}^4$ given in Example 4. After row reducing the matrix $\mathbf{A}$ for $L$ we obtained a matrix $\mathbf{B}$ in reduced row echelon form having pivots in columns 1, 3, and 5. Hence, columns 1, 3, and 5 of $\mathbf{A}$ form a basis for range$(L)$. In particular, we get the basis $\{[8, 4, -2, 6], [16, 10, -5, 15], [0, -4, 7, -7]\}$, and so $\dim(\text{range}(L)) = 3$.

From Examples 4 and 5, we see that $\dim(\ker(L)) + \dim(\text{range}(L)) = 2 + 3 = 5 = \dim(\mathbb{R}^5) = \dim(\text{domain}(L))$, for the given linear transformation $L$. We can understand why this works by examining our methods for calculating bases for the kernel and range. For $\ker(L)$, we get one basis vector for each independent variable, which corresponds to a nonpivot column of $\mathbf{A}$ after row reducing. For range$(L)$, we get one basis vector for each pivot column of $\mathbf{A}$. Together, these account for the total number of columns of $\mathbf{A}$, which is the dimension of the domain.

The fact that the number of pivots of $\mathbf{A}$ equals the number of nonzero rows in the reduced row echelon form matrix for $\mathbf{A}$ shows that $\dim(\text{range}(L)) = \text{rank}(\mathbf{A})$. This result is stated in the following theorem, which also holds when bases other than the standard bases are used (see Exercise 17).

**Theorem 5.9** *If $L: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation with matrix $\mathbf{A}$ with respect to any bases for $\mathbb{R}^n$ and $\mathbb{R}^m$, then*
**(1)** $\dim(\text{range}(L)) = \text{rank}(\mathbf{A})$
**(2)** $\dim(\ker(L)) = n - \text{rank}(\mathbf{A})$
**(3)** $\dim(\ker(L)) + \dim(\text{range}(L)) = \dim(\text{domain}(L)) = n$.

## The Dimension Theorem

The result in part (3) of Theorem 5.9 generalizes to linear transformations between any vector spaces $\mathcal{V}$ and $\mathcal{W}$, as long as the dimension of the domain is finite. We state this important theorem here, but postpone its proof until after a discussion of isomorphism in Section 5.5. An alternate proof of the Dimension Theorem that does not involve the matrix of the linear transformation is outlined in Exercise 18 of this section.

**Theorem 5.10 (Dimension Theorem)** *If $L: \mathcal{V} \to \mathcal{W}$ is a linear transformation and $\mathcal{V}$ is finite dimensional, then* range$(L)$ *is finite dimensional, and*

$$\dim(\ker(L)) + \dim(\text{range}(L)) = \dim(\mathcal{V}).$$

We have already seen that for the linear transformation in Examples 4 and 5, the dimensions of the kernel and the range add up to the dimension of the domain, as the Dimension Theorem asserts. Notice the Dimension Theorem holds for the linear transformations in Examples 1 through 3 as well.

**Example 6**

Consider $L: \mathcal{M}_{nn} \to \mathcal{M}_{nn}$ given by $L(\mathbf{A}) = \mathbf{A} + \mathbf{A}^T$. You were asked to prove that $L$ is a linear operator and to determine range$(L)$ in Exercise 13 of Section 5.1. We will again determine range$(L)$ here in order to verify that the Dimension Theorem holds.

Now, $\ker(L) = \{\mathbf{A} \in \mathcal{M}_{nn} \,|\, \mathbf{A} + \mathbf{A}^T = \mathbf{O}_n\}$. However, $\mathbf{A} + \mathbf{A}^T = \mathbf{O}_n$ if and only if $\mathbf{A} = -\mathbf{A}^T$. Hence, $\ker(L)$ is precisely the set of all skew-symmetric $n \times n$ matrices.

The range of $L$ is the set of all matrices $\mathbf{B}$ of the form $\mathbf{A} + \mathbf{A}^T$ for some $n \times n$ matrix $\mathbf{A}$. However, if $\mathbf{B} = \mathbf{A} + \mathbf{A}^T$, then $\mathbf{B}^T = \left(\mathbf{A} + \mathbf{A}^T\right)^T = \mathbf{A}^T + \mathbf{A} = \mathbf{B}$, so $\mathbf{B}$ is symmetric. Thus, range$(L) \subseteq \{$symmetric $n \times n$ matrices$\}$.

Next, if $\mathbf{B}$ is a symmetric $n \times n$ matrix, then $L(\frac{1}{2}\mathbf{B}) = \frac{1}{2}L(\mathbf{B}) = \frac{1}{2}(\mathbf{B}+\mathbf{B}^T) = \frac{1}{2}(\mathbf{B}+\mathbf{B}) = \mathbf{B}$, and so $\mathbf{B} \in \text{range}(L)$, thus proving {symmetric $n \times n$ matrices} $\subseteq \text{range}(L)$. Hence, range$(L)$ is the set of all symmetric $n \times n$ matrices.

In Exercise 14 of Section 4.6, we found that dim({skew-symmetric $n \times n$ matrices}) $= (n^2 - n)/2$ and that dim({symmetric $n \times n$ matrices}) $= \left(n^2+n\right)/2$. Notice that the Dimension Theorem holds here, since

$$\dim(\ker(L)) + \dim(\text{range}(L)) = \frac{n^2-n}{2} + \frac{n^2+n}{2} = n^2 = \dim\left(\mathcal{M}_{nn}\right).$$

## Rank of the Transpose

We can use the Range Method to prove the following Corollary of Theorem 5.9[4]:

**Corollary 5.11** *If* $\mathbf{A}$ *is any matrix, then* $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}^T)$.

*Proof.* Let $\mathbf{A}$ be an $m \times n$ matrix. Consider the linear transformation $L\colon \mathbb{R}^n \to \mathbb{R}^m$ with associated matrix $\mathbf{A}$ (using the standard bases). By the Range Method, range$(L)$ is the span of the column vectors of $\mathbf{A}$. Hence, range$(L)$ is the span of the row vectors of $\mathbf{A}^T$; that is, range$(L)$ is the row space of $\mathbf{A}^T$. Thus, $\dim(\text{range}(L)) = \text{rank}(\mathbf{A}^T)$, by the Simplified Span Method. But by Theorem 5.9, $\dim(\text{range}(L)) = \text{rank}(\mathbf{A})$. Hence, $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}^T)$. □

### Example 7

Let $\mathbf{A}$ be the matrix from Examples 4 and 5. We calculated its reduced row echelon form $\mathbf{B}$ in Example 4 and found it has 3 nonzero rows. Hence, $\text{rank}(\mathbf{A}) = 3$. Now,

$$\mathbf{A}^T = \begin{bmatrix} 8 & 4 & -2 & 6 \\ 4 & 2 & -1 & 3 \\ 16 & 10 & -5 & 15 \\ 32 & 22 & -11 & 33 \\ 0 & -4 & 7 & -7 \end{bmatrix} \text{ row reduces to } \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{7}{5} \\ 0 & 0 & 1 & -\frac{1}{5} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

showing that $\text{rank}(\mathbf{A}^T) = 3$ as well.

In some textbooks, rank$(\mathbf{A})$ is called the **row rank** of $\mathbf{A}$ and rank$(\mathbf{A}^T)$ is called the **column rank** of $\mathbf{A}$. Thus, Corollary 5.11 asserts that the row rank of $\mathbf{A}$ equals the column rank of $\mathbf{A}$.

Recall that rank$(\mathbf{A})=$ dim(row space of $\mathbf{A}$). Analogous to the concept of row space, we define the **column space** of a matrix $\mathbf{A}$ as the span of the columns of $\mathbf{A}$. In Corollary 5.11, we observed that if $L\colon \mathbb{R}^n \to \mathbb{R}^m$ with $L(\mathbf{X}) = \mathbf{AX}$ (using the standard bases), then range$(L) =$ span({columns of $\mathbf{A}$}) $=$ column space of $\mathbf{A}$, and so dim(range$(L)$) $=$ dim(column space of $\mathbf{A}$) $= \text{rank}(\mathbf{A}^T)$. With this new terminology, Corollary 5.11 asserts that dim(row space of $\mathbf{A}$) $=$ dim(column space of $\mathbf{A}$). Be careful! This statement does not imply that these *spaces* are equal, only that their *dimensions* are equal. In fact, unless $\mathbf{A}$ is square, they contain vectors of different sizes. Notice that for the matrix $\mathbf{A}$ in Example 7, the row space of $\mathbf{A}$ is a subspace of $\mathbb{R}^5$, but the column space of $\mathbf{A}$ is a subspace of $\mathbb{R}^4$.

## New Vocabulary

column rank (of a matrix)
column space (of a matrix)
Dimension Theorem
kernel (of a linear transformation)
Kernel Method
range (of a linear transformation)
Range Method
row rank (of a matrix)

[4] In Exercise 20 of Section 4.6 you were asked to prove the result in Corollary 5.11 by essentially the same method given here, only using different notation.

## Highlights

- If $L\colon \mathcal{V} \to \mathcal{W}$ is a linear transformation, the kernel of $L$ is the subset of all vectors in $\mathcal{V}$ that map to $\mathbf{0}_{\mathcal{W}}$. That is, $\ker(L) = \{\mathbf{v} \in \mathcal{V} \mid L(\mathbf{v}) = \mathbf{0}_{\mathcal{W}}\}$. The kernel of $L$ is always a subspace of $\mathcal{V}$.
- If $L\colon \mathcal{V} \to \mathcal{W}$ is a linear transformation, the range of $L$ is the subset of all vectors in $\mathcal{W}$ that are the image of some vector in $\mathcal{V}$. That is, $\text{range}(L) = \{L(\mathbf{v}) \mid \mathbf{v} \in \mathcal{V}\}$. The range of $L$ is always a subspace of $\mathcal{W}$.
- If $\mathbf{A}$ is the matrix (with respect to any bases) for a linear transformation $L\colon \mathbb{R}^n \to \mathbb{R}^m$, then $\dim(\ker(L)) = n - \text{rank}(\mathbf{A})$ and $\dim(\text{range}(L)) = \text{rank}(\mathbf{A})$.
- Kernel Method: A basis for the kernel of a linear transformation $L(\mathbf{X}) = \mathbf{AX}$ is obtained by determining the fundamental solutions for the homogeneous system $\mathbf{AX} = \mathbf{0}$. (This is accomplished by finding the reduced row echelon form matrix $\mathbf{B}$ for $\mathbf{A}$, and setting, in turn, each independent variable of the solution set of $\mathbf{BX} = \mathbf{0}$ equal to 1 and all other independent variables equal to 0.)
- Range Method: A basis for the range of a linear transformation $L(\mathbf{X}) = \mathbf{AX}$ is obtained by finding the reduced row echelon form matrix $\mathbf{B}$ for $\mathbf{A}$, and then selecting the columns of $\mathbf{A}$ that correspond to pivot columns in $\mathbf{B}$.
- Dimension Theorem: If $L\colon \mathcal{V} \to \mathcal{W}$ is a linear transformation and $\mathcal{V}$ is finite dimensional, then $\dim(\ker(L)) + \dim(\text{range}(L)) = \dim(\mathcal{V})$.
- If $\mathbf{A}$ is any matrix, $\text{rank}(\mathbf{A}) = \text{rank}\left(\mathbf{A}^T\right)$. That is, the row rank of $\mathbf{A}$ equals the column rank of $\mathbf{A}$.
- If $\mathbf{A}$ is any matrix, dim(row space of $\mathbf{A}$) = dim(column space of $\mathbf{A}$).

## Exercises for Section 5.3

**1.** Let $L\colon \mathbb{R}^4 \to \mathbb{R}^3$ be given by

$$L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 & -1 & -4 \\ 3 & 5 & 1 & -6 \\ -1 & 0 & 3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

★ **(a)** Is $[10, -5, 1, 1]$ in $\ker(L)$? Why or why not?
★ **(b)** Is $[4, 2, -5]$ in $\ker(L)$? Why or why not?
**(c)** Is $[-5, 0, 3, -2]$ in $\ker(L)$? Why or why not?
**(d)** Is $[4, 2, -5, 7]$ in $\ker(L)$? Why or why not?
★ **(e)** Is $[2, -1, 4]$ in $\text{range}(L)$? Why or why not?
**(f)** Is $[2, 6, -2]$ in $\text{range}(L)$? Why or why not?
**(g)** Is $[4, 2, -5]$ in $\text{range}(L)$? Why or why not?

**2.** Let $L\colon \mathcal{P}_3 \to \mathcal{P}_3$ be given by $L(ax^3 + bx^2 + cx + d) = (3a + 4b + 10c - d)x^3 - (2a + b - 4d)x^2 + (a + b + 2c - d)$.

★ **(a)** Is $x^3 - 5x^2 + 3x - 6$ in $\ker(L)$? Why or why not?
**(b)** Is $4x^3 - 4x^2$ in $\ker(L)$? Why or why not?
**(c)** Is $-8x^2 + 3x - 2$ in $\ker(L)$? Why or why not?
★ **(d)** Is $-4x^3 + x^2 - 1$ in $\text{range}(L)$? Why or why not?
**(e)** Is $-5x^3 + 3x - 6$ in $\text{range}(L)$? Why or why not?
**(f)** Is $5x^2 - 1$ in $\text{range}(L)$? Why or why not?

**3.** For each of the following linear transformations $L\colon \mathcal{V} \to \mathcal{W}$, find a basis for $\ker(L)$ and a basis for $\text{range}(L)$. Verify that $\dim(\ker(L)) + \dim(\text{range}(L)) = \dim(\mathcal{V})$.

★ **(a)** $L\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 & 5 \\ -2 & 3 & -13 \\ 3 & -3 & 15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

**(b)** $L\colon \mathbb{R}^5 \to \mathbb{R}^4$ given by

$$L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 4 & 8 & -8 \\ -1 & -2 & 0 & 3 & -5 \\ 1 & 2 & 3 & 3 & -1 \\ 2 & 4 & 3 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

(c) $L: \mathbb{R}^3 \to \mathbb{R}^4$ given by

$$L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 & -6 \\ -1 & 6 & 41 \\ 1 & -3 & -26 \\ 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

★ (d) $L: \mathbb{R}^4 \to \mathbb{R}^5$ given by

$$L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} -14 & -8 & -10 & 2 \\ -4 & -1 & 1 & -2 \\ -6 & 2 & 12 & -10 \\ 3 & -7 & -24 & 17 \\ 4 & 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

4. For each of the following linear transformations $L: \mathcal{V} \to \mathcal{W}$, find a basis for $\ker(L)$ and a basis for $\text{range}(L)$, and verify that $\dim(\ker(L)) + \dim(\text{range}(L)) = \dim(\mathcal{V})$:
★ (a) $L: \mathbb{R}^3 \to \mathbb{R}^2$ given by $L([x_1, x_2, x_3]) = [0, x_2]$
(b) $L: \mathbb{R}^2 \to \mathbb{R}^3$ given by $L([x_1, x_2]) = [x_1, x_1 - x_2, x_2]$
(c) $L: \mathcal{M}_{22} \to \mathcal{M}_{32}$ given by $L\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & -a_{11} \\ -a_{22} & 0 \end{bmatrix}$
★ (d) $L: \mathcal{P}_4 \to \mathcal{P}_2$ given by $L(ax^4 + bx^3 + cx^2 + dx + e) = cx^2 + dx + e$
(e) $L: \mathcal{P}_3 \to \mathcal{P}_3$ given by $L(ax^3 + bx^2 + cx + d) = cx^2 + bx + a$
★ (f) $L: \mathbb{R}^3 \to \mathbb{R}^3$ given by $L([x_1, x_2, x_3]) = [x_1, 0, x_1 - x_2 + x_3]$
★ (g) $L: \mathcal{M}_{22} \to \mathcal{M}_{22}$ given by $L(\mathbf{A}) = \mathbf{A}^T$
(h) $L: \mathcal{M}_{33} \to \mathcal{M}_{33}$ given by $L(\mathbf{A}) = \mathbf{A} + \mathbf{A}^T$
★ (i) $L: \mathcal{P}_2 \to \mathbb{R}^2$ given by $L(\mathbf{p}) = \left[\mathbf{p}(1), \mathbf{p}'(1)\right]$
(j) $L: \mathcal{P}_4 \to \mathbb{R}^3$ given by $L(\mathbf{p}) = [\mathbf{p}(0), \mathbf{p}(1), \mathbf{p}(2)]$

5. This exercise concerns the zero and identity linear transformations.
(a) Suppose that $L: \mathcal{V} \to \mathcal{W}$ is the linear transformation given by $L(\mathbf{v}) = \mathbf{0}_{\mathcal{W}}$, for all $\mathbf{v} \in \mathcal{V}$. What is $\ker(L)$? What is $\text{range}(L)$?
(b) Suppose that $L: \mathcal{V} \to \mathcal{V}$ is the linear transformation given by $L(\mathbf{v}) = \mathbf{v}$, for all $\mathbf{v} \in \mathcal{V}$. What is $\ker(L)$? What is $\text{range}(L)$?

★ 6. Consider the mapping $L: \mathcal{M}_{33} \to \mathbb{R}$ given by $L(\mathbf{A}) = \text{trace}(\mathbf{A})$ (see Exercise 13 in Section 1.4). Show that $L$ is a linear transformation. What is $\ker(L)$? What is $\text{range}(L)$? Calculate $\dim(\ker(L))$ and $\dim(\text{range}(L))$.

7. Let $\mathcal{V}$ be a vector space with basis $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$. Define $L: \mathcal{V} \to \mathcal{V}$ by $L(\mathbf{v}_1) = \mathbf{v}_2$, $L(\mathbf{v}_2) = \mathbf{v}_3, \dots, L(\mathbf{v}_{n-1}) = \mathbf{v}_n$, $L(\mathbf{v}_n) = \mathbf{v}_1$. Find $\text{range}(L)$. What is $\ker(L)$?

★ 8. Consider $L: \mathcal{P}_2 \to \mathcal{P}_4$ given by $L(\mathbf{p}) = x^2\mathbf{p}$. What is $\ker(L)$? What is $\text{range}(L)$? Verify that $\dim(\ker(L)) + \dim(\text{range}(L)) = \dim(\mathcal{P}_2)$.

9. Consider $L: \mathcal{P}_4 \to \mathcal{P}_2$ given by $L(\mathbf{p}) = \mathbf{p}''$. What is $\ker(L)$? What is $\text{range}(L)$? Verify that $\dim(\ker(L)) + \dim(\text{range}(L)) = \dim(\mathcal{P}_4)$.

★ 10. Consider $L: \mathcal{P}_n \to \mathcal{P}_n$ given by $L(\mathbf{p}) = \mathbf{p}^{(k)}$ (the $k$th derivative of $\mathbf{p}$), where $1 \le k \le n$. What is $\dim(\ker(L))$? What is $\dim(\text{range}(L))$? What happens when $k > n$?

11. Let $a$ be a fixed real number. Consider $L: \mathcal{P}_n \to \mathbb{R}$ given by $L(\mathbf{p}(x)) = \mathbf{p}(a)$ (that is, the evaluation of $\mathbf{p}$ at $x = a$). (Recall from Exercise 18 in Section 5.1 that $L$ is a linear transformation.)
(a) Show that $\left\{x - a,\ x^2 - a^2, \dots,\ x^n - a^n\right\}$ is a basis for $\ker(L)$. (Hint: What is $\text{range}(L)$?)
(b) Prove that $q(x) = 7x^3 - 9x^2 - 12x + 4$ is a linear combination of $\{x + 1, x^2 - 1, x^3 + 1\}$, without solving for the coefficients in the linear combination.

★ 12. Suppose that $L: \mathbb{R}^n \to \mathbb{R}^n$ is a linear operator given by $L(\mathbf{X}) = \mathbf{AX}$, where $|\mathbf{A}| \neq 0$. What is $\ker(L)$? What is $\text{range}(L)$?

13. Let $\mathcal{V}$ be a vector space, and let $L: \mathcal{V} \to \mathcal{V}$ be a linear operator.
(a) If $\mathcal{V}$ is finite dimensional, show that $\ker(L) = \{\mathbf{0}_{\mathcal{V}}\}$ if and only if $\text{range}(L) = \mathcal{V}$.
(b) Give an example in which $\mathcal{V}$ is infinite dimensional, $\ker(L) = \{\mathbf{0}_{\mathcal{V}}\}$, and $\text{range}(L) \neq \mathcal{V}$.

**14.** Let $L: \mathcal{V} \to \mathcal{W}$ be a linear transformation. Prove directly that $\ker(L)$ is a subspace of $\mathcal{V}$ and that $\text{range}(L)$ is a subspace of $\mathcal{W}$ using Theorem 4.2; that is, without invoking Theorem 5.8.

**15.** Let $L_1: \mathcal{V} \to \mathcal{W}$ and $L_2: \mathcal{W} \to \mathcal{X}$ be linear transformations.
- **(a)** Show that $\ker(L_1) \subseteq \ker(L_2 \circ L_1)$.
- **(b)** Show that $\text{range}(L_2 \circ L_1) \subseteq \text{range}(L_2)$.
- **(c)** If $\mathcal{V}$ is finite dimensional, prove that $\dim(\text{range}(L_2 \circ L_1)) \le \dim(\text{range}(L_1))$.

★ **16.** Give an example of a linear operator $L: \mathbb{R}^2 \to \mathbb{R}^2$ such that $\ker(L) = \text{range}(L)$.

▶ **17.** Let $L: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with $m \times n$ matrix $\mathbf{A}$ for $L$ with respect to the standard bases and $m \times n$ matrix $\mathbf{B}$ for $L$ with respect to bases $B$ and $C$.
- **(a)** Prove that $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{B})$. (Hint: Use Exercise 16 in the Review Exercises for Chapter 2.)
- **(b)** Use part (a) to finish the proof of Theorem 5.9. (Hint: Notice that Theorem 5.9 allows *any* bases to be used for $\mathbb{R}^n$ and $\mathbb{R}^m$. You can assume, from the remarks before Theorem 5.9, that the theorem is true when the standard bases are used for $\mathbb{R}^n$ and $\mathbb{R}^m$.)

**18.** This exercise outlines an alternate proof of the Dimension Theorem. Let $L: \mathcal{V} \to \mathcal{W}$ be a linear transformation with $\mathcal{V}$ finite dimensional. Assume that $\mathcal{V}$ and $\mathcal{W}$ are nontrivial vector spaces. Fig. 5.11 illustrates the relationships among the vectors referenced throughout this exercise.
- **(a)** Let $\{\mathbf{k}_1, \ldots, \mathbf{k}_s\}$ be a basis for $\ker(L)$. Show that there exist vectors $\mathbf{q}_1, \ldots, \mathbf{q}_t$ such that $\{\mathbf{k}_1, \ldots, \mathbf{k}_s, \mathbf{q}_1, \ldots, \mathbf{q}_t\}$ is a basis for $\mathcal{V}$. Express $\dim(\mathcal{V})$ in terms of $s$ and $t$.
- **(b)** Use part (a) to show that for every $\mathbf{v} \in \mathcal{V}$, there exist scalars $b_1, \ldots, b_t$ such that $L(\mathbf{v}) = b_1 L(\mathbf{q}_1) + \cdots + b_t L(\mathbf{q}_t)$.
- **(c)** Use part (b) to show that $\{L(\mathbf{q}_1), \ldots, L(\mathbf{q}_t)\}$ spans $\text{range}(L)$. Conclude that $\dim(\text{range}(L)) \le t$, and, hence, is finite.
- **(d)** Suppose that $c_1 L(\mathbf{q}_1) + \cdots + c_t L(\mathbf{q}_t) = \mathbf{0}_{\mathcal{W}}$. Prove that $c_1\mathbf{q}_1 + \cdots + c_t\mathbf{q}_t \in \ker(L)$.
- **(e)** Use part (d) to show that there are scalars $d_1, \ldots, d_s$ such that $c_1\mathbf{q}_1 + \cdots + c_t\mathbf{q}_t = d_1\mathbf{k}_1 + \cdots + d_s\mathbf{k}_s$.
- **(f)** Use part (e) and the fact that $\{\mathbf{k}_1, \ldots, \mathbf{k}_s, \mathbf{q}_1, \ldots, \mathbf{q}_t\}$ is a basis for $\mathcal{V}$ to prove that $c_1 = c_2 = \cdots = c_t = d_1 = \cdots = d_s = 0$.
- **(g)** Use parts (d) and (f) to conclude that $\{L(\mathbf{q}_1), \ldots, L(\mathbf{q}_t)\}$ is linearly independent.
- **(h)** Use parts (c) and (g) to prove that $\{L(\mathbf{q}_1), \ldots, L(\mathbf{q}_t)\}$ is a basis for $\text{range}(L)$.
- **(i)** Conclude that $\dim(\ker(L)) + \dim(\text{range}(L)) = \dim(\mathcal{V})$.

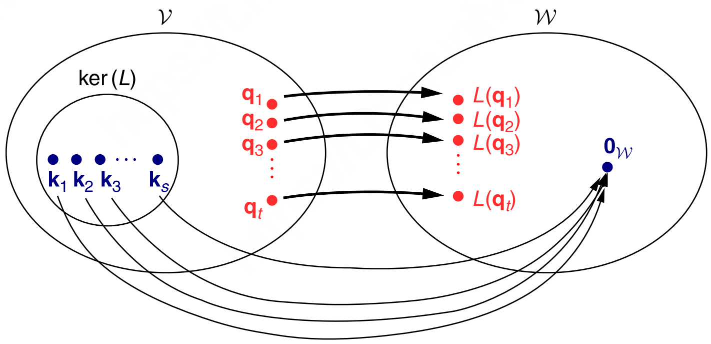


FIGURE 5.11 Images of basis elements in Exercise 18

**19.** Prove the following corollary of the Dimension Theorem: Let $L: \mathcal{V} \to \mathcal{W}$ be a linear transformation with $\mathcal{V}$ finite dimensional. Then $\dim(\ker(L)) \le \dim(\mathcal{V})$ and $\dim(\text{range}(L)) \le \dim(\mathcal{V})$.

★ **20.** True or False:
- **(a)** If $L: \mathcal{V} \to \mathcal{W}$ is a linear transformation, then $\ker(L) = \{L(\mathbf{v}) \mid \mathbf{v} \in \mathcal{V}\}$.
- **(b)** If $L: \mathcal{V} \to \mathcal{W}$ is a linear transformation, then $\text{range}(L)$ is a subspace of $\mathcal{V}$.
- **(c)** If $L: \mathcal{V} \to \mathcal{W}$ is a linear transformation and $\dim(\mathcal{V}) = n$, then $\dim(\ker(L)) = n - \dim(\text{range}(L))$.
- **(d)** If $L: \mathcal{V} \to \mathcal{W}$ is a linear transformation and $\dim(\mathcal{V}) = 5$ and $\dim(\mathcal{W}) = 3$, then the Dimension Theorem implies that $\dim(\ker(L)) = 2$.
- **(e)** If $L: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and $L(\mathbf{X}) = \mathbf{AX}$, then $\dim(\ker(L))$ equals the number of nonpivot columns in the reduced row echelon form matrix for $\mathbf{A}$.
- **(f)** If $L: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and $L(\mathbf{X}) = \mathbf{AX}$, then $\dim(\text{range}(L)) = n - \text{rank}(\mathbf{A})$.

**(g)** If $\mathbf{A}$ is a $5 \times 5$ matrix, and rank $(\mathbf{A}) = 2$, then rank $\left(\mathbf{A}^T\right) = 3$.
**(h)** If $\mathbf{A}$ is any matrix, then the row space of $\mathbf{A}$ equals the column space of $\mathbf{A}$.

# 5.4 One-to-One and Onto Linear Transformations

The kernel and the range of a linear transformation are related to the function properties one-to-one and onto. Consequently, in this section we study linear transformations that are one-to-one or onto.

## One-to-One and Onto Linear Transformations

One-to-one functions and onto functions are defined and discussed in Appendix B. In particular, Appendix B contains the usual methods for proving that a given function is, or is not, one-to-one or onto. Now, we are interested primarily in linear transformations, so we restate the definitions of *one-to-one* and *onto* specifically as they apply to this type of function.

**Definition** Let $L\colon \mathcal{V} \to \mathcal{W}$ be a linear transformation.

**(1)** $L$ is **one-to-one** if and only if distinct vectors in $\mathcal{V}$ have different images in $\mathcal{W}$. That is, $L$ is **one-to-one** if and only if, for all $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}$, $L(\mathbf{v}_1) = L(\mathbf{v}_2)$ implies $\mathbf{v}_1 = \mathbf{v}_2$.

**(2)** $L$ is **onto** if and only if every vector in the codomain $\mathcal{W}$ is the image of some vector in the domain $\mathcal{V}$. That is, $L$ is **onto** if and only if, for every $\mathbf{w} \in \mathcal{W}$, there is some $\mathbf{v} \in \mathcal{V}$ such that $L(\mathbf{v}) = \mathbf{w}$.

Notice that the two descriptions of a one-to-one linear transformation given in this definition are really contrapositives of each other.

### Example 1

**Rotation:** Recall the rotation linear operator $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ from Example 9 in Section 5.1 given by $L(\mathbf{v}) = \mathbf{A}\mathbf{v}$, where $\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. We will show that $L$ is *both one-to-one and onto*.

To show that $L$ is one-to-one, we take any two arbitrary vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in the domain $\mathbb{R}^2$, assume that $L(\mathbf{v}_1) = L(\mathbf{v}_2)$, and prove that $\mathbf{v}_1 = \mathbf{v}_2$. Now, if $L(\mathbf{v}_1) = L(\mathbf{v}_2)$, then $\mathbf{A}\mathbf{v}_1 = \mathbf{A}\mathbf{v}_2$. Because $\mathbf{A}$ is nonsingular, we can multiply both sides on the left by $\mathbf{A}^{-1}$ to obtain $\mathbf{v}_1 = \mathbf{v}_2$. Hence, $L$ is one-to-one.

To show that $L$ is onto, we must take any arbitrary vector $\mathbf{w}$ in the codomain $\mathbb{R}^2$ and show that there is some vector $\mathbf{v}$ in the domain $\mathbb{R}^2$ that maps to $\mathbf{w}$. Recall that multiplication by $\mathbf{A}^{-1}$ undoes the action of multiplication by $\mathbf{A}$, and so it must represent a *clockwise* rotation through the angle $\theta$. Hence, we can find a pre-image for $\mathbf{w}$ by rotating it *clockwise* through the angle $\theta$; that is, consider $\mathbf{v} = \mathbf{A}^{-1}\mathbf{w} \in \mathbb{R}^2$. When we apply $L$ to $\mathbf{v}$, we rotate it *counterclockwise* through the same angle $\theta$: $L(\mathbf{v}) = \mathbf{A}(\mathbf{A}^{-1}\mathbf{w}) = \mathbf{w}$, thus obtaining the original vector $\mathbf{w}$. Since $\mathbf{v}$ is in the domain and $\mathbf{v}$ maps to $\mathbf{w}$ under $L$, $L$ is onto.

### Example 2

**Differentiation:** Consider the linear transformation $L\colon \mathcal{P}_3 \to \mathcal{P}_2$ given by $L(\mathbf{p}) = \mathbf{p}'$. We will show that $L$ is *onto but not one-to-one*.

To show that $L$ is not one-to-one, we must find two different vectors $\mathbf{p}_1$ and $\mathbf{p}_2$ in the domain $\mathcal{P}_3$ that have the same image. Consider $\mathbf{p}_1 = x + 1$ and $\mathbf{p}_2 = x + 2$. Since $L(\mathbf{p}_1) = L(\mathbf{p}_2) = 1$, $L$ is not one-to-one.

To show that $L$ is onto, we must take an arbitrary vector $\mathbf{q}$ in $\mathcal{P}_2$ and find some vector $\mathbf{p}$ in $\mathcal{P}_3$ such that $L(\mathbf{p}) = \mathbf{q}$. Consider the vector $\mathbf{p} = \int \mathbf{q}(x)\,dx$ with zero constant term. Because $L(\mathbf{p}) = \mathbf{q}$, we see that $L$ is onto.

If in Example 2 we had used $\mathcal{P}_3$ for the codomain instead of $\mathcal{P}_2$, the linear transformation would not have been onto because $x^3$ would have no pre-image (why?). This provides an example of a linear transformation that is neither one-to-one nor onto. Also, Exercise 6 illustrates a linear transformation that is one-to-one but not onto. These examples, together with Examples 1 and 2, show that the concepts of one-to-one and onto are independent of each other; that is, there are linear transformations that have either property with or without the other.

Theorem B.1 in Appendix B shows that the composition of one-to-one linear transformations is one-to-one, and similarly, the composition of onto linear transformations is onto.

## Characterization by Kernel and Range

The next theorem gives an alternate way of characterizing one-to-one linear transformations and onto linear transformations.

**Theorem 5.12** *Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces, and let $L: \mathcal{V} \to \mathcal{W}$ be a linear transformation. Then:*

**(1)** *$L$ is one-to-one if and only if $\ker(L) = \{\mathbf{0}_\mathcal{V}\}$ (or, equivalently, if and only if $\dim(\ker(L)) = 0$), and*

**(2)** *If $\mathcal{W}$ is finite dimensional, then $L$ is onto if and only if $\dim(\text{range}(L)) = \dim(\mathcal{W})$.*

Thus, a linear transformation whose kernel contains a nonzero vector cannot be one-to-one.

*Proof.* First suppose that $L$ is one-to-one, and let $\mathbf{v} \in \ker(L)$. We must show that $\mathbf{v} = \mathbf{0}_\mathcal{V}$. Now, $L(\mathbf{v}) = \mathbf{0}_\mathcal{W}$. However, by Theorem 5.1, $L(\mathbf{0}_\mathcal{V}) = \mathbf{0}_\mathcal{W}$. Because $L(\mathbf{v}) = L(\mathbf{0}_\mathcal{V})$ and $L$ is one-to-one, we must have $\mathbf{v} = \mathbf{0}_\mathcal{V}$.

Conversely, suppose that $\ker(L) = \{\mathbf{0}_\mathcal{V}\}$. We must show that $L$ is one-to-one. Let $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}$, with $L(\mathbf{v}_1) = L(\mathbf{v}_2)$. We must show that $\mathbf{v}_1 = \mathbf{v}_2$. Now, $L(\mathbf{v}_1) - L(\mathbf{v}_2) = \mathbf{0}_\mathcal{W}$, implying that $L(\mathbf{v}_1 - \mathbf{v}_2) = \mathbf{0}_\mathcal{W}$. Hence, $\mathbf{v}_1 - \mathbf{v}_2 \in \ker(L)$, by definition of the kernel. Since $\ker(L) = \{\mathbf{0}_\mathcal{V}\}$, $\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{0}_\mathcal{V}$ and so $\mathbf{v}_1 = \mathbf{v}_2$.

Finally, note that, by definition, $L$ is onto if and only if $\text{range}(L) = \mathcal{W}$, and therefore part (2) of the theorem follows immediately from Theorem 4.13. □

**Example 3**

Consider the linear transformation $L: \mathcal{M}_{22} \to \mathcal{M}_{23}$ given by

$$L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a-b & 0 & c-d \\ c+d & a+b & 0 \end{bmatrix}.$$

If $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \ker(L)$, then $a - b = c - d = c + d = a + b = 0$. Solving these equations yields $a = b = c = d = 0$, and so $\ker(L)$ contains only the zero matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$; that is, $\dim(\ker(L)) = 0$. Thus, by part (1) of Theorem 5.12, $L$ is one-to-one. However, $\text{range}(L)$ is spanned by the image of a basis for $\mathcal{M}_{22}$, so $\text{range}(L)$ can have dimension at most 4. Hence, by part (2) of Theorem 5.12, $L$ is not onto. In particular, $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \notin \text{range}(L)$.

On the other hand, consider $M: \mathcal{M}_{23} \to \mathcal{M}_{22}$ given by

$$M\left(\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\right) = \begin{bmatrix} a+b & a+c \\ d+e & d+f \end{bmatrix}.$$

It is easy to see that $M$ is onto, since $M\left(\begin{bmatrix} 0 & b & c \\ 0 & e & f \end{bmatrix}\right) = \begin{bmatrix} b & c \\ e & f \end{bmatrix}$, and thus every $2 \times 2$ matrix is in $\text{range}(M)$. Thus, by part (2) of Theorem 5.12, $\dim(\text{range}(M)) = \dim(\mathcal{M}_{22}) = 4$. Then, by the Dimension Theorem, $\dim(\ker(M)) = \dim(\mathcal{M}_{23}) - \dim(\text{range}(M)) = 6 - 4 = 2$. Hence, by part (1) of Theorem 5.12, $M$ is not one-to-one. In particular, $\begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \in \ker(M)$.

Suppose that $\mathcal{V}$ and $\mathcal{W}$ are finite dimensional vector spaces and $L: \mathcal{V} \to \mathcal{W}$ is a linear transformation. If $\dim(\mathcal{V}) = \dim(\mathcal{W})$, the next result, which requires the full generality of the Dimension Theorem, asserts that we need only check that $L$ is *either* one-to-one or onto to know that $L$ has the other property as well.

**Corollary 5.13** *Let $\mathcal{V}$ and $\mathcal{W}$ be finite dimensional vector spaces with $\dim(\mathcal{V}) = \dim(\mathcal{W})$. Let $L: \mathcal{V} \to \mathcal{W}$ be a linear transformation. Then $L$ is one-to-one if and only if $L$ is onto.*

*Proof.* Let $\mathcal{V}$ and $\mathcal{W}$ be finite dimensional vector spaces with $\dim(\mathcal{V}) = \dim(\mathcal{W})$, and let $L: \mathcal{V} \to \mathcal{W}$ be a linear transformation. Then

$$\begin{aligned}
L \text{ is one-to-one} &\iff \dim(\ker(L)) = 0 && \text{by Theorem 5.12} \\
&\iff \dim(\mathcal{V}) = \dim(\text{range}(L)) && \text{by the Dimension Theorem} \\
&\iff \dim(\mathcal{W}) = \dim(\text{range}(L)) && \text{because } \dim(\mathcal{V}) = \dim(\mathcal{W}) \\
&\iff L \text{ is onto.} && \text{by Theorem 5.12}
\end{aligned}$$

□

**Example 4**

Consider $L: \mathcal{P}_2 \to \mathbb{R}^3$ given by $L(\mathbf{p}) = [\mathbf{p}(0), \mathbf{p}(1), \mathbf{p}(2)]$. Now, $\dim(\mathcal{P}_2) = \dim(\mathbb{R}^3) = 3$. Hence, by Corollary 5.13, if $L$ is either one-to-one or onto, it has the other property as well.

We will show that $L$ is one-to-one using Theorem 5.12. If $\mathbf{p} \in \ker(L)$, then $L(\mathbf{p}) = \mathbf{0}$, and so $\mathbf{p}(0) = \mathbf{p}(1) = \mathbf{p}(2) = 0$. Hence $\mathbf{p}$ is a polynomial of degree $\leq 2$ touching the $x$-axis at $x = 0$, $x = 1$, and $x = 2$. Since the graph of $\mathbf{p}$ must be either a parabola or a line, it cannot touch the $x$-axis at three distinct points unless its graph is the line $y = 0$. That is, $\mathbf{p} = \mathbf{0}$ in $\mathcal{P}_2$. Therefore, $\ker(L) = \{\mathbf{0}\}$, and $L$ is one-to-one.

Now, by Corollary 5.13, $L$ is onto. Thus, given any 3-vector $[a, b, c]$, there is some $\mathbf{p} \in \mathcal{P}_2$ such that $\mathbf{p}(0) = a$, $\mathbf{p}(1) = b$, and $\mathbf{p}(2) = c$. (This example is generalized further in Exercise 9.)

## Spanning and Linear Independence

The next theorem shows that the one-to-one property is related to linear independence, while the onto property is related to spanning.

**Theorem 5.14** *Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces, and let $L: \mathcal{V} \to \mathcal{W}$ be a linear transformation. Then:*

**(1)** *If $L$ is one-to-one, and $T$ is a linearly independent subset of $\mathcal{V}$, then $L(T)$ is linearly independent in $\mathcal{W}$.*

**(2)** *If $L$ is onto, and $S$ spans $\mathcal{V}$, then $L(S)$ spans $\mathcal{W}$.*

*Proof.* Suppose that $L$ is one-to-one, and $T$ is a linearly independent subset of $\mathcal{V}$. To prove that $L(T)$ is linearly independent in $\mathcal{W}$, it is enough to show that any finite subset of $L(T)$ is linearly independent. Suppose $\{L(\mathbf{x}_1), \ldots, L(\mathbf{x}_n)\}$ is a finite subset of $L(T)$, for vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n \in T$, and suppose $b_1 L(\mathbf{x}_1) + \cdots + b_n L(\mathbf{x}_n) = \mathbf{0}_{\mathcal{W}}$. Then, $L(b_1\mathbf{x}_1 + \cdots + b_n\mathbf{x}_n) = \mathbf{0}_{\mathcal{W}}$, implying that $b_1\mathbf{x}_1 + \cdots + b_n\mathbf{x}_n \in \ker(L)$. But since $L$ is one-to-one, Theorem 5.12 tells us that $\ker(L) = \{\mathbf{0}_{\mathcal{V}}\}$. Hence, $b_1\mathbf{x}_1 + \cdots + b_n\mathbf{x}_n = \mathbf{0}_{\mathcal{V}}$. Then, because the vectors in $T$ are linearly independent, $b_1 = b_2 = \cdots = b_n = 0$. Therefore, $\{L(\mathbf{x}_1), \ldots, L(\mathbf{x}_n)\}$ is linearly independent. Hence, $L(T)$ is linearly independent.

Now suppose that $L$ is onto, and $S$ spans $\mathcal{V}$. To prove that $L(S)$ spans $\mathcal{W}$, we must show that any vector $\mathbf{w} \in \mathcal{W}$ can be expressed as a linear combination of vectors in $L(S)$. Since $L$ is onto, there is a $\mathbf{v} \in \mathcal{V}$ such that $L(\mathbf{v}) = \mathbf{w}$. Since $S$ spans $\mathcal{V}$, there are scalars $a_1, \ldots, a_n$ and vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n \in S$ such that $\mathbf{v} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$. Thus, $\mathbf{w} = L(\mathbf{v}) = L(a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n) = a_1 L(\mathbf{v}_1) + \cdots + a_n L(\mathbf{v}_n)$. Hence, $L(S)$ spans $\mathcal{W}$. □

An almost identical proof gives the following useful generalization of part (2) of Theorem 5.14: For any linear transformation $L: \mathcal{V} \to \mathcal{W}$, and any subset $S$ of $\mathcal{V}$, $L(S)$ spans the subspace $L(\text{span}(S))$ of $\mathcal{W}$. In particular, if $S$ spans $\mathcal{V}$, then $L(S)$ spans $\text{range}(L)$. (See Exercise 11.)

**Example 5**

Consider the linear transformation $L: \mathcal{P}_2 \to \mathcal{P}_3$ given by $L(ax^2 + bx + c) = bx^3 + cx^2 + ax$. It is easy to see that $\ker(L) = \{\mathbf{0}\}$ since $L(ax^2 + bx + c) = 0x^3 + 0x^2 + 0x + 0$ only if $a = b = c = 0$, and so $L$ is one-to-one by Theorem 5.12. Consider the linearly independent set $T = \{x^2 + x, x + 1\}$ in $\mathcal{P}_2$. Notice that $L(T) = \{x^3 + x, x^3 + x^2\}$, and that $L(T)$ is linearly independent, as predicted by part (1) of Theorem 5.14.

Next, let $\mathcal{W} = \{[x, 0, z]\}$ be the $xz$–plane in $\mathbb{R}^3$. Clearly, $\dim(\mathcal{W}) = 2$. Consider $L: \mathbb{R}^3 \to \mathcal{W}$, where $L$ is the projection of $\mathbb{R}^3$ onto the $xz$-plane; that is, $L([x, y, z]) = [x, 0, z]$. It is easy to check that $S = \{[2, -1, 3], [1, -2, 0], [4, 3, -1]\}$ spans $\mathbb{R}^3$ using the Simplified Span Method. Part (2) of Theorem 5.14 then asserts that $L(S) = \{[2, 0, 3], [1, 0, 0], [4, 0, -1]\}$ spans $\mathcal{W}$. In fact, $\{[2, 0, 3], [1, 0, 0]\}$ alone spans $\mathcal{W}$, since $\dim(\text{span}(\{[2, 0, 3], [1, 0, 0]\})) = 2 = \dim(\mathcal{W})$.

In Section 5.5 we will consider isomorphisms, which are linear transformations that are simultaneously one-to-one and onto. We will see that such functions faithfully carry vector space properties from the domain to the codomain.

## New Vocabulary

one-to-one linear transformation

onto linear transformation

## Highlights

- A linear transformation $L: \mathcal{V} \to \mathcal{W}$ is one-to-one if and only if no two distinct vectors of $\mathcal{V}$ map to the same image in $\mathcal{W}$ (that is, $L(\mathbf{v}_1) = L(\mathbf{v}_2)$ implies $\mathbf{v}_1 = \mathbf{v}_2$).

- A linear transformation $L: \mathcal{V} \to \mathcal{W}$ is one-to-one if and only if $\ker(L) = \{\mathbf{0}_{\mathcal{V}}\}$ (if and only if $\dim(\ker(L)) = 0$).
- If a linear transformation $L: \mathcal{V} \to \mathcal{W}$ is one-to-one, and $T$ is a linearly independent subset of $\mathcal{V}$, then its image $L(T)$ is a linearly independent subset of $\mathcal{W}$.
- A linear transformation $L: \mathcal{V} \to \mathcal{W}$ is onto if and only if every vector in $\mathcal{W}$ is the image of some vector in $\mathcal{V}$ (that is, for every $\mathbf{w} \in \mathcal{W}$, there is some $\mathbf{v} \in \mathcal{V}$ such that $L(\mathbf{v}) = \mathbf{w}$).
- A linear transformation $L: \mathcal{V} \to \mathcal{W}$ is onto if and only if $\text{range}(L) = \mathcal{W}$ (if and only if $\dim(\text{range}(L)) = \dim(\mathcal{W})$, when $\mathcal{W}$ is finite dimensional).
- If a linear transformation $L: \mathcal{V} \to \mathcal{W}$ is onto, and $S$ spans $\mathcal{V}$, then its image $L(S)$ spans $\mathcal{W}$.
- If $L: \mathcal{V} \to \mathcal{W}$ is a linear transformation between finite dimensional vector spaces with $\dim(\mathcal{V}) = \dim(\mathcal{W})$, then $L$ is one-to-one if and only if $L$ is onto.

## Exercises for Section 5.4

1. Which of the following linear transformations are one-to-one? Which are onto? Justify your answers without using row reduction.
   - ★ (a) $L: \mathbb{R}^3 \to \mathbb{R}^4$ given by $L([x, y, z]) = [y, z, -y, 0]$
   - (b) $L: \mathbb{R}^3 \to \mathbb{R}^4$ given by $L([x, y, z]) = [x + y, y + z, x - z, x + 2y + z]$
   - ★ (c) $L: \mathbb{R}^3 \to \mathbb{R}^3$ given by $L([x, y, z]) = [2x, x + y + z, -y]$
   - (d) $L: \mathcal{P}_3 \to \mathcal{P}_2$ given by $L(ax^3 + bx^2 + cx + d) = ax^2 + bx + (c + d)$
   - ★ (e) $L: \mathcal{P}_2 \to \mathcal{P}_2$ given by $L(ax^2 + bx + c) = (a + b)x^2 + (b + c)x + (a + c)$
   - (f) $L: \mathcal{M}_{22} \to \mathcal{M}_{22}$ given by $L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} b & a - 2d \\ a + 2d & c \end{bmatrix}$
   - ★ (g) $L: \mathcal{M}_{23} \to \mathcal{M}_{22}$ given by $L\left(\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\right) = \begin{bmatrix} a & -c \\ 2e & d + f \end{bmatrix}$
   - ★ (h) $L: \mathcal{P}_2 \to \mathcal{M}_{22}$ given by $L(ax^2 + bx + c) = \begin{bmatrix} a + c & 0 \\ b - c & -3a \end{bmatrix}$
   - (i) $L: \mathbb{R}^n \to \mathbb{R}^n$ given by $L(\mathbf{x}) = \mathbf{A}\mathbf{x}$, for some nonsingular matrix $\mathbf{A}$.
   - (j) $L: \mathbb{R}^5 \to \mathbb{R}^4$ given by $L(\mathbf{x}) = \mathbf{A}\mathbf{x}$, for some matrix $\mathbf{A}$ with $\text{rank}(\mathbf{A}) = 3$.
2. Which of the following linear transformations are one-to-one? Which are onto? Justify your answers by using row reduction and Theorem 5.9 to determine the dimensions of the kernel and range.
   - ★ (a) $L: \mathbb{R}^2 \to \mathbb{R}^2$ given by $L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -4 & -3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
   - (b) $L: \mathbb{R}^3 \to \mathbb{R}^3$ given by $L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 3 & 8 & -3 \\ -2 & -7 & 7 \\ 1 & 4 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$
   - ★ (c) $L: \mathbb{R}^3 \to \mathbb{R}^3$ given by $L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -7 & 4 & -2 \\ 16 & -7 & 2 \\ 4 & -3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$
   - (d) $L: \mathbb{R}^3 \to \mathbb{R}^4$ given by $L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 5 & -4 & -2 \\ -9 & 4 & -3 \\ 6 & -3 & 1 \\ 5 & -3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$
   - (e) $L: \mathbb{R}^4 \to \mathbb{R}^4$ given by $L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} 2 & 4 & 0 & 1 \\ 3 & 5 & -4 & 2 \\ -1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$
3. In each of the following cases, the matrix for a linear transformation with respect to some ordered bases for the domain and codomain is given. Which of these linear transformations are one-to-one? Which are onto? Justify your answers by using row reduction and Theorem 5.9 to determine the dimensions of the kernel and range.

★ **(a)** $L\colon \mathcal{P}_2 \to \mathcal{P}_2$ having matrix $\begin{bmatrix} 1 & -3 & 0 \\ -4 & 13 & -1 \\ 8 & -25 & 2 \end{bmatrix}$

**(b)** $L\colon \mathcal{M}_{22} \to \mathbb{R}^3$ having matrix $\begin{bmatrix} 6 & -9 & 2 & 8 \\ 10 & -6 & 12 & 4 \\ 8 & -9 & 9 & 11 \end{bmatrix}$

★ **(c)** $L\colon \mathcal{M}_{22} \to \mathcal{P}_3$ having matrix $\begin{bmatrix} 2 & 3 & -1 & 1 \\ 5 & 2 & -4 & 7 \\ 1 & 7 & 1 & -4 \\ -2 & 19 & 7 & -19 \end{bmatrix}$

**(d)** $L\colon \mathbb{R}^3 \to \mathcal{P}_3$ having matrix $\begin{bmatrix} 6 & 10 & 8 \\ -9 & -6 & -9 \\ 2 & 12 & 9 \\ 8 & 4 & 11 \end{bmatrix}$

**4.** Suppose that $m > n$.

**(a)** Show there is no onto linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

**(b)** Show there is no one-to-one linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$.

**5.** Let $\mathbf{A}$ be a fixed $n \times n$ matrix, and consider $L\colon \mathcal{M}_{nn} \to \mathcal{M}_{nn}$ given by $L(\mathbf{B}) = \mathbf{AB} - \mathbf{BA}$.

**(a)** Show that $L$ is not one-to-one. (Hint: Consider $L(\mathbf{I}_n)$.)

**(b)** Use part (a) to show that $L$ is not onto.

**6.** Define $L\colon \mathcal{U}_3 \to \mathcal{M}_{33}$ by $L(\mathbf{A}) = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T)$. Prove that $L$ is one-to-one but is *not* onto.

★ **7.** This exercise explores the concepts of one-to-one and onto in certain cases.

**(a)** Suppose that $L\colon \mathbb{R}^6 \to \mathcal{P}_5$ is a linear transformation and that $L$ is not onto. Is $L$ one-to-one? Why or why not?

**(b)** Suppose that $L\colon \mathcal{M}_{22} \to \mathcal{P}_3$ is a linear transformation and that $L$ is not one-to-one. Is $L$ onto? Why or why not?

**8.** Define $L\colon \mathcal{P} \to \mathcal{P}$ by $L(\mathbf{p}(x)) = x\mathbf{p}(x)$.

**(a)** Show that $L$ is one-to-one but not onto.

**(b)** Explain why $L$ does not contradict Corollary 5.13.

**9.** This exercise, related to Example 4, concerns roots of polynomials.

**(a)** Let $x_1, x_2, x_3$ be distinct real numbers. Use an argument similar to that in Example 4 to show that for any given $a, b, c \in \mathbb{R}$, there is a polynomial $\mathbf{p} \in \mathcal{P}_2$ such that $\mathbf{p}(x_1) = a$, $\mathbf{p}(x_2) = b$, and $\mathbf{p}(x_3) = c$.

**(b)** For each choice of $x_1, x_2, x_3, a, b, c \in \mathbb{R}$, show that the polynomial $\mathbf{p}$ from part (a) is unique.

**(c)** Recall from algebra that a nonzero polynomial of degree $n$ can have at most $n$ roots. Use this fact to prove that if $x_1, \ldots, x_{n+1} \in \mathbb{R}$, with $x_1, \ldots, x_{n+1}$ distinct, then for any given $a_1, \ldots, a_{n+1} \in \mathbb{R}$, there is a unique polynomial $\mathbf{p} \in \mathcal{P}_n$ such that $\mathbf{p}(x_1) = a_1$, $\mathbf{p}(x_2) = a_2, \ldots, \mathbf{p}(x_n) = a_n$, and $\mathbf{p}(x_{n+1}) = a_{n+1}$.

**10.** Let $L\colon \mathcal{V} \to \mathcal{W}$ be a linear transformation between vector spaces. Suppose that for every linearly independent set $T$ in $\mathcal{V}$, $L(T)$ is linearly independent in $\mathcal{W}$. Prove that $L$ is one-to-one. (Hint: Use a proof by contradiction. Consider $T = \{\mathbf{v}\}$, with $\mathbf{v} \in \ker(L)$, $\mathbf{v} \neq \mathbf{0}$.)

**11.** Let $L\colon \mathcal{V} \to \mathcal{W}$ be a linear transformation between vector spaces, and let $S$ be a subset of $\mathcal{V}$.

**(a)** Prove that $L(S)$ spans the subspace $L(\text{span}(S))$.

**(b)** Show that if $S$ spans $\mathcal{V}$, then $L(S)$ spans range($L$).

**(c)** Show that if $L(S)$ spans $\mathcal{W}$, then $L$ is onto.

★ **12.** True or False:

**(a)** A linear transformation $L\colon \mathcal{V} \to \mathcal{W}$ is one-to-one if for all $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}$, $\mathbf{v}_1 = \mathbf{v}_2$ implies $L(\mathbf{v}_1) = L(\mathbf{v}_2)$.

**(b)** A linear transformation $L\colon \mathcal{V} \to \mathcal{W}$ is onto if for all $\mathbf{v} \in \mathcal{V}$, there is some $\mathbf{w} \in \mathcal{W}$ such that $L(\mathbf{v}) = \mathbf{w}$.

**(c)** A linear transformation $L\colon \mathcal{V} \to \mathcal{W}$ is one-to-one if $\ker(L)$ contains no vectors other than $\mathbf{0}_{\mathcal{V}}$.

**(d)** If $L$ is a linear transformation and $S$ spans the domain of $L$, then $L(S)$ spans the range of $L$.

**(e)** Suppose $\mathcal{V}$ is a finite dimensional vector space. A linear transformation $L\colon \mathcal{V} \to \mathcal{W}$ is not one-to-one if $\dim(\ker(L)) \neq 0$.

**(f)** Suppose $\mathcal{W}$ is a finite dimensional vector space. A linear transformation $L\colon \mathcal{V} \to \mathcal{W}$ is not onto if $\dim(\text{range}(L)) < \dim(\mathcal{W})$.

**(g)** If a linear transformation $L: \mathbb{R}^6 \to \mathcal{M}_{32}$ is not one-to-one, then it is not onto.
**(h)** If $L$ is a linear transformation and $T$ is a linearly independent subset of the domain of $L$, then $L(T)$ is linearly independent.
**(i)** If $L$ is a linear transformation $L: \mathcal{V} \to \mathcal{W}$, and $S$ is a subset of $\mathcal{V}$ such that $L(S)$ spans $\mathcal{W}$, then $S$ spans $\mathcal{V}$.

## 5.5 Isomorphism

In this section, we examine methods for determining whether two vector spaces are equivalent, or *isomorphic*. Isomorphism is important because if certain algebraic results are true in one of two isomorphic vector spaces, corresponding results hold true in the other as well. It is the concept of isomorphism that has allowed us to apply our techniques and formal methods to vector spaces other than $\mathbb{R}^n$.

### Isomorphisms: Invertible Linear Transformations

We restate here the definition from Appendix B for the inverse of a function as it applies to linear transformations.

**Definition** Let $L: \mathcal{V} \to \mathcal{W}$ be a linear transformation. Then $L$ is an **invertible linear transformation** if and only if there is a function $M: \mathcal{W} \to \mathcal{V}$ such that $(M \circ L)(\mathbf{v}) = \mathbf{v}$, for all $\mathbf{v} \in \mathcal{V}$, and $(L \circ M)(\mathbf{w}) = \mathbf{w}$, for all $\mathbf{w} \in \mathcal{W}$. Such a function $M$ is called an **inverse** of $L$.

If the inverse $M$ of $L: \mathcal{V} \to \mathcal{W}$ exists, then it is unique by Theorem B.3 and is usually denoted by $L^{-1}: \mathcal{W} \to \mathcal{V}$.

**Definition** A linear transformation $L: \mathcal{V} \to \mathcal{W}$ that is both one-to-one and onto is called an **isomorphism** from $\mathcal{V}$ to $\mathcal{W}$.

The next result shows that the previous two definitions actually refer to the same class of linear transformations.

**Theorem 5.15** *Let $L: \mathcal{V} \to \mathcal{W}$ be a linear transformation. Then $L$ is an isomorphism if and only if $L$ is an invertible linear transformation. Moreover, if $L$ is invertible, then $L^{-1}$ is also a linear transformation.*

Notice that Theorem 5.15 also asserts that whenever $L$ is an isomorphism, $L^{-1}$ is an isomorphism as well because $L^{-1}$ is an invertible linear transformation (with $L$ as its inverse).

*Proof.* The "if and only if" part of Theorem 5.15 follows directly from Theorem B.2. Thus, we only need to prove the last assertion in Theorem 5.15. That is, suppose $L: \mathcal{V} \to \mathcal{W}$ is invertible (and thus, an isomorphism) with inverse $L^{-1}$. We need to prove $L^{-1}$ is a linear transformation. To do this, we must show both of the following properties hold:

**(1)** $L^{-1}(\mathbf{w}_1 + \mathbf{w}_2) = L^{-1}(\mathbf{w}_1) + L^{-1}(\mathbf{w}_2)$, for all $\mathbf{w}_1, \mathbf{w}_2 \in \mathcal{W}$.
**(2)** $L^{-1}(c\mathbf{w}) = cL^{-1}(\mathbf{w})$, for all $c \in \mathbb{R}$, and for all $\mathbf{w} \in \mathcal{W}$.

**Property (1)**: Since $L$ is an isomorphism, $L$ is onto. Hence for $\mathbf{w}_1, \mathbf{w}_2 \in \mathcal{W}$ there exist $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}$ such that $L(\mathbf{v}_1) = \mathbf{w}_1$ and $L(\mathbf{v}_2) = \mathbf{w}_2$. But then,

$$\begin{aligned} L^{-1}(\mathbf{w}_1 + \mathbf{w}_2) &= L^{-1}(L(\mathbf{v}_1) + L(\mathbf{v}_2)) \\ &= L^{-1}(L(\mathbf{v}_1 + \mathbf{v}_2)) && \text{since } L \text{ is a linear transformation} \\ &= \mathbf{v}_1 + \mathbf{v}_2 && \text{since } L \text{ and } L^{-1} \text{ are inverses} \\ &= L^{-1}(\mathbf{w}_1) + L^{-1}(\mathbf{w}_2) && \text{since } L \text{ is one-to-one} \end{aligned}$$

**Property (2)**: Again, since $L$ is an isomorphism, for $\mathbf{w} \in \mathcal{W}$ there exists $\mathbf{v} \in \mathcal{V}$ such that $L(\mathbf{v}) = \mathbf{w}$. But then,

$$\begin{aligned} L^{-1}(c\mathbf{w}) &= L^{-1}(cL(\mathbf{v})) \\ &= L^{-1}(L(c\mathbf{v})) && \text{since } L \text{ is a linear transformation} \\ &= c\mathbf{v} && \text{since } L \text{ and } L^{-1} \text{ are inverses} \end{aligned}$$

$$= cL^{-1}(\mathbf{w}) \qquad \text{since } L \text{ is one-to-one}$$

Because both properties (1) and (2) hold, $L^{-1}$ is a linear transformation. □

**Example 1**

Recall the rotation linear operator $L: \mathbb{R}^2 \to \mathbb{R}^2$ with

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

given in Example 9 in Section 5.1. In Example 1 in Section 5.4 we proved that $L$ is both one-to-one and onto. Hence, $L$ is an isomorphism and has an inverse, $L^{-1}$. Because $L$ represents a *counterclockwise* rotation of vectors through the angle $\theta$, then $L^{-1}$ must represent a *clockwise* rotation through the angle $\theta$, as we saw in Example 1 of Section 5.4. Equivalently, $L^{-1}$ can be thought of as a *counterclockwise* rotation through the angle $-\theta$. Thus,

$$L^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Of course, $L^{-1}$ is also an isomorphism.

The next theorem gives a simple method for determining whether a linear transformation between finite dimensional vector spaces is an isomorphism.

**Theorem 5.16** *Let $\mathcal{V}$ and $\mathcal{W}$ both be nontrivial finite dimensional vector spaces with ordered bases $B$ and $C$, respectively, and let $L: \mathcal{V} \to \mathcal{W}$ be a linear transformation. Then $L$ is an isomorphism if and only if the matrix representation $\mathbf{A}_{BC}$ for $L$ with respect to $B$ and $C$ is nonsingular. Moreover, if $L$ is an isomorphism, the matrix for $L^{-1}$ with respect to $C$ and $B$ is $(\mathbf{A}_{BC})^{-1}$.*

To prove one half of Theorem 5.16, let $L$ be an isomorphism, and let $\mathbf{A}_{BC}$ be the matrix for $L$ with respect to $B$ and $C$, and let $\mathbf{D}_{CB}$ be the matrix for $L^{-1}$ with respect to $C$ and $B$. Theorem 5.7 then shows that $\mathbf{D}_{CB}\mathbf{A}_{BC} = \mathbf{I}_n$, with $n = \dim(\mathcal{V})$, and $\mathbf{A}_{BC}\mathbf{D}_{CB} = \mathbf{I}_k$, with $k = \dim(\mathcal{W})$. Applying Exercise 21 in Section 2.4 twice, we find that $n \le k$ and $k \le n$, and hence $n = k$. Thus, $\mathbf{D}_{CB} = (\mathbf{A}_{BC})^{-1}$, so $\mathbf{A}_{BC}$ is nonsingular. The proof of the converse is straightforward, and you are asked to give the details in Exercise 8. Notice, in particular, that the matrix for any isomorphism must be a square matrix.

**Example 2**

Consider $L: \mathbb{R}^3 \to \mathbb{R}^3$ given by $L(\mathbf{v}) = \mathbf{A}\mathbf{v}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now, $\mathbf{A}$ is nonsingular ($|\mathbf{A}| = 1 \neq 0$). Hence, by Theorem 5.16, $L$ is an isomorphism. Geometrically, $L$ represents a shear in the $z$-direction (see Table 5.1).

Theorem B.4 in Appendix B shows that the composition of isomorphisms results in an isomorphism. In particular, the inverse of the composition $L_2 \circ L_1$ is $L_1^{-1} \circ L_2^{-1}$. That is, the transformations must be undone in *reverse* order to arrive at the correct inverse. (Compare this with part (3) of Theorem 2.12 for matrix multiplication.)

When an isomorphism exists between two vector spaces, properties from the domain are carried over to the codomain by the isomorphism. In particular, the following theorem, which follows immediately from Theorem 5.14, shows that spanning sets map to spanning sets, and linearly independent sets map to linearly independent sets.

**Theorem 5.17** *Suppose $L: \mathcal{V} \to \mathcal{W}$ is an isomorphism. Let $S$ span $\mathcal{V}$ and let $T$ be a linearly independent subset of $\mathcal{V}$. Then $L(S)$ spans $\mathcal{W}$ and $L(T)$ is linearly independent.*

## Isomorphic Vector Spaces

**Definition** Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces. Then $\mathcal{V}$ is **isomorphic** to $\mathcal{W}$, denoted $\mathcal{V} \cong \mathcal{W}$, if and only if there exists an isomorphism $L\colon \mathcal{V} \to \mathcal{W}$.

If $\mathcal{V} \cong \mathcal{W}$, there is some isomorphism $L\colon \mathcal{V} \to \mathcal{W}$. Then by Theorem 5.15, $L^{-1}\colon \mathcal{W} \to \mathcal{V}$ is also an isomorphism, so $\mathcal{W} \cong \mathcal{V}$. Hence, we usually speak of such $\mathcal{V}$ and $\mathcal{W}$ as being *isomorphic to each other*.

Also notice that if $\mathcal{V} \cong \mathcal{W}$ and $\mathcal{W} \cong \mathcal{X}$, then there are isomorphisms $L_1\colon \mathcal{V} \to \mathcal{W}$ and $L_2\colon \mathcal{W} \to \mathcal{X}$. But then $L_2 \circ L_1\colon \mathcal{V} \to \mathcal{X}$ is an isomorphism, and so $\mathcal{V} \cong \mathcal{X}$. In other words, two vector spaces such as $\mathcal{V}$ and $\mathcal{X}$ that are both isomorphic to the same vector space $\mathcal{W}$ are isomorphic to each other.

### Example 3

Consider $L_1\colon \mathbb{R}^4 \to \mathcal{P}_3$ given by $L_1([a,b,c,d]) = ax^3 + bx^2 + cx + d$ and $L_2\colon \mathcal{M}_{22} \to \mathcal{P}_3$ given by $L_2\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ax^3 + bx^2 + cx + d$. $L_1$ and $L_2$ are certainly both isomorphisms. Hence, $\mathbb{R}^4 \cong \mathcal{P}_3$ and $\mathcal{M}_{22} \cong \mathcal{P}_3$. Thus, the composition $L_2^{-1} \circ L_1\colon \mathbb{R}^4 \to \mathcal{M}_{22}$ is also an isomorphism, and so $\mathbb{R}^4 \cong \mathcal{M}_{22}$. Notice that all of these vector spaces have dimension 4.

Next, we show that finite dimensional vector spaces $\mathcal{V}$ and $\mathcal{W}$ must have the same dimension for an isomorphism to exist between them.

**Theorem 5.18** *Suppose $\mathcal{V} \cong \mathcal{W}$ and $\mathcal{V}$ is finite dimensional. Then $\mathcal{W}$ is finite dimensional and* $\dim(\mathcal{V}) = \dim(\mathcal{W})$.

*Proof.* Since $\mathcal{V} \cong \mathcal{W}$, there is an isomorphism $L\colon \mathcal{V} \to \mathcal{W}$. Let $\dim(\mathcal{V}) = n$, and let $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for $\mathcal{V}$. By Theorem 5.17, $L(B) = \{L(\mathbf{v}_1), \ldots, L(\mathbf{v}_n)\}$ both spans $\mathcal{W}$ and is linearly independent, and so must be a basis for $\mathcal{W}$. Also, because $L$ is a one-to-one function, $|L(B)| = |B| = n$. Therefore, $\dim(\mathcal{V}) = \dim(\mathcal{W})$. □

Theorem 5.18 implies that there is no possible isomorphism from, say, $\mathbb{R}^3$ to $\mathcal{P}_4$ or from $\mathcal{M}_{22}$ to $\mathbb{R}^3$, because the dimensions of the spaces do not agree. Notice that Theorem 5.18 gives another confirmation of the fact that any matrix for an isomorphism must be square.

## Isomorphism of *n*-Dimensional Vector Spaces

Example 3 hints that any two finite dimensional vector spaces of the same dimension are isomorphic. This result, which is one of the most important in all linear algebra, is a corollary of the next theorem.

**Theorem 5.19** *If $\mathcal{V}$ is any $n$-dimensional vector space, then $\mathcal{V} \cong \mathbb{R}^n$.*

*Proof.* Suppose that $\mathcal{V}$ is a vector space with $\dim(\mathcal{V}) = n$. If we can find an isomorphism $L\colon \mathcal{V} \to \mathbb{R}^n$, then $\mathcal{V} \cong \mathbb{R}^n$, and we will be done. Let $B = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ be an ordered basis for $\mathcal{V}$. Consider the mapping $L(\mathbf{v}) = [\mathbf{v}]_B$, for all $\mathbf{v} \in \mathcal{V}$. Now, $L$ is a linear transformation by Example 4 in Section 5.1. Also,

$$\mathbf{v} \in \ker(L) \iff [\mathbf{v}]_B = [0, \ldots, 0] \iff \mathbf{v} = 0\mathbf{v}_1 + \cdots + 0\mathbf{v}_n \iff \mathbf{v} = \mathbf{0}.$$

Hence, $\ker(L) = \{\mathbf{0}_{\mathcal{V}}\}$, and $L$ is one-to-one.

If $\mathbf{a} = [a_1, \ldots, a_n] \in \mathbb{R}^n$, then $L(a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n) = [a_1, \ldots, a_n]$, showing that $\mathbf{a} \in \text{range}(L)$. Hence, $L$ is onto, and so $L$ is an isomorphism. □

In particular, Theorem 5.19 tells us that $\mathcal{P}_n \cong \mathbb{R}^{n+1}$ and that $\mathcal{M}_{mn} \cong \mathbb{R}^{mn}$. Also, the proof of Theorem 5.19 illustrates that coordinatization of vectors in an $n$-dimensional vector space $\mathcal{V}$ automatically gives an isomorphism of $\mathcal{V}$ with $\mathbb{R}^n$.

By the remarks before Example 3, Theorem 5.19 implies the following converse of Theorem 5.18:

**Corollary 5.20** *Any two n-dimensional vector spaces $\mathcal{V}$ and $\mathcal{W}$ are isomorphic. That is, if* $\dim(\mathcal{V}) = \dim(\mathcal{W})$, *then* $\mathcal{V} \cong \mathcal{W}$.

For example, suppose that $\mathcal{V}$ and $\mathcal{W}$ are both vector spaces with $\dim(\mathcal{V}) = \dim(\mathcal{W}) = 47$. Then by Corollary 5.20, $\mathcal{V} \cong \mathcal{W}$ and by Theorem 5.19, $\mathcal{V} \cong \mathcal{W} \cong \mathbb{R}^{47}$.

## Isomorphism and the Methods

We now have the means to justify the use of the Simplified Span Method and the Independence Test Method on vector spaces other than $\mathbb{R}^n$. Suppose $\mathcal{V} \cong \mathbb{R}^n$. By using the coordinatization isomorphism or its inverse as the linear transformation $L$ in Theorem 5.17, we see that spanning sets in $\mathcal{V}$ are mapped to spanning sets in $\mathbb{R}^n$, and vice versa. Similarly, linearly independent sets in $\mathcal{V}$ are mapped to linearly independent sets in $\mathbb{R}^n$, and vice versa. This is illustrated in the following example.

### Example 4

Consider the subset $S = \left\{x^3 - 2x^2 + x - 2,\ \ x^3 + x^2 + x + 1,\ \ x^3 - 5x^2 + x - 5,\ x^3 - x^2 - x + 1\right\}$ of $\mathcal{P}_3$. We use the coordinatization isomorphism $L\colon \mathcal{P}_3 \to \mathbb{R}^4$ with respect to the standard basis of $\mathcal{P}_3$ to obtain $L(S) = \{[1,-2,1,-2], [1,1,1,1], [1,-5,1,-5], [1,-1,-1,1]\}$, a subset of $\mathbb{R}^4$ corresponding to $S$. Row reducing

$$\begin{bmatrix} 1 & -2 & 1 & -2 \\ 1 & 1 & 1 & 1 \\ 1 & -5 & 1 & -5 \\ 1 & -1 & -1 & 1 \end{bmatrix} \text{ to obtain } \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

shows, by the Simplified Span Method, that $\text{span}(\{[1,-2,1,-2], [1,1,1,1], [1,-5,1,-5], [1,-1,-1,1]\}) = \text{span}(\{[1,0,0,1], [0,1,0,1], [0,0,1,-1]\})$. Since $L^{-1}$ is an isomorphism, Theorem 5.17 shows that $L^{-1}(\{[1,0,0,1], [0,1,0,1], [0,0,1,-1]\}) = \{x^3+1,\ x^2+1,\ x-1\}$ spans the same subspace of $\mathcal{P}_3$ that $S$ does. That is, $\text{span}(\{x^3+1,\ x^2+1,\ x-1\}) = \text{span}(S)$.

Similarly, row reducing

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & 1 & -5 & -1 \\ 1 & 1 & 1 & -1 \\ -2 & 1 & -5 & 1 \end{bmatrix} \text{ to obtain } \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

shows, by the Independence Test Method, that $\{[1,-2,1,-2], [1,1,1,1], [1,-1,-1,1]\}$ is a linearly independent subset of $\mathbb{R}^4$, and that $[1,-5,1,-5] = 2[1,-2,1,-2] - [1,1,1,1] + 0[1,-1,-1,1]$. Since $L^{-1}$ is an isomorphism, Theorem 5.17 shows us that $L^{-1}(\{[1,-2,1,-2], [1,1,1,1], [1,-1,-1,1]\}) = \left\{x^3 - 2x^2 + x - 2, x^3 + x^2 + x + 1, x^3 - x^2 - x + 1\right\}$ is a linearly independent subset of $\mathcal{P}_3$. The fact that $L^{-1}$ is a linear transformation also assures us that $x^3 - 5x^2 + x - 5 = 2\left(x^3 - 2x^2 + x - 2\right) - \left(x^3 + x^2 + x + 1\right) + 0\left(x^3 - x^2 - x + 1\right)$.

In addition to preserving dimension, spanning and linear independence, isomorphisms keep intact most other properties of vector spaces and the linear transformations between them. In particular, the next theorem shows that when we coordinatize the domain and codomain of a linear transformation, the kernel and the range are preserved.

**Theorem 5.21** *Let $L\colon \mathcal{V} \to \mathcal{W}$ be a linear transformation between nontrivial finite dimensional vector spaces, and let $L_1\colon \mathcal{V} \to \mathbb{R}^n$ and $L_2\colon \mathcal{W} \to \mathbb{R}^m$ be coordinatization isomorphisms with respect to some ordered bases B and C for $\mathcal{V}$ and $\mathcal{W}$, respectively. Let $M = L_2 \circ L \circ L_1^{-1}\colon \mathbb{R}^n \to \mathbb{R}^m$, so that $M([\mathbf{v}]_B) = [L(\mathbf{v})]_C$. Then,*

**(1)** $L_1^{-1}(\ker(M)) = \ker(L) \subseteq \mathcal{V}$,
**(2)** $L_2^{-1}(\text{range}(M)) = \text{range}(L) \subseteq \mathcal{W}$,
**(3)** $\dim(\ker(M)) = \dim(\ker(L))$, *and*
**(4)** $\dim(\text{range}(M)) = \dim(\text{range}(L))$.

Fig. 5.12 illustrates the situation in Theorem 5.21. The linear transformation $M$ in Theorem 5.21 is merely a "$\mathbb{R}^n \to \mathbb{R}^m$" version of $L$, using coordinatized vectors instead of the actual vectors in $\mathcal{V}$ and $\mathcal{W}$. Because $L_1^{-1}$ and $L_2^{-1}$ are isomorphisms, parts (1) and (2) of the theorem show that the subspace $\ker(L)$ of $\mathcal{V}$ is isomorphic to the subspace $\ker(M)$ of $\mathbb{R}^n$, and that the subspace $\text{range}(L)$ of $\mathcal{W}$ is isomorphic to the subspace $\text{range}(M)$ of $\mathbb{R}^m$. Parts (3) and (4) of the theorem follow directly from parts (1) and (2) because isomorphic finite dimensional vector spaces must have the same dimension. You are asked to prove a more general version of Theorem 5.21 as well as other related statements in Exercises 16 and 17.

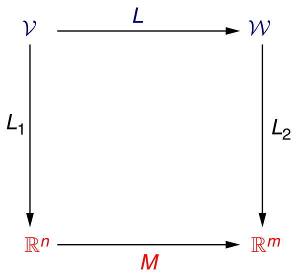


**FIGURE 5.12** The linear transformations $L$ and $M$ and the isomorphisms $L_1$ and $L_2$ in Theorem 5.21

The importance of Theorem 5.21 is that it justifies our use of the Kernel Method and the Range Method of Section 5.3 when vector spaces other than $\mathbb{R}^n$ are involved. Suppose that we want to find $\ker(L)$ and $\text{range}(L)$ for a given linear transformation $L\colon \mathcal{V} \to \mathcal{W}$. We begin by coordinatizing the domain $\mathcal{V}$ and codomain $\mathcal{W}$ using coordinatization isomorphisms $L_1$ and $L_2$ as in Theorem 5.21. The mapping $M$ created in Theorem 5.21 is thus an equivalent "$\mathbb{R}^n \to \mathbb{R}^m$" version of $L$. By applying the Kernel and Range Methods to $M$, we can find bases for $\ker(M)$ and $\text{range}(M)$. However, parts (1) and (2) of the theorem assure us that $\ker(L)$ is isomorphic to $\ker(M)$, and, similarly, that $\text{range}(L)$ is isomorphic to $\text{range}(M)$. Therefore, by reversing the coordinatizations, we can find bases for $\ker(L)$ and $\text{range}(L)$. In fact, this is exactly the approach that was used without justification in Section 5.3 to determine bases for the kernel and range for linear transformations involving vector spaces other than $\mathbb{R}^n$.

## Proving the Dimension Theorem Using Isomorphism

Recall the Dimension Theorem:

**(Dimension Theorem)** If $L\colon \mathcal{V} \to \mathcal{W}$ is a linear transformation and $\mathcal{V}$ is finite dimensional, then $\text{range}(L)$ is finite dimensional, and

$$\dim(\ker(L)) + \dim(\text{range}(L)) = \dim(\mathcal{V}).$$

In Section 5.3, we stated the Dimension Theorem in its full generality, but only *proved* it for linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$. We now supply the general proof, assuming that the special case for linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$ has already been proved.

*Proof.* The theorem is obviously true if $\mathcal{V}$ is the trivial vector space. Suppose $B$ is a finite, nonempty ordered basis for $\mathcal{V}$. Then, by the comments directly after Theorem 5.14 regarding spanning sets and range, $\text{range}(L)$ is spanned by the finite set $L(B)$, and so $\text{range}(L)$ is finite dimensional. Since $L$ does not interact at all with the vectors in $\mathcal{W}$ outside of $\text{range}(L)$, we can consider adjusting $L$ so that its codomain is just the subspace $\text{range}(L)$ of $\mathcal{W}$. That is, without loss of generality, we can let $\mathcal{W} = \text{range}(L)$. Hence, we can assume that $\mathcal{W}$ is finite dimensional.

Let $L_1\colon \mathcal{V} \to \mathbb{R}^n$ and $L_2\colon \mathcal{W} \to \mathbb{R}^m$ be coordinatization transformations with respect to some ordered bases for $\mathcal{V}$ and $\mathcal{W}$, respectively. Applying the special case of the Dimension Theorem to the linear transformation $L_2 \circ L \circ L_1^{-1}\colon \mathbb{R}^n \to \mathbb{R}^m$, we get

$$\begin{aligned}
\dim(\mathcal{V}) = n = \dim(\mathbb{R}^n) &= \dim\left(\text{domain}\left(L_2 \circ L \circ L_1^{-1}\right)\right) \\
&= \dim(\ker(L_2 \circ L \circ L_1^{-1})) + \dim(\text{range}(L_2 \circ L \circ L_1^{-1})) \\
&= \dim(\ker(L)) + \dim(\text{range}(L)), \qquad \text{by parts (2) and (4) of Theorem 5.21.}
\end{aligned}$$

□

So far, we have proved many important results concerning the concepts of one-to-one, onto, and isomorphism. For convenience, these and other useful properties from the exercises are summarized in Table 5.2.

**TABLE 5.2 Conditions on linear transformations that are one-to-one, onto, or isomorphisms**

Let $L: \mathcal{V} \to \mathcal{W}$ be a linear transformation, and let $B$ be a basis for $\mathcal{V}$.

| | | |
|---|---|---|
| **$L$ is *one-to-one*** | | |
| $\iff$ | $\ker(L) = \{\mathbf{0}_{\mathcal{V}}\}$ | Theorem 5.12 |
| $\iff$ | $\dim(\ker(L)) = 0$ | Theorem 5.12 |
| $\iff$ | the image of every linearly independent set in $\mathcal{V}$ is linearly independent in $\mathcal{W}$ | Theorem 5.14 and Exercise 10 in Section 5.4 |
| **$L$ is *onto*** | | |
| $\iff$ | $\text{range}(L) = \mathcal{W}$ | Definition |
| $\iff$ | $\dim(\text{range}(L)) = \dim(\mathcal{W})$ | Theorem 4.13* |
| $\iff$ | the image of *every* spanning set for $\mathcal{V}$ is a spanning set for $\mathcal{W}$ | Theorem 5.14 |
| $\iff$ | the image of *some* spanning set for $\mathcal{V}$ is a spanning set for $\mathcal{W}$ | Exercise 11 in Section 5.4 |
| **$L$ is an *isomorphism*** | | |
| $\iff$ | $L$ is both one-to-one and onto | Definition |
| $\iff$ | $L$ is invertible (that is, $L^{-1}: \mathcal{W} \to \mathcal{V}$ exists) | Theorem 5.15 |
| $\iff$ | the matrix for $L$ (with respect to *every* pair of ordered bases for $\mathcal{V}$ and $\mathcal{W}$) is nonsingular | Theorem 5.16* |
| $\iff$ | the matrix for $L$ (with respect to *some* pair of ordered bases for $\mathcal{V}$ and $\mathcal{W}$) is nonsingular | Theorem 5.16* |
| $\iff$ | the images of vectors in $B$ are distinct and $L(B)$ is a basis for $\mathcal{W}$ | Exercise 13 |
| $\iff$ | $L$ is one-to-one and $\dim(\mathcal{V}) = \dim(\mathcal{W})$ | Corollary 5.13* |
| $\iff$ | $L$ is onto and $\dim(\mathcal{V}) = \dim(\mathcal{W})$ | Corollary 5.13* |
| **Furthermore, if $L: \mathcal{V} \to \mathcal{W}$ is an isomorphism, then** | | |
| (1) | $\dim(\mathcal{V}) = \dim(\mathcal{W})$ | Theorem 5.18* |
| (2) | $L^{-1}$ is an isomorphism from $\mathcal{W}$ to $\mathcal{V}$ | Theorem 5.15 |
| (3) | for any subspace $\mathcal{Y}$ of $\mathcal{V}$, $\dim(\mathcal{Y}) = \dim(L(\mathcal{Y}))$ | Exercise 15* |

* *True only in the finite dimensional case.*

## New Vocabulary

inverse of a linear transformation
invertible linear transformation
isomorphic vector spaces
isomorphism

## Highlights

- A linear transformation $L: \mathcal{V} \to \mathcal{W}$ is invertible if and only if there is a function $M: \mathcal{W} \to \mathcal{V}$ such that $M \circ L$ and $L \circ M$ are the identity linear operators on $\mathcal{V}$ and $\mathcal{W}$, respectively.
- If a linear transformation $L: \mathcal{V} \to \mathcal{W}$ is invertible, then its inverse $L^{-1}$ is also a linear transformation.
- A linear transformation $L$ is an isomorphism if and only if $L$ is both one-to-one and onto.
- A linear transformation $L: \mathcal{V} \to \mathcal{W}$ is an isomorphism if and only if $L$ is invertible.
- Suppose $\mathcal{V}$ and $\mathcal{W}$ are nontrivial finite dimensional vector spaces. Then a linear transformation $L: \mathcal{V} \to \mathcal{W}$ is an isomorphism if and only if the matrix $\mathbf{A}_{BC}$ for $L$ (with respect to any ordered bases $B$ and $C$) is nonsingular. Moreover, if $L$ is an isomorphism, then the matrix for $L^{-1}$ with respect to $C$ and $B$ is $(\mathbf{A}_{BC})^{-1}$.

- If $L: \mathcal{V} \to \mathcal{W}$ is an isomorphism, and $T$ is a linearly independent subset of $\mathcal{V}$, its image $L(T)$ is linearly independent in $\mathcal{W}$.
- If $L: \mathcal{V} \to \mathcal{W}$ is an isomorphism, and $S$ spans $\mathcal{V}$, then its image $L(S)$ spans $\mathcal{W}$.
- If $L: \mathcal{V} \to \mathcal{W}$ is an isomorphism, with $\dim(\mathcal{V})$ and $\dim(\mathcal{W})$ finite, and $\mathcal{Y}$ is a subspace of $\mathcal{V}$, then $\dim(\mathcal{Y}) = \dim(L(\mathcal{Y}))$ (that is, the dimension of every subspace of $\mathcal{V}$ is equal to the dimension of its image under $L$).
- Finite dimensional vector spaces $\mathcal{V}$ and $\mathcal{W}$ are isomorphic if and only if $\dim(\mathcal{V}) = \dim(\mathcal{W})$.
- All $n$-dimensional vector spaces are isomorphic to $\mathbb{R}^n$ (and to each other).
- The Simplified Span Method and the Independence Test Method can be justified for sets of vectors in any $n$-dimensional vector space $\mathcal{V}$ by applying a coordinatization isomorphism from $\mathcal{V}$ to $\mathbb{R}^n$.
- The Kernel Method and the Range Method can be justified for any linear transformation $L: \mathcal{V} \to \mathcal{W}$, with $\dim(\mathcal{V}) = n$ and $\dim(\mathcal{W}) = m$, by applying coordinatization isomorphisms between $\mathcal{V}$ and $\mathbb{R}^n$ and between $\mathcal{W}$ and $\mathbb{R}^m$.

## Exercises for Section 5.5

**1.** Each part of this exercise gives matrices for linear operators $L_1$ and $L_2$ on $\mathbb{R}^3$ with respect to the standard basis. For each part, do the following:

**(i)** Show that $L_1$ and $L_2$ are isomorphisms.
**(ii)** Find $L_1^{-1}$ and $L_2^{-1}$.
**(iii)** Calculate $L_2 \circ L_1$ directly.
**(iv)** Calculate $(L_2 \circ L_1)^{-1}$ by inverting the appropriate matrix.
**(v)** Calculate $L_1^{-1} \circ L_2^{-1}$ directly from your answer to (ii) and verify that the answer agrees with the result you obtained in (iv).

★ **(a)** $L_1: \begin{bmatrix} 0 & -2 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $L_2: \begin{bmatrix} 1 & 0 & 0 \\ -2 & 0 & 1 \\ 0 & -3 & 0 \end{bmatrix}$

★ **(c)** $L_1: \begin{bmatrix} -9 & 2 & 1 \\ -6 & 1 & 1 \\ 5 & 0 & -2 \end{bmatrix}$, $L_2: \begin{bmatrix} -4 & 2 & 1 \\ -3 & 1 & 0 \\ -5 & 2 & 1 \end{bmatrix}$

**(b)** $L_1: \begin{bmatrix} 0 & 1 & 1 \\ 3 & 1 & 2 \\ 4 & 2 & 3 \end{bmatrix}$, $L_2: \begin{bmatrix} 8 & 0 & 1 \\ 5 & 4 & -1 \\ 1 & -2 & 1 \end{bmatrix}$

**(d)** $L_1: \frac{1}{21}\begin{bmatrix} -4 & 20 & 5 \\ 20 & 5 & -4 \\ 5 & -4 & 20 \end{bmatrix}$, $L_2: \frac{1}{7}\begin{bmatrix} 1 & 3 & 5 \\ 6 & 4 & -5 \\ 6 & -3 & 2 \end{bmatrix}$

**2.** Show that $L: \mathcal{M}_{mn} \to \mathcal{M}_{nm}$ given by $L(\mathbf{A}) = \mathbf{A}^T$ is an isomorphism. (Recall that we have already shown that $L$ is a linear transformation.)

**3.** Let $\mathbf{A}$ be a fixed nonsingular $n \times n$ matrix.

**(a)** Show that $L_1: \mathcal{M}_{nn} \to \mathcal{M}_{nn}$ given by $L_1(\mathbf{B}) = \mathbf{AB}$ is an isomorphism. (Hint: Be sure to show first that $L_1$ is a linear transformation.)
**(b)** Show that $L_2: \mathcal{M}_{nn} \to \mathcal{M}_{nn}$ given by $L_2(\mathbf{B}) = \mathbf{ABA}^{-1}$ is an isomorphism.
**(c)** Show that $L_3: \mathcal{M}_{nn} \to \mathcal{M}_{nn}$ given by $L_2(\mathbf{B}) = \mathbf{ABA}^T$ is an isomorphism.

**4.** Show that $L: \mathcal{P}_n \to \mathcal{P}_n$ given by $L(\mathbf{p}) = \mathbf{p} + \mathbf{p}'$ is an isomorphism. (Hint: First show that $L$ is a linear transformation.)

**5.** Let $R: \mathbb{R}^2 \to \mathbb{R}^2$ be the operator that reflects a vector through the line $y = x$; that is, $R([a, b]) = [b, a]$.

★ **(a)** Find the matrix for $R$ with respect to the standard basis for $\mathbb{R}^2$.
**(b)** Show that $R$ is an isomorphism.
**(c)** Prove that $R^{-1} = R$ using the matrix from part (a).
**(d)** Give a geometric explanation for the result in part (c).

**6.** Prove that the change of basis process is essentially an isomorphism; that is, if $B$ and $C$ are two different finite bases for a vector space $\mathcal{V}$, with $\dim(\mathcal{V}) = n$, then the mapping $L: \mathbb{R}^n \to \mathbb{R}^n$ given by $L([\mathbf{v}]_B) = [\mathbf{v}]_C$ is an isomorphism. (Hint: First show that $L$ is a linear transformation.)

**7.** Let $\mathcal{V}$, $\mathcal{W}$, $\mathcal{X}$ and $\mathcal{Y}$ be vector spaces. Let $L_1: \mathcal{W} \to \mathcal{X}$ and $L_2: \mathcal{W} \to \mathcal{X}$ be linear transformations. Let $M: \mathcal{V} \to \mathcal{W}$ and $Q: \mathcal{X} \to \mathcal{Y}$ be isomorphisms.

**(a)** If $Q \circ L_1 = Q \circ L_2$, show that $L_1 = L_2$.
**(b)** If $L_1 \circ M = L_2 \circ M$, show that $L_1 = L_2$.

► **8.** Finish the proof of Theorem 5.16 by showing that if $B$ and $C$ are ordered bases for nontrivial finite dimensional vector spaces $\mathcal{V}$ and $\mathcal{W}$, respectively, and if $L: \mathcal{V} \to \mathcal{W}$ is a linear transformation with matrix $\mathbf{A}_{BC}$ nonsingular, then $L$ is an isomorphism.

**9.** This exercise explores isomorphisms between various vector spaces.

**(a)** Explain why $\mathcal{M}_{mn} \cong \mathcal{M}_{nm}$.
**(b)** Explain why $\mathcal{P}_{4n+3} \cong \mathcal{M}_{4,n+1}$.
**(c)** Explain why the subspace of upper triangular matrices in $\mathcal{M}_{nn}$ is isomorphic to $\mathbb{R}^{n(n+1)/2}$. Is the subspace still isomorphic to $\mathbb{R}^{n(n+1)/2}$ if *upper* is replaced by *lower*?

**10.** Let $\mathcal{V}$ be a vector space. Show that a linear operator $L\colon \mathcal{V} \to \mathcal{V}$ is an isomorphism if and only if $L \circ L$ is an isomorphism.

**11.** Let $\mathcal{V}$ be a nontrivial vector space. Suppose that $L\colon \mathcal{V} \to \mathcal{V}$ is a linear operator.
**(a)** If $L \circ L$ is the zero transformation, show that $L$ is not an isomorphism.
**(b)** If $L \circ L = L$ and $L$ is not the identity transformation, show that $L$ is not an isomorphism.
**(c)** Suppose $\mathcal{V}$ is finite dimensional, $\dim(\mathcal{V})$ is odd, and $I$ is the identity transformation on $\mathcal{V}$. Show that $L \circ L \neq -I$.
**(d)** Suppose $\mathcal{V} = \mathbb{R}^2$ and $I$ is the identity transformation on $\mathbb{R}^2$. Prove that there are an infinite number of isomorphisms such that $L \circ L = -I$. Do this by giving a description of all such isomorphisms. (Hint: First show that $L(\mathbf{e}_1)$ is not a scalar multiple of $\mathbf{e}_1$. Then consider the possible images of the ordered basis $\{\mathbf{e}_1, L(\mathbf{e}_1)\}$ for $\mathbb{R}^2$.)

**12.** Let $L\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator with matrix $\mathbf{A}$ (using the standard basis for $\mathbb{R}^n$). Prove that $L$ is an isomorphism if and only if the columns of $\mathbf{A}$ are linearly independent.

**13.** Let $L\colon \mathcal{V} \to \mathcal{W}$ be a linear transformation between vector spaces, and let $B$ be a basis for $\mathcal{V}$.
**(a)** Show that if $L$ is an isomorphism, then $L(B)$ is a basis for $\mathcal{W}$.
**(b)** Prove that if $L(B)$ is a basis for $\mathcal{W}$, and the images of vectors in $B$ are distinct, then $L$ is an isomorphism. (Hint: Use Exercise 11(c) in Section 5.4 to show $L$ is onto. Then show $\ker(L) = \{\mathbf{0}_{\mathcal{V}}\}$ using a proof by contradiction.)
**(c)** Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ by $T(\mathbf{X}) = \begin{bmatrix} 3 & 5 & 3 \\ 1 & 2 & 1 \end{bmatrix}\mathbf{X}$, and let $B$ be the standard basis in $\mathbb{R}^3$. Show that $T(B)$ is a basis for $\mathbb{R}^2$, but $T$ is not an isomorphism.
**(d)** Explain why part (c) does not provide a counterexample to part (b).

**14.** Let $L\colon \mathcal{V} \to \mathcal{W}$ be an isomorphism between finite dimensional vector spaces, and let $B$ be a basis for $\mathcal{V}$. Show that for all $\mathbf{v} \in \mathcal{V}$, $[\mathbf{v}]_B = [L(\mathbf{v})]_{L(B)}$. (Hint: Use the fact from Exercise 13(a) that $L(B)$ is a basis for $\mathcal{W}$.)

► **15.** Let $L\colon \mathcal{V} \to \mathcal{W}$ be an isomorphism, with $\mathcal{V}$ finite dimensional. If $\mathcal{Y}$ is any subspace of $\mathcal{V}$, prove that $\dim(L(\mathcal{Y})) = \dim(\mathcal{Y})$. (Note: Exercises 16 and 17 will use this exercise to help prove the general Dimension Theorem. Therefore, you should not use the Dimension Theorem in your solution to this problem. Instead, you may use any results given before the Dimension Theorem, along with Theorem 5.14, and part (a) of Exercise 11 in Section 5.4, since the proofs of the latter two results did not use the Dimension Theorem.)

**16.** Suppose $T\colon \mathcal{V} \to \mathcal{W}$ is a linear transformation, and $T_1\colon \mathcal{X} \to \mathcal{V}$, and $T_2\colon \mathcal{W} \to \mathcal{Y}$ are isomorphisms.
**(a)** Prove that $\ker(T_2 \circ T) = \ker(T)$.
**(b)** Prove that $\text{range}(T \circ T_1) = \text{range}(T)$.
► **(c)** Prove that $T_1(\ker(T \circ T_1)) = \ker(T)$.
**(d)** If $\mathcal{V}$ is finite dimensional, show that $\dim(\ker(T)) = \dim(\ker(T \circ T_1))$. (Hint: Use part (c) and Exercise 15.)
► **(e)** Prove that $\text{range}(T_2 \circ T) = T_2(\text{range}(T))$.
**(f)** If $\mathcal{V}$ is finite dimensional, show that $\dim(\text{range}(T)) = \dim(\text{range}(T_2 \circ T))$. (Hint: Use part (e) and Exercise 15.)

**17.** Suppose $L\colon \mathcal{V} \to \mathcal{W}$ is a linear transformation, and that $L_1\colon \mathcal{V} \to \mathbb{R}^n$ and $L_2\colon \mathcal{W} \to \mathbb{R}^m$ are isomorphisms. Let $M = L_2 \circ L \circ L_1^{-1}$.
► **(a)** Use part (c) of Exercise 16 with $T = L_2 \circ L$ and $T_1 = L_1^{-1}$ to prove that $L_1^{-1}(\ker(M)) = \ker(L_2 \circ L)$.
► **(b)** Use part (a) of this exercise together with part (a) of Exercise 16 to prove that $L_1^{-1}(\ker(M)) = \ker(L)$.
► **(c)** Use part (b) of this exercise together with Exercise 15 to prove that $\dim(\ker(M)) = \dim(\ker(L))$.
**(d)** Use part (e) of Exercise 16 to prove that $L_2^{-1}(\text{range}(M)) = \text{range}\left(L \circ L_1^{-1}\right)$. (Hint: Let $T = L \circ L_1^{-1}$ and $T_2 = L_2$. Then apply $L_2^{-1}$ to both sides.)
**(e)** Use part (d) of this exercise together with part (b) of Exercise 16 to prove that $L_2^{-1}(\text{range}(M)) = \text{range}(L)$.
**(f)** Use part (e) of this exercise together with Exercise 15 to prove that $\dim(\text{range}(M)) = \dim(\text{range}(L))$.

**18.** We show in this exercise that any isomorphism from $\mathbb{R}^2$ to $\mathbb{R}^2$ is the composition of certain types of reflections, contractions/dilations, and shears. (See Exercise 11 in Section 5.1 for the definition of a shear.) Note that if $a \neq 0$,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{ad-bc}{a} \end{bmatrix} \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{bmatrix},$$

and if $c \neq 0$,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{bc-ad}{c} \end{bmatrix}\begin{bmatrix} 1 & \frac{d}{c} \\ 0 & 1 \end{bmatrix}.$$

**(a)** Use the given equations to show that every nonsingular $2 \times 2$ matrix can be expressed as a product of matrices, each of which is in one of the following forms:

$$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}, \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}, \text{ or } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

**(b)** Show that when $k \geq 0$, multiplying either of the first two matrices in part (a) times the vector $[x, y]$ represents a contraction/dilation along the $x$-coordinate or the $y$-coordinate.

**(c)** Show that when $k < 0$, multiplying either of the first two matrices in part (a) times the vector $[x, y]$ represents a contraction/dilation along the $x$-coordinate or the $y$-coordinate, followed by a reflection through one of the axes. $\left(\text{Hint: } \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -k & 0 \\ 0 & 1 \end{bmatrix}.\right)$

**(d)** Explain why multiplying either of the third or fourth matrices in part (a) times $[x, y]$ represents a shear.

**(e)** Explain why multiplying the last matrix in part (a) times $[x, y]$ represents a reflection through the line $y = x$.

**(f)** Using parts (a) through (e), show that any isomorphism from $\mathbb{R}^2$ to $\mathbb{R}^2$ is the composition of a finite number of the following linear operators: reflection through an axis, reflection through $y = x$, contraction/dilation of the $x$- or $y$-coordinate, shear in the $x$- or $y$-direction.

**19.** Express the linear transformation $L: \mathbb{R}^2 \to \mathbb{R}^2$ that rotates the plane 45° in a counterclockwise direction as a composition of the transformations described in part (f) of Exercise 18.

**20.** **(a)** For the vector space $\mathcal{V}$ given in Example 8 of Section 4.1, find a basis for $\mathcal{V}$ and verify that $\dim(\mathcal{V}) = 2$.

**(b)** For the vector space $\mathcal{V}$ given in Example 8 of Section 4.1, show that $L: \mathcal{V} \to \mathbb{R}^2$ given by $L(\mathbf{v}) = \mathbf{v} + [1, -2]$ is an isomorphism. (Note: We have already shown in Exercise 37 of Section 5.1 that $L$ is a linear transformation.)

**(c)** For the vector space $\mathcal{V}$ given in Example 7 of Section 4.1, show that $L: \mathcal{V} \to \mathbb{R}$ given by $L(v) = \ln(v)$ is an isomorphism. (Note: We have already shown in Exercise 20(a) of Section 5.1 that $L$ is a linear transformation.)

★ **21.** True or False:

**(a)** If the inverse $L^{-1}$ of a linear transformation $L$ exists, then $L^{-1}$ is also a linear transformation.

**(b)** A linear transformation is an isomorphism if and only if it is invertible.

**(c)** If $L: \mathcal{V} \to \mathcal{V}$ is a linear operator on a nontrivial finite dimensional vector space $\mathcal{V}$ with ordered basis $B$, and the matrix for $L$ with respect to $B$ is $\mathbf{A}_{BB}$, then $L$ is an isomorphism if and only if $|\mathbf{A}_{BB}| = 0$.

**(d)** If $L: \mathcal{V} \to \mathcal{W}$ is a linear transformation, then $L$ is one-to-one if and only if $L$ is onto.

**(e)** If $L: \mathcal{V} \to \mathcal{W}$ is a linear transformation and $M: \mathcal{X} \to \mathcal{V}$ is an isomorphism, then $\ker(L \circ M) = \ker(L)$.

**(f)** If $L: \mathcal{V} \to \mathcal{W}$ is a linear transformation and $M: \mathcal{X} \to \mathcal{V}$ is an isomorphism, then $\text{range}(L \circ M) = \text{range}(L)$.

**(g)** If $L: \mathcal{V} \to \mathcal{W}$ is an isomorphism and $\mathbf{w}_1, \ldots, \mathbf{w}_n \in \mathcal{W}$, then for every set of scalars $a_1, \ldots, a_n$, $L^{-1}(a_1\mathbf{w}_1 + \cdots + a_n\mathbf{w}_n) = a_1 L^{-1}(\mathbf{w}_1) + \cdots + a_n L^{-1}(\mathbf{w}_n)$.

**(h)** $\mathbb{R}^{28} \cong \mathcal{P}_{27} \cong \mathcal{M}_{74}$.

## 5.6 Diagonalization of Linear Operators

In Section 3.4, we examined a method for diagonalizing certain square matrices. In this section, we generalize this process to diagonalize certain linear operators. In doing so, we will also, finally, accomplish two goals: first, verifying that fundamental eigenvectors for distinct eigenvalues are linearly independent (thus validating Step 5 of the Diagonalization Method in Section 3.4), and second, establishing that the geometric multiplicity of an eigenvalue is always less than or equal to its algebraic multiplicity (as asserted in Section 3.4).

### Eigenvalues, Eigenvectors, and Eigenspaces for Linear Operators

We define eigenvalues, eigenvectors, and eigenspaces for linear operators in a manner analogous to their definitions for matrices.

**Definition** Let $L: \mathcal{V} \to \mathcal{V}$ be a linear operator. A real number $\lambda$ is said to be an **eigenvalue** of $L$ if and only if there is a nonzero vector $\mathbf{v} \in \mathcal{V}$ such that $L(\mathbf{v}) = \lambda\mathbf{v}$. Also, any nonzero vector $\mathbf{v}$ such that $L(\mathbf{v}) = \lambda\mathbf{v}$ is said to be an **eigenvector** for $L$ corresponding to the eigenvalue $\lambda$.

If $\lambda$ is an eigenvalue for $L$, then $E_\lambda$, the **eigenspace of** $\lambda$, is defined to be the set of all eigenvectors for $L$ corresponding to $\lambda$, together with the zero vector $\mathbf{0}_\mathcal{V}$ of $\mathcal{V}$. That is, $E_\lambda = \{\mathbf{v} \in \mathcal{V} \mid L(\mathbf{v}) = \lambda\mathbf{v}\}$.

If $L$ is a linear operator on $\mathbb{R}^n$ given by multiplication by a square matrix $\mathbf{A}$ (that is, $L(\mathbf{v}) = \mathbf{Av}$), then the eigenvalues and eigenvectors for $L$ are merely the eigenvalues and eigenvectors of the matrix $\mathbf{A}$, since $L(\mathbf{v}) = \lambda\mathbf{v}$ if and only if $\mathbf{Av} = \lambda\mathbf{v}$. Hence, all of the results regarding eigenvalues and eigenvectors for matrices in Section 3.4 apply to this type of operator. In fact, this concept generalizes to a relationship between a linear operator and its matrix with respect to any ordered basis.

**Theorem 5.22** *Let $L: \mathcal{V} \to \mathcal{V}$ be a linear operator on a nontrivial finite dimensional vector space, and let $\mathbf{A}_{BB}$ be the matrix for $L$ with respect to some ordered basis $B$ for $\mathcal{V}$. Then,*

**(1)** *$\lambda$ is an eigenvalue for $L$ if and only if $\lambda$ is an eigenvalue for $\mathbf{A}_{BB}$.*

**(2)** *$\mathbf{v}$ is an eigenvector for $L$ corresponding to the eigenvalue $\lambda$ if and only if $[\mathbf{v}]_B$ is an eigenvector for $\mathbf{A}_{BB}$ corresponding to the eigenvalue $\lambda$.*

Theorem 5.22 essentially says that the coordinatization process preserves eigenvalues, eigenvectors, and eigenspaces. You are asked to prove Theorem 5.22 in Exercise 9.

Also, it can easily be shown that the eigenspace of a linear operator $L: \mathcal{V} \to \mathcal{V}$ is a subspace of the vector space $\mathcal{V}$. (See Exercise 10.)

We next consider an example involving a linear operator that is not given directly by matrix multiplication.

**Example 1**

Consider $L: \mathcal{M}_{nn} \to \mathcal{M}_{nn}$ given by $L(\mathbf{A}) = \mathbf{A}^T$. Then every nonzero $n \times n$ symmetric matrix $\mathbf{S}$ is an eigenvector for $L$ corresponding to the eigenvalue $\lambda = 1$ because $L(\mathbf{S}) = \mathbf{S}^T = \mathbf{S}$ (since $\mathbf{S}$ is symmetric) $= 1\mathbf{S}$. In fact, it is easy to see that the eigenspace $E_1$ equals the set of all $n \times n$ symmetric matrices. Similarly, every nonzero skew-symmetric $n \times n$ matrix $\mathbf{V}$ is an eigenvector for $L$ corresponding to the eigenvalue $\lambda_2 = -1$ because $L(\mathbf{V}) = \mathbf{V}^T = -\mathbf{V}$. Hence, clearly, the eigenspace $E_{-1}$ is the set of all $n \times n$ skew-symmetric matrices.

## The Characteristic Polynomial of a Linear Operator

Next, we define the characteristic polynomial of a linear operator.

**Definition** Let $L$ be a linear operator on a nontrivial finite dimensional vector space $\mathcal{V}$. Suppose $\mathbf{A}$ is the matrix representation of $L$ with respect to some ordered basis $B$ for $\mathcal{V}$. Then the **characteristic polynomial of** $L$, $p_L(x)$, is defined to be $p_\mathbf{A}(x)$.

It appears at first glance as though the characteristic polynomial for a linear operator may be dependent upon the choice of the basis $B$. However, if $L$ is a linear operator on a nontrivial finite dimensional vector space $\mathcal{V}$, and $B$ and $C$ are two ordered bases for $\mathcal{V}$, then Theorem 5.6 implies that $\mathbf{A}_{CC} = \mathbf{PA}_{BB}\mathbf{P}^{-1}$, where $\mathbf{P}$ is the transition matrix from $B$ to $C$. Hence, $\mathbf{A}_{CC}$ and $\mathbf{A}_{BB}$ are similar matrices. By Exercise 6 in Section 3.4, similar matrices have the same characteristic polynomial. Therefore it does not matter which ordered basis is used to find the characteristic polynomial for $L$.

Theorem 5.22 implies that the roots of $p_L(x)$ are the eigenvalues of $L$.

**Example 2**

Consider $L: \mathcal{P}_2 \to \mathcal{P}_2$ determined by $L(\mathbf{p}(x)) = x^2\mathbf{p}''(x) + (3x-2)\mathbf{p}'(x) + 5\mathbf{p}(x)$. You can check that $L(x^2) = 13x^2 - 4x$, $L(x) = 8x - 2$, and $L(1) = 5$. Thus, the matrix representation of $L$ with respect to the standard basis $S = (x^2, x, 1)$ is

$$\mathbf{A} = \begin{bmatrix} 13 & 0 & 0 \\ -4 & 8 & 0 \\ 0 & -2 & 5 \end{bmatrix}.$$

Hence,

$$p_L(x) = p_{\mathbf{A}}(x) = \begin{vmatrix} x-13 & 0 & 0 \\ 4 & x-8 & 0 \\ 0 & 2 & x-5 \end{vmatrix} = (x-13)(x-8)(x-5),$$

since this is the determinant of a lower triangular matrix. The eigenvalues of $L$ are the roots of $p_L(x)$, namely $\lambda_1 = 13$, $\lambda_2 = 8$, and $\lambda_3 = 5$.

## Criteria for Diagonalization

Given a linear operator $L$ on a finite dimensional vector space $\mathcal{V}$, we would like to find a basis $B$ for $\mathcal{V}$ such that the matrix for $L$ with respect to $B$ is diagonal. But, just as not every square matrix can be diagonalized, neither can every linear operator.

**Definition** A linear operator $L$ on a nontrivial finite dimensional vector space $\mathcal{V}$ is **diagonalizable** if and only if the matrix representation of $L$ with respect to some ordered basis for $\mathcal{V}$ is a diagonal matrix.

The next two results indicate precisely which linear operators are diagonalizable.

**Theorem 5.23** *Let $L$ be a linear operator on a nontrivial finite dimensional vector space $\mathcal{V}$, and let $B$ be an ordered basis for $\mathcal{V}$. Then $L$ is diagonalizable if and only if $\mathbf{A}_{BB}$ is diagonalizable.*

Notice in Theorem 5.23 that $\mathbf{A}_{BB}$ is diagonalizable but not necessarily diagonal.

Theorem 5.23 asserts that $L$ is diagonalizable precisely when $\mathbf{A}_{BB}$ is diagonalizable for any basis $B$ for $\mathcal{V}$. (You are asked to prove Theorem 5.23 in Exercise 11.) As we will see, diagonalizing $L$ essentially amounts to diagonalizing $\mathbf{A}_{BB}$. Another consequence of Theorem 5.23 is that if $B$ and $C$ are two ordered bases for $\mathcal{V}$, then $\mathbf{A}_{BB}$ is diagonalizable if and only if $\mathbf{A}_{CC}$ is diagonalizable. (See Exercise 13.)

**Theorem 5.24** *Let $L$ be a linear operator on a nontrivial $n$-dimensional vector space $\mathcal{V}$. Then $L$ is diagonalizable if and only if there is a set of $n$ linearly independent eigenvectors for $L$.*

The idea behind this theorem is that, if $L$ is diagonalizable, then the matrix representation for $L$ with respect to some ordered basis for $\mathcal{V}$ is a diagonal matrix. That particular ordered basis is a set of $n$ linearly independent eigenvectors for $L$. You are asked to provide the details of the proof of Theorem 5.24 in Exercise 12.

### Example 3

Let $L: \mathcal{M}_{22} \to \mathcal{M}_{22}$ be given by $L(\mathbf{A}) = \mathbf{A}^T$. In Example 1, we saw that 1 and $-1$ are eigenvalues for $L$ and that the eigenspace $E_1$ is the set of symmetric $2 \times 2$ matrices. This eigenspace has basis

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

The eigenspace $E_{-1}$ is the set of skew-symmetric $2 \times 2$ matrices. A basis for $E_{-1}$ is

$$\left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}.$$

Using all four vectors in the order listed above gives a linearly independent set $B$, which is an ordered basis for $\mathcal{M}_{22}$. Hence, by Theorem 5.24, $L$ is diagonalizable. The matrix for $L$ with respect to $B$ is

$$\mathbf{A}_{BB} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Note that the matrix for $L$ with respect to the standard basis for $\mathcal{M}_{22}$ is

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

By Theorem 5.23, $\mathbf{A}$ is a diagonalizable matrix. We can verify this by noting that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{A}_{BB}$, where

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

the matrix whose columns are the standard coordinatization of the eigenvectors for $L$ in the ordered basis for $\mathcal{M}_{22}$.

**Example 4**

Consider the linear operator $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ that rotates the plane counterclockwise through an angle of $\frac{\pi}{4}$. Now, every nonzero vector $\mathbf{v}$ is moved to $L(\mathbf{v})$, which is not parallel to $\mathbf{v}$, since $L(\mathbf{v})$ forms a 45° angle with $\mathbf{v}$. Hence, $L$ has *no* eigenvectors, and so a set of two linearly independent eigenvectors cannot be found for $L$. Therefore, by Theorem 5.24, $L$ is not diagonalizable.

## Linear Independence of Eigenvectors

Theorem 5.24 asserts that finding enough *linearly independent* eigenvectors is crucial to the diagonalization process. The next theorem gives a condition under which a set of eigenvectors is guaranteed to be linearly independent.

**Theorem 5.25** *Let $L$ be a linear operator on a vector space $\mathcal{V}$, and let $\lambda_1, \ldots, \lambda_t$ be distinct eigenvalues for $L$. If $\mathbf{v}_1, \ldots, \mathbf{v}_t$ are eigenvectors for $L$ corresponding to $\lambda_1, \ldots, \lambda_t$, respectively, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_t\}$ is linearly independent. That is, eigenvectors corresponding to distinct eigenvalues are linearly independent.*

*Proof.* We proceed by induction on $t$.

**Base Step**: Suppose that $t = 1$. Any eigenvector $\mathbf{v}_1$ for $\lambda_1$ is nonzero, so $\{\mathbf{v}_1\}$ is linearly independent.

**Inductive Step**: Let $\lambda_1, \ldots, \lambda_{k+1}$ be distinct eigenvalues for $L$, and let $\mathbf{v}_1, \ldots, \mathbf{v}_{k+1}$ be corresponding eigenvectors. Our inductive hypothesis is that the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent. We must prove that $\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}\}$ is linearly independent. Suppose that $a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k + a_{k+1}\mathbf{v}_{k+1} = \mathbf{0}_\mathcal{V}$. Showing that $a_1 = a_2 = \cdots = a_k = a_{k+1} = 0$ will finish the proof. Now,

$$\begin{aligned} & L\left(a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k + a_{k+1}\mathbf{v}_{k+1}\right) = L(\mathbf{0}_\mathcal{V}) \\ \Longrightarrow\quad & a_1L(\mathbf{v}_1) + \cdots + a_kL(\mathbf{v}_k) + a_{k+1}L(\mathbf{v}_{k+1}) = L(\mathbf{0}_\mathcal{V}) \\ \Longrightarrow\quad & a_1\lambda_1\mathbf{v}_1 + \cdots + a_k\lambda_k\mathbf{v}_k + a_{k+1}\lambda_{k+1}\mathbf{v}_{k+1} = \mathbf{0}_\mathcal{V}. \end{aligned}$$

Multiplying both sides of the original equation $a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k + a_{k+1}\mathbf{v}_{k+1} = \mathbf{0}_\mathcal{V}$ by $\lambda_{k+1}$ yields

$$a_1\lambda_{k+1}\mathbf{v}_1 + \cdots + a_k\lambda_{k+1}\mathbf{v}_k + a_{k+1}\lambda_{k+1}\mathbf{v}_{k+1} = \mathbf{0}_\mathcal{V}.$$

Subtracting the last two equations containing $\lambda_{k+1}$ gives

$$a_1(\lambda_1 - \lambda_{k+1})\mathbf{v}_1 + \cdots + a_k(\lambda_k - \lambda_{k+1})\mathbf{v}_k = \mathbf{0}_\mathcal{V}.$$

Hence, our inductive hypothesis implies that

$$a_1(\lambda_1 - \lambda_{k+1}) = \cdots = a_k(\lambda_k - \lambda_{k+1}) = 0.$$

Since the eigenvalues $\lambda_1, \ldots, \lambda_{k+1}$ are distinct, none of the factors $\lambda_i - \lambda_{k+1}$ in these equations can equal zero, for $1 \le i \le k$. Thus, $a_1 = a_2 = \cdots = a_k = 0$. Finally, plugging these values into the earlier equation $a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k + a_{k+1}\mathbf{v}_{k+1} = \mathbf{0}_{\mathcal{V}}$ gives $a_{k+1}\mathbf{v}_{k+1} = \mathbf{0}_{\mathcal{V}}$. Since $\mathbf{v}_{k+1} \neq \mathbf{0}_{\mathcal{V}}$, we must have $a_{k+1} = 0$ as well. □

**Example 5**

Recall the linear operator $L: \mathcal{P}_2 \to \mathcal{P}_2$ from Example 2 determined by $L(\mathbf{p}(x)) = x^2\mathbf{p}''(x) + (3x-2)\mathbf{p}'(x) + 5\mathbf{p}(x)$. In that example, we saw that the matrix for $L$ with respect to the standard basis is

$$\mathbf{A} = \begin{bmatrix} 13 & 0 & 0 \\ -4 & 8 & 0 \\ 0 & -2 & 5 \end{bmatrix},$$

and that the eigenvalues for $L$ are $\lambda_1 = 13$, $\lambda_2 = 8$, and $\lambda_3 = 5$. A quick check verifies that $[5, -4, 1]$, $[0, -3, 2]$, and $[0, 0, 1]$ are corresponding eigenvectors for $\mathbf{A}$, respectively, for the distinct eigenvalues $\lambda_1$, $\lambda_2$, and $\lambda_3$. Therefore, by Theorem 5.25, the set $B = \{5x^2 - 4x + 1, -3x + 2, 1\}$, the corresponding set of eigenvectors for $L$, is linearly independent. This is easy to see, since each polynomial in $B$ has a different degree. In fact, since $\dim(\mathcal{P}_2) = 3$, this set $B$ is a basis for $\mathcal{P}_2$.

Also note that $L$ is diagonalizable by Theorem 5.24, since there are 3 linearly independent eigenvectors for $L$ and $\dim(\mathcal{P}_2) = 3$. In fact, the matrix for $L$ with respect to the ordered basis $B$ is diagonal with the eigenvalues 13, 8, and 5 on the main diagonal.

As illustrated in Example 5, Theorems 5.23 and 5.25 combine to prove the following:

**Corollary 5.26** *If $L$ is a linear operator on an $n$-dimensional vector space and $L$ has $n$ distinct eigenvalues, then $L$ is diagonalizable.*

The converse to this corollary is false, since it is possible to get $n$ linearly independent eigenvectors from fewer than $n$ eigenvalues (see Exercise 6).

As mentioned earlier, one of the major purposes of this section is to justify Step 5 of the Diagonalization Method in Section 3.4. The next theorem, a generalization of Theorem 5.25, essentially accomplishes that goal. The proof is left as Exercises 19 and 20.

**Theorem 5.27** *Let $L: \mathcal{V} \to \mathcal{V}$ be a linear operator on a finite dimensional vector space, and let $B_1, B_2, \ldots, B_k$ be bases for eigenspaces $E_{\lambda_1}, \ldots, E_{\lambda_k}$ for $L$, where $\lambda_1, \ldots, \lambda_k$ are distinct eigenvalues for $L$. Then $B_i \cap B_j = \emptyset$ for $1 \le i < j \le k$, and $B_1 \cup B_2 \cup \cdots \cup B_k$ is a linearly independent subset of $\mathcal{V}$.*

Theorem 5.27 asserts that for a given operator on a finite dimensional vector space, the bases for distinct eigenspaces are disjoint, and the union of two or more bases from distinct eigenspaces always constitutes a linearly independent set.[5] This is precisely what we assumed in Step 5 of the Diagonalization Method for matrices back in Section 3.4. Specifically, we assumed that if we have $n$ fundamental eigenvectors for a matrix $\mathbf{A}$, then the matrix $\mathbf{P}$ whose columns are these eigenvectors is nonsingular; that is, these fundamental eigenvectors are linearly independent. Hence, Theorem 5.27 justifies Step 5 of the Diagonalization Method.

**Example 6**

Suppose $L: \mathcal{M}_{22} \to \mathcal{M}_{22}$ is the operator from Example 3 given by $L(\mathbf{A}) = \mathbf{A}^T$. Recall that the bases for the eigenspaces $E_1$ and $E_{-1}$ are

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \text{ and } \left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\},$$

respectively. Note that the intersection of these two bases is the empty set, and that combining these two bases results in a linearly independent set, as promised by Theorem 5.27.

[5] The conclusion of Theorem 5.27 is also true for infinite dimensional vector spaces having an infinite set of eigenvalues and eigenspaces. However, for simplicity, we will not consider the infinite dimensional case here.

## Method for Diagonalizing a Linear Operator

Theorem 5.27 suggests a method for diagonalizing a given linear operator $L: \mathcal{V} \to \mathcal{V}$, when possible. This method, outlined below, illustrates how to find a basis $B$ so that the matrix for $L$ with respect to $B$ is diagonal. First, choose a basis $C$ for $\mathcal{V}$. Next, find the matrix for $L$ with respect to $C$, and then use the Diagonalization Method of Section 3.4 on this matrix to obtain a basis $Z$ of eigenvectors in $\mathbb{R}^n$. Finally, the desired basis $B$ for $\mathcal{V}$ consists of the vectors in $\mathcal{V}$ whose coordinatization with respect to $C$ are the vectors in $Z$.

In the case where $\mathcal{V} = \mathbb{R}^n$, we can use the standard basis for $C$, in which case the method amounts to finding the matrix for $L$ with respect to the standard basis and then simply applying the Diagonalization Method to that matrix to find a basis of eigenvectors for $\mathbb{R}^n$.

**Method for Diagonalizing a Linear Operator (if possible) (Generalized Diagonalization Method)**
Let $L: \mathcal{V} \to \mathcal{V}$ be a linear operator on a nontrivial $n$-dimensional vector space $\mathcal{V}$.

**Step 1:** Find a basis $C$ for $\mathcal{V}$ (if $\mathcal{V} = \mathbb{R}^n$, we can use the standard basis), and calculate the matrix representation $\mathbf{A}_{CC}$ of $L$ with respect to $C$.
**Step 2:** Apply the Diagonalization Method of Section 3.4 to $\mathbf{A}_{CC}$ in order to obtain all of the eigenvalues $\lambda_1, \ldots, \lambda_k$ of $\mathbf{A}_{CC}$ and a basis in $\mathbb{R}^n$ for each eigenspace $E_{\lambda_i}$ of $\mathbf{A}_{CC}$ (by solving an appropriate homogeneous system if necessary). If the union of the bases of the $E_{\lambda_i}$ contains fewer than $n$ elements, then $L$ is not diagonalizable, and we stop. Otherwise, let $Z = (\mathbf{w}_1, \ldots, \mathbf{w}_n)$ be an ordered basis for $\mathbb{R}^n$ consisting of the union of the bases for the $E_{\lambda_i}$.
**Step 3:** Reverse the $C$-coordinatization isomorphism on the vectors in $Z$ to obtain an ordered basis $B = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ for $\mathcal{V}$; that is, $[\mathbf{v}_i]_C = \mathbf{w}_i$.
The matrix representation for $L$ with respect to $B$ is the diagonal matrix $\mathbf{D}$ whose $(i,i)$ entry $d_{ii}$ is the eigenvalue for $L$ corresponding to $\mathbf{v}_i$. In most practical situations, the transition matrix $\mathbf{P}$ from $B$- to $C$-coordinates is useful; $\mathbf{P}$ is the $n \times n$ matrix whose columns are $[\mathbf{v}_1]_C, \ldots, [\mathbf{v}_n]_C$—that is, $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n$. Note that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}_{CC}\mathbf{P}$.

If we have a linear operator $L$ on $\mathbb{R}^n$ and use the standard basis for $C$, then the $C$-coordinatization isomorphism in this method is merely the identity mapping. Also, if $L$ is given by $L(\mathbf{v}) = \mathbf{A}\mathbf{v}$ for some $n \times n$ matrix $\mathbf{A}$, then this method is practically identical to the Diagonalization Method of Section 3.4.

In the next example, the linear operator does not have $\mathbb{R}^n$ as its domain. Hence, Steps 1 and 3 of the process require additional work.

### Example 7

Let $L: \mathcal{M}_{22} \to \mathcal{M}_{22}$ be the linear operator given by

$$L(\mathbf{A}) = \begin{bmatrix} 23 & 50 \\ -10 & -22 \end{bmatrix}\mathbf{A}.$$

We will follow the Generalized Diagonalization Method to find an ordered basis $B$ for $\mathcal{M}_{22}$ such that the matrix $\mathbf{A}_{BB}$ for $L$ is diagonal.

**Step 1**: We let $C$ be the standard basis for $\mathcal{M}_{22}$. Direct computation of the images of the vectors in $C$ produces the following matrix representation for $L$:

$$\mathbf{A}_{CC} = \begin{bmatrix} 23 & 0 & 50 & 0 \\ 0 & 23 & 0 & 50 \\ -10 & 0 & -22 & 0 \\ 0 & -10 & 0 & -22 \end{bmatrix}.$$

**Step 2**: We apply the Diagonalization Method of Section 3.4 to the matrix $\mathbf{A}_{CC}$.[6]

**D-Step 1**: Direct computation yields $p_{\mathbf{A}_{CC}}(x) = x^4 - 2x^3 - 11x^2 + 12x + 36 = (x-3)^2(x+2)^2$.

**D-Step 2**: The eigenvalues[7] for $\mathbf{A}_{CC}$ are $\lambda_1 = 3$ and $\lambda_2 = -2$.

**D-Step 3**: We compute fundamental eigenvectors for each eigenvalue.

[6] In order to minimize any confusion between the steps of the Generalized Diagonalization Method in this section with steps in the Method from Section 3.4, we have labeled the steps from the Diagonalization Method of Section 3.4 as D-Step 1, D-Step 2, etc.
[7] See Exercise 18 for a general principle involving linear operators of the type considered here. It provides a different approach to computing the eigenvalues.

For $\lambda_1 = 3$: Row reducing $[3\mathbf{I}_4 - \mathbf{A}_{CC} \mid \mathbf{0}]$ produces the fundamental eigenvectors $[-5, 0, 2, 0]$ and $[0, -5, 0, 2]$.
For $\lambda_2 = -2$: Row reducing $[-2\mathbf{I}_4 - \mathbf{A}_{CC} \mid \mathbf{0}]$ produces the fundamental eigenvectors $[-2, 0, 1, 0]$ and $[0, -2, 0, 1]$.

**D-Step 4**: We have 4 fundamental eigenvectors, and so $\mathbf{A}_{CC}$ *can* be diagonalized!
The desired ordered basis in $\mathbb{R}^4$ is formed using these 4 fundamental eigenvectors:

$$Z = ([-5, 0, 2, 0], [0, -5, 0, 2], [-2, 0, 1, 0], [0, -2, 0, 1]).$$

**Step 3**: The ordered basis $B$ for $\mathcal{M}_{22}$ corresponding to $Z$ is

$$B = \left( \begin{bmatrix} -5 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -5 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2 \\ 0 & 1 \end{bmatrix} \right).$$

The matrix for $L$ with respect to $B$ is the diagonal matrix

$$\mathbf{D} = \mathbf{A}_{BB} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}.$$

Finally, the transition matrix from $B$-coordinates to $C$-coordinates (standard) is the $4 \times 4$ matrix whose columns are the vectors in $Z$.

## Geometric and Algebraic Multiplicity

As we have seen, the number of eigenvectors in a basis for each eigenspace is crucial in determining whether a given linear operator is diagonalizable, and so we often need to consider the dimension of each eigenspace.

**Definition** Let $L$ be a linear operator on a finite dimensional vector space, and let $\lambda$ be an eigenvalue for $L$. Then the dimension of the eigenspace $E_\lambda$ is called the **geometric multiplicity of** $\lambda$.

### Example 8

In Examples 1, 3, and 6 we examined the transpose linear operator on $\mathcal{M}_{22}$ having eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$. In those examples, we found $\dim(E_{\lambda_1}) = 3$ and $\dim(E_{\lambda_2}) = 1$. Hence, the geometric multiplicity of $\lambda_1$ is 3 and the geometric multiplicity of $\lambda_2$ is 1.

In Example 7, we studied a different linear operator on $\mathcal{M}_{22}$ having eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -2$. In that example, we found $\dim(E_{\lambda_1}) = 2$ and $\dim(E_{\lambda_2}) = 2$. Hence, the geometric multiplicity of $\lambda_1$ is 2 and the geometric multiplicity of $\lambda_2$ is 2.

We define the algebraic multiplicity of a linear operator in a manner analogous to the matrix-related definition in Section 3.4.

**Definition** Let $L$ be a linear operator on a finite dimensional vector space, and let $\lambda$ be an eigenvalue for $L$. Suppose that $(x - \lambda)^k$ is the highest power of $(x - \lambda)$ that divides $p_L(x)$. Then $k$ is called the **algebraic multiplicity of** $\lambda$.

The next theorem accomplishes another major goal of this section by validating the relationship suggested in Section 3.4 between the algebraic and geometric multiplicities of an eigenvalue.

**Theorem 5.28** *Let L be a linear operator on a nontrivial finite dimensional vector space $\mathcal{V}$, and let $\lambda$ be an eigenvalue for L. Then*

$$1 \le (\textit{geometric multiplicity of } \lambda) \le (\textit{algebraic multiplicity of } \lambda).$$

The proof of Theorem 5.28 uses the following lemma:

**Lemma 5.29** *Let* $\mathbf{A}$ *be an* $n \times n$ *matrix symbolically represented by* $\mathbf{A} = \begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{O} & \mathbf{D} \end{bmatrix}$, *where* $\mathbf{B}$ *is an* $m \times m$ *submatrix,* $\mathbf{C}$ *is an* $m \times (n-m)$ *submatrix,* $\mathbf{O}$ *is an* $(n-m) \times m$ *zero submatrix, and* $\mathbf{D}$ *is an* $(n-m) \times (n-m)$ *submatrix. Then,* $|\mathbf{A}| = |\mathbf{B}| \cdot |\mathbf{D}|$.

Lemma 5.29 follows from Exercise 14 in Section 3.2. (We suggest you complete that exercise if you have not already done so.)

*Proof.* Proof of Theorem 5.28: Let $\mathcal{V}$, $L$, and $\lambda$ be as given in the statement of the theorem, and let $k$ represent the geometric multiplicity of $\lambda$. By definition, the eigenspace $E_\lambda$ must contain at least one nonzero vector, and thus $k = \dim(E_\lambda) \geq 1$. Thus, the first inequality in the theorem is proved.

Next, choose a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ for $E_\lambda$ and expand it to an ordered basis $B = (\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_n)$ for $\mathcal{V}$. Let $\mathbf{A}$ be the matrix representation for $L$ with respect to $B$. Notice that for $1 \leq i \leq k$, the $i$th column of $\mathbf{A} = [L(\mathbf{v}_i)]_B = [\lambda \mathbf{v}_i]_B = \lambda[\mathbf{v}_i]_B = \lambda \mathbf{e}_i$. Thus, $\mathbf{A}$ has the form

$$\mathbf{A} = \begin{bmatrix} \lambda \mathbf{I}_k & \mathbf{C} \\ \mathbf{O} & \mathbf{D} \end{bmatrix},$$

where $\mathbf{C}$ is a $k \times (n-k)$ submatrix, $\mathbf{O}$ is an $(n-k) \times k$ zero submatrix, and $\mathbf{D}$ is an $(n-k) \times (n-k)$ submatrix.

The form of $\mathbf{A}$ makes it straightforward to calculate the characteristic polynomial of $L$:

$$\begin{aligned} p_L(x) = p_{\mathbf{A}}(x) = |x\mathbf{I}_n - \mathbf{A}| &= \left| x\mathbf{I}_n - \begin{bmatrix} \lambda \mathbf{I}_k & \mathbf{C} \\ \mathbf{O} & \mathbf{D} \end{bmatrix} \right| \\ &= \begin{vmatrix} (x-\lambda)\mathbf{I}_k & -\mathbf{C} \\ \mathbf{O} & x\mathbf{I}_{n-k} - \mathbf{D} \end{vmatrix} \\ &= |(x-\lambda)\mathbf{I}_k| \cdot |x\mathbf{I}_{n-k} - \mathbf{D}| \qquad \text{by Lemma 5.29} \\ &= (x-\lambda)^k \cdot p_{\mathbf{D}}(x). \end{aligned}$$

Let $l$ be the number of factors of $x - \lambda$ in $p_{\mathbf{D}}(x)$. (Note that $l \geq 0$, with $l = 0$ if $p_{\mathbf{D}}(\lambda) \neq 0$.) Then, altogether, $(x-\lambda)^{k+l}$ is the largest power of $x - \lambda$ that divides $p_L(x)$. Hence,

$$\text{geometric multiplicity of } \lambda = k \leq k + l = \text{algebraic multiplicity of } \lambda.$$

□

**Example 9**

Consider the linear operator $L\colon \mathbb{R}^4 \to \mathbb{R}^4$ given by

$$L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} 5 & 2 & 0 & 1 \\ -2 & 1 & 0 & -1 \\ 4 & 4 & 3 & 2 \\ 16 & 0 & -8 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

In Exercise 3(a), you are asked to verify that $p_L(x) = (x-3)^3(x+5)$. Thus, the eigenvalues for $L$ are $\lambda_1 = 3$ and $\lambda_2 = -5$. Notice that the algebraic multiplicity of $\lambda_1$ is 3 and the algebraic multiplicity of $\lambda_2$ is 1.

Next we find the eigenspaces of $\lambda_1$ and $\lambda_2$ by solving appropriate homogeneous systems. Let $\mathbf{A}$ be the matrix for $L$. For $\lambda_1 = 3$, we solve $(3\mathbf{I}_4 - \mathbf{A})\mathbf{v} = \mathbf{0}$ using row reduction to obtain the basis $\{[1, -1, 2, 0], [1, -2, 0, 2]\}$ for $E_3$. (See Exercise 3(b).) Therefore, the geometric multiplicity of $\lambda_1$ is 2, which is less than its algebraic multiplicity.

In Exercise 3(c), you are asked to solve an appropriate system to show that the eigenspace for $\lambda_2 = -5$ has dimension 1, with basis $\{[1, -1, 2, -8]\}$ for $E_{-5}$. Thus, the geometric multiplicity of $\lambda_2$ is 1. Hence, the geometric and algebraic multiplicities of $\lambda_2$ are equal.

The eigenvalue $\lambda_2$ in Example 9 also illustrates the principle that if the algebraic multiplicity of an eigenvalue is 1, then its geometric multiplicity must also be 1. This follows immediately from Theorem 5.28.

## Multiplicities and Diagonalization

Theorem 5.28 gives us a way to use algebraic and geometric multiplicities to determine whether a linear operator is diagonalizable. Let $L\colon \mathcal{V} \to \mathcal{V}$ be a linear operator, with $\dim(\mathcal{V}) = n$. Then $p_L(x)$ has degree $n$. Therefore, the sum of the algebraic multiplicities for all eigenvalues can be at most $n$. Now, for $L$ to be diagonalizable, $L$ must have $n$ linearly independent eigenvectors by Theorem 5.24. This can only happen if the sum of the geometric multiplicities of all eigenvalues for $L$ equals $n$. Theorem 5.28 then forces the geometric multiplicity of every eigenvalue to equal its algebraic multiplicity (why?). We have therefore proven the following alternate characterization of diagonalizability:

**Theorem 5.30** *Let $L: \mathcal{V} \to \mathcal{V}$ be a linear operator with $\dim(\mathcal{V}) = n \geq 1$. Then $L$ is diagonalizable if and only if both of the following conditions hold: (1) the sum of the algebraic multiplicities over all eigenvalues of $L$ equals $n$, and, (2) the geometric multiplicity of each eigenvalue equals its algebraic multiplicity.*

Theorem 5.30 gives another justification that the transpose operator $L$ on $\mathcal{M}_{22}$ in Examples 1, 3, and 6 is diagonalizable. First, the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$ have algebraic multiplicities 3 and 1, respectively, and $3 + 1 = 4 = \dim(\mathcal{M}_{22})$. Also, the eigenvalues respectively have geometric multiplicities 3 and 1, which equal their algebraic multiplicities. These conditions ensure that $L$ is diagonalizable.

### Example 10

Theorem 5.30 shows that the operator on $\mathbb{R}^4$ in Example 9 is not diagonalizable because the geometric multiplicity of $\lambda_1 = 3$ is 2, while its algebraic multiplicity is 3.

### Example 11

Let $L: \mathbb{R}^3 \to \mathbb{R}^3$ be a rotation about the $z$-axis through an angle of $\frac{\pi}{3}$. Then the matrix for $L$ with respect to the standard basis is

$$\mathbf{A} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

as described in Table 5.1. Using $\mathbf{A}$, we calculate $p_L(x) = x^3 - 2x^2 + 2x - 1 = (x-1)(x^2 - x + 1)$, where the quadratic factor has no real roots. Therefore, $\lambda = 1$ is the only eigenvalue, and its algebraic multiplicity is 1. Hence, by Theorem 5.30, $L$ is not diagonalizable because the sum of the algebraic multiplicities of its eigenvalues equals 1, which is less than $\dim(\mathbb{R}^3) = 3$.

## The Cayley-Hamilton Theorem

We conclude this section with an interesting relationship between a matrix and its characteristic polynomial. If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is any polynomial and $\mathbf{A}$ is an $n \times n$ matrix, we define $p(\mathbf{A})$ to be the $n \times n$ matrix given by $p(\mathbf{A}) = a_n\mathbf{A}^n + a_{n-1}\mathbf{A}^{n-1} + \cdots + a_1\mathbf{A} + a_0\mathbf{I}_n$.

**Theorem 5.31** **(Cayley-Hamilton Theorem)** *Let $\mathbf{A}$ be an $n \times n$ matrix, and let $p_{\mathbf{A}}(x)$ be its characteristic polynomial. Then $p_{\mathbf{A}}(\mathbf{A}) = \mathbf{O}_n$.*

The Cayley-Hamilton Theorem is an important result in advanced linear algebra. We have placed its proof in Appendix A for the interested reader.

### Example 12

Let $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 4 & -1 \end{bmatrix}$. Then $p_{\mathbf{A}}(x) = x^2 - 2x - 11$ (verify!). The Cayley-Hamilton Theorem states that $p_{\mathbf{A}}(\mathbf{A}) = \mathbf{O}_2$. To check this, note that

$$p_{\mathbf{A}}(\mathbf{A}) = \mathbf{A}^2 - 2\mathbf{A} - 11\mathbf{I}_2 = \begin{bmatrix} 17 & 4 \\ 8 & 9 \end{bmatrix} - \begin{bmatrix} 6 & 4 \\ 8 & -2 \end{bmatrix} - \begin{bmatrix} 11 & 0 \\ 0 & 11 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

♦ **Application:** You have now covered the prerequisites for Section 8.9, "Differential Equations."

## New Vocabulary

algebraic multiplicity (of an eigenvalue)
Cayley-Hamilton Theorem
characteristic polynomial (for a linear operator)
diagonalizable linear operator
eigenspace (for an eigenvalue of a linear operator)
eigenvalue of a linear operator
eigenvector of a linear operator
Generalized Diagonalization Method (for a linear operator)
geometric multiplicity (of an eigenvalue)

## Highlights

- Let $L: \mathcal{V} \to \mathcal{V}$ be a linear operator on a nontrivial finite dimensional vector space $\mathcal{V}$, and let $\mathbf{A}_{BB}$ be the matrix for $L$ with respect to some ordered basis $B$ for $\mathcal{V}$. Then, $\lambda$ is an eigenvalue for $L$ if and only if $\lambda$ is an eigenvalue for $\mathbf{A}_{BB}$.
- Let $L: \mathcal{V} \to \mathcal{V}$ be a linear operator on a nontrivial finite dimensional vector space $\mathcal{V}$, and let $\mathbf{A}_{BB}$ be the matrix for $L$ with respect to some ordered basis $B$ for $\mathcal{V}$. Then, $\mathbf{v}$ is an eigenvector for $L$ corresponding to the eigenvalue $\lambda$ if and only if $[\mathbf{v}]_B$ is an eigenvector for $\mathbf{A}_{BB}$ corresponding to the eigenvalue $\lambda$.
- Let $\mathcal{V}$ be a nontrivial finite dimensional vector space. Then a linear operator $L: \mathcal{V} \to \mathcal{V}$ is diagonalizable if and only if the matrix $\mathbf{A}_{BB}$ for $L$ with respect to some ordered basis $B$ for $\mathcal{V}$ is a diagonal matrix.
- Let $L$ be a linear operator on a nontrivial finite dimensional vector space $\mathcal{V}$, and let $B$ be an ordered basis for $L$. Then $L$ is diagonalizable if and only if $\mathbf{A}_{BB}$ is diagonalizable.
- Let $\mathcal{V}$ be a nontrivial $n$-dimensional vector space. Then a linear operator $L: \mathcal{V} \to \mathcal{V}$ is diagonalizable if and only if $L$ has $n$ linearly independent eigenvectors.
- If $L: \mathcal{V} \to \mathcal{V}$ is a linear operator, eigenvectors for $L$ corresponding to distinct eigenvalues are linearly independent. More generally, the union of bases for distinct eigenspaces of $L$ is a linearly independent set.
- If $\mathcal{V}$ is a nontrivial $n$-dimensional vector space, and a linear operator $L: \mathcal{V} \to \mathcal{V}$ has $n$ distinct eigenvalues, then $L$ is diagonalizable.
- If $\mathcal{V}$ is a nontrivial finite dimensional vector space, $L: \mathcal{V} \to \mathcal{V}$ is a linear operator having matrix $\mathbf{A}$ (with respect to some ordered basis for $\mathcal{V}$), and $\mathbf{A}$ is diagonalizable, then the Diagonalization Method of Section 3.4 can be used to find the eigenvalues of $L$ and a basis of eigenvectors for $L$.
- Let $\mathcal{V}$ be a finite dimensional vector space. Then if $L: \mathcal{V} \to \mathcal{V}$ is a linear operator having eigenvalue $\lambda$, the geometric multiplicity of $\lambda$ is $\dim(E_\lambda)$, and the algebraic multiplicity of $\lambda$ is the highest power of $(x - \lambda)$ that divides the characteristic polynomial $p_L(x)$. Also, $1 \leq$ (geometric multiplicity of $\lambda$) $\leq$ (algebraic multiplicity of $\lambda$).
- Let $\mathcal{V}$ be a nontrivial $n$-dimensional vector space. Then a linear operator $L: \mathcal{V} \to \mathcal{V}$ is diagonalizable if and only if both of the following conditions hold: (1) the sum of all the algebraic multiplicities of all the eigenvalues of $L$ is equal to $n$, and, (2) the geometric multiplicity of each eigenvalue equals its algebraic multiplicity.
- If $\mathbf{A}$ is an $n \times n$ matrix with characteristic polynomial $p_{\mathbf{A}}(x)$, then $p_{\mathbf{A}}(\mathbf{A}) = \mathbf{O}_n$. That is, every matrix is a "root" of its characteristic polynomial (Cayley-Hamilton Theorem).

## Exercises for Section 5.6

1. For each of the following, let $L$ be a linear operator on $\mathbb{R}^n$ represented by the given matrix with respect to the standard basis. Find all eigenvalues for $L$, and find a basis for the eigenspace corresponding to each eigenvalue. Compare the geometric and algebraic multiplicities of each eigenvalue.

★ (a) $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 33 & -24 \\ 40 & -29 \end{bmatrix}$

★ (c) $\begin{bmatrix} 7 & 1 & 2 \\ -11 & -2 & -3 \\ -24 & -3 & -7 \end{bmatrix}$

★ (d) $\begin{bmatrix} 2 & 0 & 0 \\ 4 & -3 & -6 \\ -4 & 5 & 8 \end{bmatrix}$

(e) $\begin{bmatrix} 5 & -5 & -9 \\ 5 & -3 & -9 \\ 1 & -2 & -2 \end{bmatrix}$

(f) $\begin{bmatrix} -1 & -3 & -8 & 4 \\ -18 & 5 & 46 & 13 \\ 4 & -2 & -13 & -2 \\ -6 & 0 & 10 & 6 \end{bmatrix}$

2. Each of the following represents a linear operator $L$ on a vector space $\mathcal{V}$. Let $C$ be the standard basis in each case, and let $\mathbf{A}$ be the matrix representation of $L$ with respect to $C$. Follow Steps 1 and 2 of the Generalized Diagonalization Method to determine whether $L$ is diagonalizable. If $L$ is diagonalizable, finish the method by performing Step 3. In particular, find the following:
   (i) An ordered basis $B$ for $\mathcal{V}$ consisting of eigenvectors for $L$
   (ii) The diagonal matrix $\mathbf{D}$ that is the matrix representation of $L$ with respect to $B$
   (iii) The transition matrix $\mathbf{P}$ from $B$ to $C$
   Finally, check your work by verifying that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$.
   (a) $L\colon \mathbb{R}^4 \to \mathbb{R}^4$ given by $L([x_1, x_2, x_3, x_4]) = [x_4, x_1, x_2, x_1]$
   ★ (b) $L\colon \mathcal{P}_2 \to \mathcal{P}_2$ given by $L(\mathbf{p}(x)) = (x-1)\mathbf{p}'(x)$
   (c) $L\colon \mathcal{P}_2 \to \mathcal{P}_2$ given by $L(\mathbf{p}(x)) = x^2\mathbf{p}''(x) + (x+1)\mathbf{p}'(x) - 3\mathbf{p}(x)$
   ★ (d) $L\colon \mathcal{P}_2 \to \mathcal{P}_2$ given by $L(\mathbf{p}(x)) = (x-3)^2\mathbf{p}''(x) + x\mathbf{p}'(x) - 5\mathbf{p}(x)$
   ★ (e) $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $L$ is the counterclockwise rotation about the origin through an angle of $\frac{\pi}{3}$ radians
   (f) $L\colon \mathcal{M}_{22} \to \mathcal{M}_{22}$ given by $L(\mathbf{K}) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\mathbf{K} - \mathbf{K}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$
   (g) $L\colon \mathcal{M}_{22} \to \mathcal{M}_{22}$ given by $L(\mathbf{K}) = \mathbf{K} + \mathbf{K}^T$
   ★ (h) $L\colon \mathcal{M}_{22} \to \mathcal{M}_{22}$ given by $L(\mathbf{K}) = \begin{bmatrix} -4 & 3 \\ -10 & 7 \end{bmatrix}\mathbf{K}$
3. Consider the linear operator $L\colon \mathbb{R}^4 \to \mathbb{R}^4$ from Example 9.
   (a) Verify that $p_L(x) = (x-3)^3(x+5) = x^4 - 4x^3 - 18x^2 + 108x - 135$. (Hint: Use a cofactor expansion along the third column.)
   (b) Show that $\{[1, -1, 2, 0], [1, -2, 0, 2]\}$ is a basis for the eigenspace $E_3$ for $L$ by solving an appropriate homogeneous system.
   (c) Show that $\{[1, -1, 2, -8]\}$ is a basis for the eigenspace $E_{-5}$ for $L$ by solving an appropriate homogeneous system.
4. Let $L\colon \mathcal{P}_2 \to \mathcal{P}_2$ be the translation operator given by $L(\mathbf{p}(x)) = \mathbf{p}(x+a)$, for some (fixed) real number $a$.
   ★ (a) Find all eigenvalues for $L$ when $a = 1$, and find a basis for each eigenspace.
   (b) Find all eigenvalues for $L$ when $a$ is an arbitrary nonzero number, and find a basis for each eigenspace.
5. Let $\mathbf{A}$ be an $n \times n$ upper triangular matrix with all main diagonal entries equal. Show that $\mathbf{A}$ is diagonalizable if and only if $\mathbf{A}$ is a diagonal matrix.
6. Explain why Example 7 provides a counterexample to the converse of Corollary 5.26.
7. This exercise concerns particular algebraic and geometric multiplicities.
   ★ (a) Give an example of a $3 \times 3$ upper triangular matrix having an eigenvalue $\lambda$ with algebraic multiplicity 3 and geometric multiplicity 1.
   (b) Give an example of a $3 \times 3$ upper triangular matrix having an eigenvalue $\lambda$ with algebraic multiplicity 3 and geometric multiplicity 2.
   ★ (c) Give an example of a $3 \times 3$ upper triangular matrix, one of whose eigenvalues has algebraic multiplicity 2 and geometric multiplicity 2.
8. This exercise explores properties of eigenvalues for isomorphisms.
   (a) Suppose that $L$ is a linear operator on a nontrivial finite dimensional vector space. Prove $L$ is an isomorphism if and only if 0 is not an eigenvalue for $L$.
   (b) Let $L$ be an isomorphism from a vector space to itself. Suppose that $\lambda$ is an eigenvalue for $L$ having eigenvector $\mathbf{v}$. Prove that $\mathbf{v}$ is an eigenvector for $L^{-1}$ corresponding to the eigenvalue $1/\lambda$.
▶ 9. The purpose of this exercise is to prove Theorem 5.22. Let $L\colon \mathcal{V} \to \mathcal{V}$ be a linear operator on a nontrivial finite dimensional vector space, and let $\mathbf{A}_{BB}$ be the matrix for $L$ with respect to some ordered basis $B$ for $\mathcal{V}$.
   (a) Prove that $\lambda$ is an eigenvalue for $L$ if and only if $\lambda$ is an eigenvalue for $\mathbf{A}_{BB}$.
   (b) Prove that $\mathbf{v}$ is an eigenvector for $L$ corresponding to the eigenvalue $\lambda$ if and only if $[\mathbf{v}]_B$ is an eigenvector for $\mathbf{A}_{BB}$ corresponding to the eigenvalue $\lambda$.
10. Let $L\colon \mathcal{V} \to \mathcal{V}$ be a linear operator on a nontrivial vector space, and let $\lambda$ be an eigenvalue for $L$. Prove that the eigenspace $E_\lambda$ is a subspace of $\mathcal{V}$.
▶ 11. The purpose of this exercise is to prove Theorem 5.23. Let $L$ be a linear operator on a nontrivial finite dimensional vector space $\mathcal{V}$, and let $B$ be an ordered basis for $\mathcal{V}$. Let $\mathbf{A}_{BB}$ be the matrix for $L$ with respect to $B$.

**(a)** Suppose $\mathbf{A}_{BB}$ is a diagonalizable matrix. Prove that there is an ordered basis $C$ for $\mathcal{V}$ such that the matrix representation of $L$ with respect to $C$ is diagonal, and hence that $L$ is a diagonalizable operator.

**(b)** Prove the converse to part (a). That is, show that if $L$ is a diagonalizable operator, then $\mathbf{A}_{BB}$ is a diagonalizable matrix.

► **12.** Prove Theorem 5.24.

**13.** Let $L$ be a linear operator on a nontrivial finite dimensional vector space $\mathcal{V}$, and let $B$ and $C$ be ordered bases for $\mathcal{V}$. Prove that if $\mathbf{A}_{BB}$ is a diagonalizable matrix, then $\mathbf{A}_{CC}$ is a diagonalizable matrix.

**14.** Let $\mathbf{A}$ be an $n \times n$ matrix. Suppose that $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$ of eigenvectors for $\mathbf{A}$ with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Show that $|\mathbf{A}| = \lambda_1\lambda_2\cdots\lambda_n$.

**15.** Let $L$ be a linear operator on an $n$-dimensional vector space, with $\{\lambda_1, \ldots, \lambda_k\}$ equal to the set of all distinct eigenvalues for $L$. Show that $\boldsymbol{\Sigma}_{i=1}^{k}$(geometric multiplicity of $\lambda_i$) $\leq n$.

**16.** Let $L$ be a linear operator on a nontrivial finite dimensional vector space $\mathcal{V}$. Show that if $L$ is diagonalizable, then every root of $p_L(x)$ is real.

**17.** Let $\mathbf{A}$ and $\mathbf{B}$ be commuting $n \times n$ matrices.

**(a)** Show that if $\lambda$ is an eigenvalue for $\mathbf{A}$ and $\mathbf{v} \in E_\lambda$ (the eigenspace for $\mathbf{A}$ associated with $\lambda$), then $\mathbf{Bv} \in E_\lambda$.

**(b)** Prove that if $\mathbf{A}$ has $n$ distinct eigenvalues, then $\mathbf{B}$ is diagonalizable.

**18.** This exercise concerns certain linear operators on $\mathcal{M}_{nn}$ defined as multiplication by a fixed matrix $\mathbf{A}$.

**(a)** Let $\mathbf{A}$ be a fixed $2 \times 2$ matrix with distinct eigenvalues $\lambda_1$ and $\lambda_2$. Show that the linear operator $L\colon \mathcal{M}_{22} \to \mathcal{M}_{22}$ given by $L(\mathbf{K}) = \mathbf{AK}$ is diagonalizable with eigenvalues $\lambda_1$ and $\lambda_2$, each having multiplicity 2. (Hint: Use eigenvectors for $\mathbf{A}$ to help create eigenvectors for $L$.)

**(b)** Generalize part (a) as follows: Let $\mathbf{A}$ be a fixed diagonalizable $n \times n$ matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_k$. Show that the linear operator $L\colon \mathcal{M}_{nn} \to \mathcal{M}_{nn}$ given by $L(\mathbf{K}) = \mathbf{AK}$ is diagonalizable with eigenvalues $\lambda_1, \ldots, \lambda_k$. In addition, show that, for each $i$, the geometric multiplicity of $\lambda_i$ for $L$ is $n$ times the geometric multiplicity of $\lambda_i$ for $\mathbf{A}$.

► **19.** Let $L\colon \mathcal{V} \to \mathcal{V}$ be a linear operator on a finite dimensional vector space $\mathcal{V}$. Suppose that $\lambda_1$ and $\lambda_2$ are distinct eigenvalues for $L$ and that $B_1$ and $B_2$ are bases for the eigenspaces $E_{\lambda_1}$ and $E_{\lambda_2}$ for $L$. Prove that $B_1 \cap B_2$ is empty. (This establishes the first half of Theorem 5.27.)

► **20.** Let $L\colon \mathcal{V} \to \mathcal{V}$ be a linear operator on a finite dimensional vector space $\mathcal{V}$. Suppose that $\lambda_1, \ldots, \lambda_n$ are distinct eigenvalues for $L$ and that $B_i = \{\mathbf{v}_{i1}, \ldots, \mathbf{v}_{ik_i}\}$ is a basis for the eigenspace $E_{\lambda_i}$, for $1 \leq i \leq n$. The goal of this exercise is to show that $B = \cup_{i=1}^{n} B_i$ is linearly independent. Suppose that $\boldsymbol{\Sigma}_{i=1}^{n}\boldsymbol{\Sigma}_{j=1}^{k_i} a_{ij}\mathbf{v}_{ij} = \mathbf{0}$.

**(a)** Let $\mathbf{u}_i = \boldsymbol{\Sigma}_{j=1}^{k_i} a_{ij}\mathbf{v}_{ij}$. Show that $\mathbf{u}_i \in E_{\lambda_i}$.

**(b)** Note that $\boldsymbol{\Sigma}_{i=1}^{n}\mathbf{u}_i = \mathbf{0}$. Use Theorem 5.25 to show that $\mathbf{u}_i = \mathbf{0}$, for $1 \leq i \leq n$.

**(c)** Conclude that $a_{ij} = 0$, for $1 \leq i \leq n$ and $1 \leq j \leq k_i$.

**(d)** Explain why parts (a) through (c) prove that $B$ is linearly independent. (This establishes the second half of Theorem 5.27.)

**21.** Verify that the Cayley-Hamilton Theorem holds for the matrix $\mathbf{A}$ for $L$ with respect to the standard basis from Example 3. (Be sure to show your work for finding $p_{\mathbf{A}}(x)$.)

► **22.** Explain the flaw in the following "proof" for the Cayley-Hamilton Theorem:
Proof: Plugging in $\mathbf{A}$ for $x$ in $p_{\mathbf{A}}(x) = |x\mathbf{I} - \mathbf{A}|$, we get $p_{\mathbf{A}}(\mathbf{A}) = |\mathbf{AI} - \mathbf{A}| = |\mathbf{O}| = 0$.

★ **23.** True or False:

**(a)** If $L\colon \mathcal{V} \to \mathcal{V}$ is a linear operator and $\lambda$ is an eigenvalue for $L$, then $E_\lambda = \{\lambda L(\mathbf{v}) \mid \mathbf{v} \in \mathcal{V}\}$.

**(b)** If $L$ is a linear operator on a nontrivial finite dimensional vector space $\mathcal{V}$ and $\mathbf{A}$ is a matrix for $L$ with respect to some ordered basis for $\mathcal{V}$, then $p_L(x) = p_{\mathbf{A}}(x)$.

**(c)** If $\dim(\mathcal{V}) = 5$, a linear operator $L$ on $\mathcal{V}$ is diagonalizable when $L$ has 5 linearly independent eigenvectors.

**(d)** Eigenvectors for a given linear operator $L$ are linearly independent if and only if they correspond to distinct eigenvalues of $L$.

**(e)** If $L$ is a linear operator on a nontrivial finite dimensional vector space, then the union of bases for distinct eigenspaces for $L$ is a linearly independent set.

**(f)** If $L\colon \mathbb{R}^6 \to \mathbb{R}^6$ is a diagonalizable linear operator, then the union of bases for all the distinct eigenspaces of $L$ is actually a basis for $\mathbb{R}^6$.

**(g)** If $L$ is a diagonalizable linear operator on a finite dimensional vector space $\mathcal{V}$, the Generalized Diagonalization Method produces a basis $B$ for $\mathcal{V}$ so that the matrix for $L$ with respect to $B$ is diagonal.

**(h)** If $L$ is a linear operator on a finite dimensional vector space $\mathcal{V}$ and $\lambda$ is an eigenvalue for $L$, then the algebraic multiplicity of $\lambda$ is never greater than the geometric multiplicity of $\lambda$.

(i) If $\dim(\mathcal{V}) = 7$ and $L: \mathcal{V} \to \mathcal{V}$ is a linear operator, then $L$ is diagonalizable whenever the sum of the algebraic multiplicities of all the eigenvalues equals 7.

(j) If $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$, then $(1\mathbf{I}_2 - \mathbf{A})(4\mathbf{I}_2 - \mathbf{A}) = \mathbf{O}_2$.

## Review Exercises for Chapter 5

1. Which of the following are linear transformations? Prove your answer is correct.
   ★ (a) $f: \mathbb{R}^3 \to \mathbb{R}^3$ given by $f([x, y, z]) = [4z - y, 3x + 1, 2y + 5x]$
   (b) $g: \mathbb{R}^3 \to \mathbb{R}^3$ given by $g([x, y, z]) = [2\sin(x), \cos(y) - 1, xyz]$
   (c) $h: \mathcal{P}_3 \to \mathcal{M}_{23}$ given by $h(\mathbf{p}(x)) = \begin{bmatrix} \mathbf{p}(1) & \mathbf{p}(2) & \mathbf{p}(3) \\ \mathbf{p}'(1) & \mathbf{p}'(2) & \mathbf{p}'(3) \end{bmatrix}$, where $\mathbf{p}'(x)$ is the derivative of $\mathbf{p}(x)$.
2. Find the image of $[-3, 2]$ under the linear transformation that rotates every vector $[x, y]$ in $\mathbb{R}^2$ counterclockwise about the origin through $\theta = 4\pi/3$. Use 3 decimal places in your answer.
★ 3. Let $\mathbf{B}$ and $\mathbf{C}$ be fixed $n \times n$ matrices, with $\mathbf{B}$ nonsingular. Show that the mapping $f: \mathcal{M}_{nn} \to \mathcal{M}_{nn}$ given by $f(\mathbf{A}) = \mathbf{CAB}^{-1}$ is a linear operator.
★ 4. Suppose $L: \mathbb{R}^3 \to \mathbb{R}^3$ is a linear operator and $L([1, 0, 0]) = [-3, 2, 4]$, $L([0, 1, 0]) = [5, -1, 3]$, and $L([0, 0, 1]) = [-4, 0, -2]$. Find $L([6, 2, -7])$. Find $L([x, y, z])$, for any $[x, y, z] \in \mathbb{R}^3$.
5. Let $L_1: \mathcal{V} \to \mathcal{W}$ and $L_2: \mathcal{W} \to \mathcal{X}$ be linear transformations. Suppose $\mathcal{V}'$ is a subspace of $\mathcal{V}$ and $\mathcal{X}'$ is a subspace of $\mathcal{X}$.
   (a) Prove that $(L_2 \circ L_1)(\mathcal{V}')$ is a subspace of $\mathcal{X}$.
   ★ (b) Prove that $(L_2 \circ L_1)^{-1}(\mathcal{X}')$ is a subspace of $\mathcal{V}$.
6. For each of the following linear transformations $L: \mathcal{V} \to \mathcal{W}$, find the matrix $\mathbf{A}_{BC}$ for $L$ with respect to the given bases $B$ for $\mathcal{V}$ and $C$ for $\mathcal{W}$ using the method of Theorem 5.5:
   ★ (a) $L: \mathbb{R}^3 \to \mathbb{R}^2$ given by $L([x, y, z]) = [3y + 2z, 4x - 7y]$ with $B = ([-5, -3, -2], [3, 0, 1], [5, 2, 2])$ and $C = ([4, 3], [-3, -2])$
   (b) $L: \mathcal{M}_{22} \to \mathcal{P}_2$ given by $L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (2b - c + 3a)x^2 + (4d - a)x + (2b - 3d + 5c)$ with $B = \left(\begin{bmatrix} 3 & 4 \\ -7 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix}, \begin{bmatrix} -2 & -2 \\ 3 & -2 \end{bmatrix}, \begin{bmatrix} -6 & -3 \\ 3 & -4 \end{bmatrix}\right)$ and $C = (2x^2 - 2x + 1, 7x^2 - 6x + 2, -6x^2 + x + 5)$
7. In each case, find the matrix $\mathbf{A}_{DE}$ for the given linear transformation $L: \mathcal{V} \to \mathcal{W}$ with respect to the given bases $D$ and $E$ by first finding the matrix for $L$ with respect to the standard bases $B$ and $C$ for $\mathcal{V}$ and $\mathcal{W}$, respectively, and then using the method of Theorem 5.6.
   (a) $L: \mathbb{R}^4 \to \mathbb{R}^3$ given by $L([a, b, c, d]) = [2a - b + 3c, 3d + a - 4b, a + 2d]$ with $D = ([3, -2, -2, 3], [2, -1, -1, 2], [-4, 7, 3, 0], [-2, 2, 1, 1])$ and $E = ([-2, -1, 2], [3, -2, 2], [-6, 2, -1])$
   ★ (b) $L: \mathcal{P}_2 \to \mathcal{M}_{22}$ given by $L(ax^2 + bx + c) = \begin{bmatrix} 6a - b - c & 3b + 2c \\ 2a - 4c & a - 5b + c \end{bmatrix}$ with
   $D = (-5x^2 + 2x + 5, 3x^2 - x - 1, -2x^2 + x + 3)$ and $E = \left(\begin{bmatrix} 3 & 2 \\ 2 & 5 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}, \begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix}\right)$
8. Find the matrix with respect to the standard bases for the composition $L_3 \circ L_2 \circ L_1: \mathbb{R}^3 \to \mathbb{R}^3$ if $L_1$ is a reflection through the $xz$-plane, $L_2$ is a rotation about the $z$-axis of 90°, and $L_3$ is a projection onto the $xy$-plane.
9. Suppose $L: \mathbb{R}^3 \to \mathbb{R}^3$ is the linear operator whose matrix with respect to the standard basis $B$ for $\mathbb{R}^3$ is

$$\mathbf{A}_{BB} = \frac{1}{41}\begin{bmatrix} 23 & 36 & 12 \\ 36 & -31 & -24 \\ -12 & 24 & 49 \end{bmatrix}.$$

   ★ (a) Find $p_{\mathbf{A}_{BB}}(x)$. (Be sure to incorporate $\frac{1}{41}$ correctly into your calculations.)
   (b) Find all eigenvalues for $\mathbf{A}_{BB}$ and fundamental eigenvectors for each eigenvalue.
   (c) Combine the fundamental eigenvectors to form a basis $C$ for $\mathbb{R}^3$.
   (d) Find $\mathbf{A}_{CC}$. (Hint: Use $\mathbf{A}_{BB}$ and the transition matrix $\mathbf{P}$ from $C$ to $B$.)
   (e) Use $\mathbf{A}_{CC}$ to give a geometric description of the operator $L$, as was done in Example 6 of Section 5.2.

**10.** Consider the linear transformation $L: \mathbb{R}^4 \rightarrow \mathbb{R}^4$ given by

$$L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} 3 & 1 & -3 & 5 \\ 2 & 1 & -1 & 2 \\ 2 & 3 & 5 & -6 \\ 1 & 4 & 10 & -13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

★ **(a)** Find a basis for $\ker(L)$ and a basis for $\text{range}(L)$.
**(b)** Verify that $\dim(\ker(L)) + \dim(\text{range}(L)) = \dim(\mathbb{R}^4)$.
**(c)** Is $[-18, 26, -4, 2]$ in $\ker(L)$? Is $[-18, 26, -6, 2]$ in $\ker(L)$? Why or why not?
**(d)** Is $[8, 3, -11, -23]$ in $\text{range}(L)$? Why or why not?

**11.** For $L: \mathcal{M}_{32} \rightarrow \mathcal{P}_3$ given by

$$\begin{aligned} L\left(\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\right) = {} & (a + 2b - c - d - 4e)x^3 \\ & + (2a + 4b + 5c + 12d + 13e + 7f)x^2 \\ & + (-2a - 4b - 2d + 2e - 2f)x \\ & + (a + 2b - 2c - 3d - 7e - f), \end{aligned}$$

find a basis for $\ker(L)$ and a basis for $\text{range}(L)$, and verify that $\dim(\ker(L)) + \dim(\text{range}(L)) = \dim(\mathcal{M}_{32})$.

★ **12.** Let $\mathcal{V}, \mathcal{W}, \mathcal{X}$ be finite dimensional vector spaces, and let $L_1: \mathcal{V} \rightarrow \mathcal{W}$ and $L_2: \mathcal{W} \rightarrow \mathcal{X}$ be linear transformations.
**(a)** Show that $\dim(\ker(L_1)) \leq \dim(\ker(L_2 \circ L_1))$.
**(b)** Find linear transformations $L_1, L_2: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ for which $\dim(\ker(L_1)) < \dim(\ker(L_2 \circ L_1))$.

**13.** Let $\mathbf{A}$ be a fixed $m \times n$ matrix, and let $L: \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $M: \mathbb{R}^m \rightarrow \mathbb{R}^n$ be given by $L(\mathbf{X}) = \mathbf{AX}$ and $M(\mathbf{Y}) = \mathbf{A}^T\mathbf{Y}$.
**(a)** Prove that $\dim(\ker(L)) - \dim(\ker(M)) = n - m$.
**(b)** Prove that if $L$ is onto then $M$ is one-to-one.
**(c)** Is the converse to part (b) true? Prove or disprove.

**14.** Consider $L: \mathcal{P}_3 \rightarrow \mathcal{M}_{22}$ given by $L\left(ax^3 + bx^2 + cx + d\right) = \begin{bmatrix} a - d & 2b \\ b & c + d \end{bmatrix}$.
**(a)** Without using row reduction, determine whether $L$ is one-to-one and whether $L$ is onto.
**(b)** What is $\dim(\ker(L))$? What is $\dim(\text{range}(L))$?

**15.** In each case, use row reduction to determine whether the given linear transformation $L$ is one-to-one and whether $L$ is onto, and find $\dim(\ker(L))$ and $\dim(\text{range}(L))$.
★ **(a)** $L: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2 & -1 & 1 \\ -11 & 3 & -3 \\ 13 & -8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$
**(b)** $L: \mathbb{R}^4 \rightarrow \mathcal{P}_2$ having matrix $\begin{bmatrix} 5 & 5 & 3 & 7 \\ 3 & 3 & 2 & 4 \\ 2 & 1 & -1 & 0 \end{bmatrix}$ with respect to the standard bases for $\mathbb{R}^4$ and $\mathcal{P}_2$

**16.** This exercise concerns properties of certain linear transformations.
**(a)** Prove that any linear transformation from $\mathcal{P}_3$ to $\mathbb{R}^3$ is not one-to-one.
**(b)** Prove that any linear transformation from $\mathcal{P}_2$ to $\mathcal{M}_{22}$ is not onto.

**17.** Let $L: \mathcal{V} \rightarrow \mathcal{W}$ be a linear transformation.
**(a)** Suppose $L$ is one-to-one and $L(\mathbf{v}_1) = cL(\mathbf{v}_2)$ with $c \neq 0$ for some vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}$. Show that $\mathbf{v}_1 = c\mathbf{v}_2$, and explain why this result agrees with part (1) of Theorem 5.14.
**(b)** Suppose $L$ is onto and $\mathbf{w} \in \mathcal{W}$. Let $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}$ and suppose that $L(a\mathbf{v}_1 + b\mathbf{v}_2) \neq \mathbf{w}$ for all $a, b \in \mathbb{R}$. Prove that $\{\mathbf{v}_1, \mathbf{v}_2\}$ does not span $\mathcal{V}$. (Hint: Use part (2) of Theorem 5.14.)

**18.** Consider the linear operators $L_1$ and $L_2$ on $\mathbb{R}^4$ having the given matrices with respect to the standard basis:

$$L_1\colon \begin{bmatrix} 3 & 6 & 1 & 1 \\ 5 & 2 & -2 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & -1 & -2 & -1 \end{bmatrix}, \quad L_2\colon \begin{bmatrix} 9 & 8 & 5 & 4 \\ 9 & 13 & 4 & 7 \\ 5 & 9 & 2 & 5 \\ -5 & -2 & -2 & 0 \end{bmatrix}.$$

★ **(a)** Show that $L_1$ and $L_2$ are isomorphisms.
★ **(b)** Calculate the matrices for $L_2 \circ L_1$, $L_1^{-1}$, and $L_2^{-1}$.
**(c)** Verify that the matrix for $(L_2 \circ L_1)^{-1}$ agrees with the matrix for $L_1^{-1} \circ L_2^{-1}$.

**19.** This exercise involves isomorphisms related to a particular type of shear.
**(a)** Show that a shear in the $z$-direction with factor $k$ (see Table 5.1 in Section 5.2) is an isomorphism from $\mathbb{R}^3$ to itself.
**(b)** Calculate the inverse isomorphism of the shear in part (a). Describe the effect of the inverse geometrically.

**20.** Consider the subspace $\mathcal{W}$ of $\mathcal{M}_{nn}$ consisting of all $n \times n$ symmetric matrices, and let $\mathbf{B}$ be a fixed $n \times n$ nonsingular matrix.
**(a)** Prove that if $\mathbf{A} \in \mathcal{W}$, then $\mathbf{B}^T\mathbf{A}\mathbf{B} \in \mathcal{W}$.
**(b)** Prove that the linear operator on $\mathcal{W}$ given by $L(\mathbf{A}) = \mathbf{B}^T\mathbf{A}\mathbf{B}$ is an isomorphism. (Hint: Show either that $L$ is one-to-one or that $L$ is onto, and then use Corollary 5.13.)

**21.** Consider the subspace $\mathcal{W}$ of $\mathcal{P}_4$ consisting of all polynomials of the form $ax^4 + bx^3 + cx^2$, for some $a, b, c \in \mathbb{R}$.
★ **(a)** Prove that $L\colon \mathcal{W} \to \mathcal{P}_3$ given by $L(\mathbf{p}) = \mathbf{p}' + \mathbf{p}''$ is one-to-one.
**(b)** Is $L$ an isomorphism from $\mathcal{W}$ to $\mathcal{P}_3$?
**(c)** Find a vector in $\mathcal{P}_3$ that is not in range($L$).

**22.** For each of the following, let $S$ be the standard basis, and let $L$ be the indicated linear operator with respect to $S$.
**(i)** Find all eigenvalues for $L$, and a basis of fundamental eigenvectors for each eigenspace.
**(ii)** Compare the geometric and algebraic multiplicities of each eigenvalue, and determine whether $L$ is diagonalizable.
**(iii)** If $L$ is diagonalizable, find an ordered basis $B$ of eigenvectors for $L$, a diagonal matrix $\mathbf{D}$ that is the matrix for $L$ with respect to the basis $B$, and the transition matrix $\mathbf{P}$ from $B$ to $S$.

★ **(a)** $L\colon \mathbb{R}^3 \to \mathbb{R}^3$ having matrix $\begin{bmatrix} -9 & 18 & -16 \\ 32 & -63 & 56 \\ 44 & -84 & 75 \end{bmatrix}$

**(b)** $L\colon \mathbb{R}^3 \to \mathbb{R}^3$ having matrix $\begin{bmatrix} 4 & -2 & 4 \\ 32 & -18 & 34 \\ 14 & -8 & 15 \end{bmatrix}$

★ **(c)** $L\colon \mathbb{R}^3 \to \mathbb{R}^3$ having matrix $\begin{bmatrix} -97 & 20 & 12 \\ -300 & 63 & 36 \\ -300 & 60 & 39 \end{bmatrix}$

**(d)** $L\colon \mathcal{P}_3 \to \mathcal{P}_3$ given by $L(\mathbf{p}(x)) = \mathbf{p}(x+1) - \mathbf{p}(x-1)$

**23.** Show that $L\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by reflection through the plane determined by the linearly independent vectors $[a, b, c]$ and $[d, e, f]$ is diagonalizable, and state a diagonal matrix $\mathbf{D}$ that is similar to the matrix for $L$ with respect to the standard basis for $\mathbb{R}^3$, as well as a basis of eigenvectors for $L$. (Hint: Use Exercise 8(a) in Section 3.1 to find a vector that is orthogonal to both $[a, b, c]$ and $[d, e, f]$. Then, follow the strategy outlined in the last paragraph of Example 6 in Section 5.2.)

**24.** Verify that the Cayley-Hamilton Theorem holds for the matrix in Example 9 of Section 5.6. (Hint: See part (a) of Exercise 3 in Section 5.6.)

**25.** Let $\mathcal{V}$ and $\mathcal{W}$ be finite dimensional vector spaces, let $\mathcal{Y}$ be a subspace of $\mathcal{W}$, and let $L\colon \mathcal{V} \to \mathcal{W}$ be a linear transformation. Define $L_1\colon L^{-1}(\mathcal{Y}) \to \mathcal{W}$ as the restriction of $L$ to $L^{-1}(\mathcal{Y})$. That is, for every $\mathbf{v} \in L^{-1}(\mathcal{Y})$, $L_1(\mathbf{v}) = L(\mathbf{v})$.
**(a)** Prove that $\dim(L^{-1}(\mathcal{Y})) = \dim(\text{range}(L_1)) + \dim(\ker(L_1))$.
**(b)** Show that $\text{range}(L_1) \subseteq \mathcal{Y}$.

**(c)** Prove that $\ker(L) \subseteq L^{-1}(\mathcal{Y})$.
**(d)** Prove that $\ker(L_1) = \ker(L)$. (Note: From part (c), both are subsets of $L^{-1}(\mathcal{Y})$.)
**(e)** Use parts (a), (b), and (d) to show that

$$\dim(L^{-1}(\mathcal{Y})) \leq \dim(\ker(L)) + \dim(\mathcal{Y}).$$

**26.** Suppose $L_1$ and $L_2$ are linear operators on $\mathbb{R}^n$.
**(a)** Show that $\ker(L_2 \circ L_1) = L_1^{-1}(\ker(L_2))$.
**(b)** Use part (a) of this exercise and part (e) of Exercise 25 to prove that[8]

$$\dim(\ker(L_2 \circ L_1)) \leq \dim(\ker(L_1)) + \dim(\ker(L_2)).$$

**(c)** Suppose $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices with $\text{rank}(\mathbf{A}) = k$ and $\text{rank}(\mathbf{B}) = m$. Use part (b) to prove that $\text{rank}(\mathbf{BA}) \geq k + m - n$. (Hint: Let $L_1$ and $L_2$ be the linear operators on $\mathbb{R}^n$ given by $L_1(\mathbf{v}) = \mathbf{Av}$ and $L_2(\mathbf{v}) = \mathbf{Bv}$.)

**27.** Suppose $\mathbf{A}$ is a diagonalizable $n \times n$ matrix with $n$ distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. Suppose that $p_{\mathbf{A}}(x) = (x - \lambda_1)q(x)$, for some polynomial $q(x)$.
**(a)** Prove that every nonzero column of $q(\mathbf{A})$ is an eigenvector for $\mathbf{A}$ corresponding to $\lambda_1$. (Hint: Use the Cayley-Hamilton Theorem on $\mathbf{A}$ to show that $\mathbf{A}q(\mathbf{A}) = \lambda_1 q(\mathbf{A})$.)
**(b)** Show that $q(\mathbf{A}) = (\mathbf{A} - \lambda_2\mathbf{I}) \cdots (\mathbf{A} - \lambda_n\mathbf{I})$.
**(c)** For $1 \leq i \leq n$, let $L_i$ be the linear operator on $\mathbb{R}^n$ given by $L_i(\mathbf{v}) = (\mathbf{A} - \lambda_i\mathbf{I})\mathbf{v}$. Show that, for each $i$, $\dim(\ker(L_i)) = 1$. (Hint: Show that $\ker(L_i)$ is the eigenspace $E_{\lambda_i}$.)
**(d)** Let $L$ be the linear operator on $\mathbb{R}^n$ given by $L(\mathbf{v}) = q(\mathbf{A})\mathbf{v}$. Use parts (b) and (c), and the footnote to part (b) of Exercise 26 to prove that $\dim(\ker(L)) \leq n - 1$.
**(e)** Use part (d) to show that $q(\mathbf{A})$ is not the zero matrix, and therefore has at least one nonzero column.
**(f)** Use parts (a), (c), and (e) to explain why each nonzero column of $q(\mathbf{A})$, by itself, is a basis for the eigenspace $E_{\lambda_1}$.

**28.** Let $\mathbf{A} = \begin{bmatrix} -8 & -9 & -5 \\ 2 & 3 & 1 \\ 12 & 12 & 8 \end{bmatrix}$. Use the result in part (f) of Exercise 27 (rather than row reduction) to find a basis for each eigenspace of $\mathbf{A}$, and then construct the matrix $\mathbf{P}$ such that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$ is a diagonal matrix.

★ **29.** True or False:
**(a)** There is only one linear transformation $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $L(\mathbf{i}) = \mathbf{j}$ and $L(\mathbf{j}) = \mathbf{i}$.
**(b)** There is only one linear transformation $L\colon \mathbb{R}^3 \to \mathbb{R}^2$ such that $L(\mathbf{i}) = \mathbf{j}$ and $L(\mathbf{j}) = \mathbf{i}$.
**(c)** The matrix with respect to the standard basis for a clockwise rotation about the origin through an angle of 45° in $\mathbb{R}^2$ is $\left(\frac{\sqrt{2}}{2}\right)\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.
**(d)** If $L\colon \mathcal{V} \to \mathcal{W}$ is a linear transformation and $\mathcal{Y}$ is a subspace of $\mathcal{V}$, then $T\colon \mathcal{Y} \to \mathcal{W}$ given by $T(\mathbf{y}) = L(\mathbf{y})$ for all $\mathbf{y} \in \mathcal{Y}$ is a linear transformation.
**(e)** Let $\mathbf{B}$ be a fixed $m \times n$ matrix, and let $L\colon \mathbb{R}^n \to \mathbb{R}^m$ be given by $L(\mathbf{X}) = \mathbf{BX}$. Then $\mathbf{B}$ is the matrix for $L$ with respect to the standard bases for $\mathbb{R}^n$ and $\mathbb{R}^m$.
**(f)** If $L\colon \mathcal{V} \to \mathcal{W}$ is a linear transformation between nontrivial finite dimensional vector spaces, and if $\mathbf{A}_{BC}$ and $\mathbf{A}_{DE}$ are matrices for $L$ with respect to the bases $B$ and $D$ for $\mathcal{V}$ and $C$ and $E$ for $\mathcal{W}$, then $\mathbf{A}_{BC}$ and $\mathbf{A}_{DE}$ are similar matrices.
**(g)** There is a linear operator $L$ on $\mathbb{R}^5$ such that $\ker(L) = \text{range}(L)$.
**(h)** If $\mathbf{A}$ is an $m \times n$ matrix and $L\colon \mathbb{R}^n \to \mathbb{R}^m$ is the linear transformation $L(\mathbf{X}) = \mathbf{AX}$, then $\dim(\text{range}(L)) = \dim(\text{row space of } \mathbf{A})$.
**(i)** If $\mathbf{A}$ is an $m \times n$ matrix and $L\colon \mathbb{R}^n \to \mathbb{R}^m$ is the linear transformation $L(\mathbf{X}) = \mathbf{AX}$, then $\text{range}(L) = \text{column space of } \mathbf{A}$.
**(j)** The Dimension Theorem shows that if $L\colon \mathcal{V} \to \mathcal{W}$ is a linear transformation and $\mathcal{V}$ is finite dimensional, then $\mathcal{W}$ is also finite dimensional.
**(k)** A linear transformation $L\colon \mathcal{V} \to \mathcal{W}$ is one-to-one if and only if $\ker(L)$ is empty.

[8] More generally, it can be proved by induction that, if $L_1, \ldots, L_k$ are linear operators on $\mathbb{R}^n$, then $\dim(\ker(L_k \circ \cdots \circ L_1)) \leq \sum_{i=1}^{k} \dim(\ker(L_i))$.

**(l)** If $\mathcal{V}$ is a finite dimensional vector space, then a linear transformation $L: \mathcal{V} \to \mathcal{W}$ is one-to-one if and only if $\dim(\text{range}(L)) = \dim(\mathcal{V})$.

**(m)** Every linear transformation is either one-to-one or onto or both.

**(n)** If $\mathcal{V}$ is a finite dimensional vector space and $L: \mathcal{V} \to \mathcal{W}$ is an onto linear transformation, then $\mathcal{W}$ is finite dimensional.

**(o)** If $L: \mathcal{V} \to \mathcal{W}$ is a one-to-one linear transformation and $T$ is a linearly independent subset of $\mathcal{V}$, then $L(T)$ is a linearly independent subset of $\mathcal{W}$.

**(p)** If $L: \mathcal{V} \to \mathcal{W}$ is a one-to-one and onto function between vector spaces, then $L$ is a linear transformation.

**(q)** If $\mathcal{V}$ and $\mathcal{W}$ are nontrivial finite dimensional vector spaces, and $L: \mathcal{V} \to \mathcal{W}$ is a linear transformation, then $L$ is an isomorphism if and only if the matrix for $L$ with respect to some bases for $\mathcal{V}$ and $\mathcal{W}$ is square.

**(r)** If $L: \mathbb{R}^3 \to \mathbb{R}^3$ is the isomorphism that reflects vectors through the plane $2x + 3y - z = 0$, then $L^{-1} = L$.

**(s)** Every nontrivial vector space $\mathcal{V}$ is isomorphic to $\mathbb{R}^n$ for some $n$.

**(t)** If $\mathcal{W}_1$ and $\mathcal{W}_2$ are two planes through the origin in $\mathbb{R}^3$, then there exists an isomorphism $L: \mathcal{W}_1 \to \mathcal{W}_2$.

**(u)** If $L: \mathcal{V} \to \mathcal{W}$ is a linear transformation and $M: \mathcal{W} \to \mathcal{X}$ is an isomorphism, then $\ker(M \circ L) = \ker(L)$.

**(v)** If $L: \mathcal{V} \to \mathcal{W}$ is a linear transformation and $M: \mathcal{W} \to \mathcal{X}$ is an isomorphism, then $\text{range}(M \circ L) = \text{range}(L)$.

**(w)** If $\mathbf{A}$ is an $n \times n$ matrix and $\lambda$ is an eigenvalue for $\mathbf{A}$, then $E_\lambda$ is the kernel of the linear operator on $\mathbb{R}^n$ whose matrix with respect to the standard basis is $(\lambda \mathbf{I}_n - \mathbf{A})$.

**(x)** If $L$ is a linear operator on an $n$-dimensional vector space $\mathcal{V}$ such that $L$ has $n$ distinct eigenvalues, then the algebraic multiplicity for each eigenvalue is 1.

**(y)** If $L$ is a linear operator on a nontrivial finite dimensional vector space $\mathcal{V}$, $x^2$ is a factor of $p_L(x)$, and $\dim(E_0) = 1$, then $L$ is not diagonalizable.

**(z)** If $L$ is a linear operator on a nontrivial finite dimensional vector space $\mathcal{V}$ and $B_1, \ldots, B_k$ are bases for $k$ different eigenspaces for $L$, then $B_1 \cup B_2 \cup \cdots \cup B_k$ is a basis for a subspace of $\mathcal{V}$.

Chapter 6

# Orthogonality

**Geometry Is Never Pointless**

Linear algebra exists at the crossroads between algebra and geometry. Yet, in our study of abstract vector spaces in Chapters 4 and 5, we often concentrated on the algebra at the expense of the geometry. But the underlying geometry is important also. For example, in our study of general vector spaces and linear transformations, we avoided the dot product because it is not defined in every vector space. Therefore, we could not discuss lengths of vectors or angles in general vector spaces as we can in $\mathbb{R}^n$. But geometric properties of $\mathbb{R}^n$, such as orthogonality, are derived from the length and dot product of vectors.

However, in this chapter, we restrict our attention to $\mathbb{R}^n$ and present some additional structures and properties related to the dot product, re-examining them in the light of the more general vector space properties of Chapters 4 and 5. In particular, we examine special bases for $\mathbb{R}^n$ whose vectors are mutually orthogonal, and introduce orthogonal complements of subspaces of $\mathbb{R}^n$. Finally, we use orthogonality to diagonalize any symmetric matrix. These new levels of understanding will place additional applications within our reach.

## 6.1 Orthogonal Bases and the Gram-Schmidt Process

In this section, we investigate orthogonality of vectors in more detail. Our main goal is the Gram-Schmidt Process, a method for constructing a basis of mutually orthogonal vectors for any nontrivial subspace of $\mathbb{R}^n$.

### Orthogonal and Orthonormal Vectors

**Definition** Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a subset of $k$ distinct vectors of $\mathbb{R}^n$. Then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is an **orthogonal set of vectors** if and only if the dot product of any two distinct vectors in this set is zero—that is, if and only if $\mathbf{v}_i \cdot \mathbf{v}_j = 0$, for $1 \le i, j \le k, i \ne j$. Also, $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is an **orthonormal set of vectors** if and only if it is an orthogonal set and all its vectors are unit vectors (that is, $\|\mathbf{v}_i\| = 1$, for $1 \le i \le k$).

In particular, any set containing a single vector is orthogonal, and any set containing a single unit vector is orthonormal.

**Example 1**

In $\mathbb{R}^3$, $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is an orthogonal set because $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$. In fact, this is an orthonormal set, since we also have $\|\mathbf{i}\| = \|\mathbf{j}\| = \|\mathbf{k}\| = 1$.

In $\mathbb{R}^4$, $\{[1, 0, -1, 0], [3, 0, 3, 0]\}$ is an orthogonal set because $[1, 0, -1, 0] \cdot [3, 0, 3, 0] = 0$. If we normalize each vector (that is, divide each of these vectors by its length), we create the orthonormal set of vectors

$$\left\{\left[\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, 0\right], \left[\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0\right]\right\}.$$

The next theorem is proved in the same manner as Result 7 in Section 1.3.

**Theorem 6.1** *Let $T = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be an orthogonal set of nonzero vectors in $\mathbb{R}^n$. Then $T$ is a linearly independent set.*

Notice that the orthogonal sets in Example 1 are indeed linearly independent.

### Orthogonal and Orthonormal Bases

Theorem 6.1 assures us that any orthogonal set of nonzero vectors in $\mathbb{R}^n$ is linearly independent, so any such set forms a basis for some subspace of $\mathbb{R}^n$.

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**Definition** A basis $B$ for a subspace $\mathcal{W}$ of $\mathbb{R}^n$ is an **orthogonal basis** for $\mathcal{W}$ if and only if $B$ is an orthogonal set. Similarly, a basis $B$ for $\mathcal{W}$ is an **orthonormal basis** for $\mathcal{W}$ if and only if $B$ is an orthonormal set.

The following corollary follows immediately from Theorem 6.1:

**Corollary 6.2** *If $B$ is an orthogonal set of $n$ nonzero vectors in $\mathbb{R}^n$, then $B$ is an orthogonal basis for $\mathbb{R}^n$. Similarly, if $B$ is an orthonormal set of $n$ vectors in $\mathbb{R}^n$, then $B$ is an orthonormal basis for $\mathbb{R}^n$.*

### Example 2

Consider the following subset of $\mathbb{R}^3$: $\{[1,0,-1],[-1,4,-1],[2,1,2]\}$. Because every pair of distinct vectors in this set is orthogonal (verify!), this is an orthogonal set. By Corollary 6.2, this is also an orthogonal basis for $\mathbb{R}^3$. Normalizing each vector, we obtain the following orthonormal basis for $\mathbb{R}^3$:

$$\left\{\left[\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}}\right],\left[-\frac{1}{3\sqrt{2}},\frac{4}{3\sqrt{2}},-\frac{1}{3\sqrt{2}}\right],\left[\frac{2}{3},\frac{1}{3},\frac{2}{3}\right]\right\}.$$

One of the advantages of using an orthogonal or orthonormal basis is that it is easy to coordinatize vectors with respect to that basis.

**Theorem 6.3** *If $B=(\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_k)$ is a nonempty ordered orthogonal basis for a subspace $\mathcal{W}$ of $\mathbb{R}^n$, and if $\mathbf{v}$ is any vector in $\mathcal{W}$, then*

$$[\mathbf{v}]_B=\left[\frac{(\mathbf{v}\cdot\mathbf{v}_1)}{(\mathbf{v}_1\cdot\mathbf{v}_1)},\frac{(\mathbf{v}\cdot\mathbf{v}_2)}{(\mathbf{v}_2\cdot\mathbf{v}_2)},\ldots,\frac{(\mathbf{v}\cdot\mathbf{v}_k)}{(\mathbf{v}_k\cdot\mathbf{v}_k)}\right]=\left[\frac{(\mathbf{v}\cdot\mathbf{v}_1)}{||\mathbf{v}_1||^2},\frac{(\mathbf{v}\cdot\mathbf{v}_2)}{||\mathbf{v}_2||^2},\ldots,\frac{(\mathbf{v}\cdot\mathbf{v}_k)}{||\mathbf{v}_k||^2}\right].$$

*In particular, if $B$ is an ordered orthonormal basis for $\mathcal{W}$, then $[\mathbf{v}]_B=[\mathbf{v}\cdot\mathbf{v}_1,\mathbf{v}\cdot\mathbf{v}_2,\ldots,\mathbf{v}\cdot\mathbf{v}_k]$.*

*Proof.* Suppose that $[\mathbf{v}]_B=[a_1,a_2,\ldots,a_k]$, where $a_1,a_2,\ldots,a_k\in\mathbb{R}$. We must show that $a_i=(\mathbf{v}\cdot\mathbf{v}_i)/(\mathbf{v}_i\cdot\mathbf{v}_i)$, for $1\le i\le k$. Now, $\mathbf{v}=a_1\mathbf{v}_1+a_2\mathbf{v}_2+\cdots+a_k\mathbf{v}_k$. Hence,

$$\begin{aligned}\mathbf{v}\cdot\mathbf{v}_i&=(a_1\mathbf{v}_1+a_2\mathbf{v}_2+\cdots+a_i\mathbf{v}_i+\cdots+a_k\mathbf{v}_k)\cdot\mathbf{v}_i\\&=a_1(\mathbf{v}_1\cdot\mathbf{v}_i)+a_2(\mathbf{v}_2\cdot\mathbf{v}_i)+\cdots+a_i(\mathbf{v}_i\cdot\mathbf{v}_i)+\cdots+a_k(\mathbf{v}_k\cdot\mathbf{v}_i)\\&=a_1(0)+a_2(0)+\cdots+a_i(\mathbf{v}_i\cdot\mathbf{v}_i)+\cdots+a_k(0)\qquad\text{because } B \text{ is orthogonal}\\&=a_i(\mathbf{v}_i\cdot\mathbf{v}_i).\end{aligned}$$

Thus, $a_i=(\mathbf{v}\cdot\mathbf{v}_i)/(\mathbf{v}_i\cdot\mathbf{v}_i)=(\mathbf{v}\cdot\mathbf{v}_i)/||\mathbf{v}_i||^2$. In the special case when $B$ is orthonormal, $\|\mathbf{v}_i\|=1$, and so $a_i=\mathbf{v}\cdot\mathbf{v}_i$. □

### Example 3

Consider the ordered orthogonal basis $B=(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3)$ for $\mathbb{R}^3$ from Example 2, where $\mathbf{v}_1=[1,0,-1]$, $\mathbf{v}_2=[-1,4,-1]$, and $\mathbf{v}_3=[2,1,2]$. Let $\mathbf{v}=[-1,5,3]$. We will use Theorem 6.3 to find $[\mathbf{v}]_B$.

Now, $\mathbf{v}\cdot\mathbf{v}_1=-4$, $\mathbf{v}\cdot\mathbf{v}_2=18$, and $\mathbf{v}\cdot\mathbf{v}_3=9$. Also, $\mathbf{v}_1\cdot\mathbf{v}_1=2$, $\mathbf{v}_2\cdot\mathbf{v}_2=18$, and $\mathbf{v}_3\cdot\mathbf{v}_3=9$. Hence,

$$[\mathbf{v}]_B=\left[\frac{(\mathbf{v}\cdot\mathbf{v}_1)}{(\mathbf{v}_1\cdot\mathbf{v}_1)},\frac{(\mathbf{v}\cdot\mathbf{v}_2)}{(\mathbf{v}_2\cdot\mathbf{v}_2)},\ldots,\frac{(\mathbf{v}\cdot\mathbf{v}_k)}{(\mathbf{v}_k\cdot\mathbf{v}_k)}\right]=\left[\frac{-4}{2},\frac{18}{18},\frac{9}{9}\right]=[-2,1,1].$$

Similarly, suppose $C=(\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3)$ is the ordered orthonormal basis for $\mathbb{R}^3$ from Example 2; that is, $\mathbf{w}_1=\left[\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}}\right]$, $\mathbf{w}_2=\left[-\frac{1}{3\sqrt{2}},\frac{4}{3\sqrt{2}},-\frac{1}{3\sqrt{2}}\right]$, and $\mathbf{w}_3=\left[\frac{2}{3},\frac{1}{3},\frac{2}{3}\right]$. Again, let $\mathbf{v}=[-1,5,3]$. Then $\mathbf{v}\cdot\mathbf{w}_1=-2\sqrt{2}$, $\mathbf{v}\cdot\mathbf{w}_2=3\sqrt{2}$, and $\mathbf{v}\cdot\mathbf{w}_3=3$. By Theorem 6.3, $[\mathbf{v}]_C=\left[-2\sqrt{2},3\sqrt{2},3\right]$. These coordinates can be verified by checking that

$$[-1,5,3]=-2\sqrt{2}\left[\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}}\right]+3\sqrt{2}\left[-\frac{1}{3\sqrt{2}},\frac{4}{3\sqrt{2}},-\frac{1}{3\sqrt{2}}\right]+3\left[\frac{2}{3},\frac{1}{3},\frac{2}{3}\right].$$

## The Gram-Schmidt Process: Finding an Orthogonal Basis for a Subspace of $\mathbb{R}^n$

We have just seen that it is convenient to work with an orthogonal basis whenever possible. Now, suppose $\mathcal{W}$ is a subspace of $\mathbb{R}^n$ with basis $B = \{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$. There is a straightforward way to replace $B$ with an orthogonal basis for $\mathcal{W}$. This is known as the **Gram-Schmidt Process**.

**Method for Finding an Orthogonal Basis for the Span of a Linearly Independent Subset (Gram-Schmidt Process)**

Let $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ be a linearly independent subset of $\mathbb{R}^n$. We create a new set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ of vectors as follows:

$$\begin{aligned}
&\text{Let} \quad \mathbf{v}_1 = \mathbf{w}_1.\\
&\text{Let} \quad \mathbf{v}_2 = \mathbf{w}_2 - \left(\frac{\mathbf{w}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1.\\
&\text{Let} \quad \mathbf{v}_3 = \mathbf{w}_3 - \left(\frac{\mathbf{w}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{w}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2.\\
&\qquad\qquad \vdots\\
&\text{Let} \quad \mathbf{v}_k = \mathbf{w}_k - \left(\frac{\mathbf{w}_k \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{w}_k \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 - \cdots - \left(\frac{\mathbf{w}_k \cdot \mathbf{v}_{k-1}}{\mathbf{v}_{k-1} \cdot \mathbf{v}_{k-1}}\right)\mathbf{v}_{k-1}.
\end{aligned}$$

Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an orthogonal basis for $\text{span}(\{\mathbf{w}_1, \ldots, \mathbf{w}_k\})$.

The justification that the Gram-Schmidt Process is valid is given in the following theorem:

**Theorem 6.4** *Let $B = \{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ be a basis for a subspace $\mathcal{W}$ of $\mathbb{R}^n$. Then the set $T = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ obtained by applying the Gram-Schmidt Process to $B$ is an orthogonal basis for $\mathcal{W}$.*

*Hence, any nontrivial subspace $\mathcal{W}$ of $\mathbb{R}^n$ has an orthogonal basis.*

*Proof.* Let $\mathcal{W}$, $B$, and $T$ be as given in the statement of the theorem. To prove that $T$ is an orthogonal basis for $\mathcal{W}$, we must prove three statements about $T$.

**(1)** $T \subseteq \mathcal{W}$.
**(2)** Every vector in $T$ is nonzero.
**(3)** $T$ is an orthogonal set.

Theorem 6.1 will then show that $T$ is linearly independent, and since $|T| = k = \dim(\mathcal{W})$, $T$ is an orthogonal basis for $\mathcal{W}$.

We proceed by induction, proving for each $i$, $1 \le i \le k$, that

**(1′)** $\{\mathbf{v}_1, \ldots, \mathbf{v}_i\} \subseteq \text{span}(\{\mathbf{w}_1, \ldots, \mathbf{w}_i\})$,
**(2′)** $\mathbf{v}_i \neq \mathbf{0}$,
**(3′)** $\{\mathbf{v}_1, \ldots, \mathbf{v}_i\}$ is an orthogonal set.

Obviously, once the induction is complete, properties (1), (2), and (3) will be established for $T$, and the theorem will be proved.

**Base Step**: Since $\mathbf{v}_1 = \mathbf{w}_1 \in B$, it is clear that $\{\mathbf{v}_1\} \subseteq \text{span}(\{\mathbf{w}_1\})$, $\mathbf{v}_1 \neq \mathbf{0}$, and $\{\mathbf{v}_1\}$ is an orthogonal set.

**Inductive Step**: The inductive hypothesis asserts that $\{\mathbf{v}_1, \ldots, \mathbf{v}_i\}$ is an orthogonal subset of $\text{span}(\{\mathbf{w}_1, \ldots, \mathbf{w}_i\})$ consisting of nonzero vectors. We need to prove (1′), (2′), and (3′) for $\{\mathbf{v}_1, \ldots, \mathbf{v}_{i+1}\}$.

To establish (1′), we only need to prove that $\mathbf{v}_{i+1} \in \text{span}(\{\mathbf{w}_1, \ldots, \mathbf{w}_{i+1}\})$, since we already know from the inductive hypothesis that $\{\mathbf{v}_1, \ldots, \mathbf{v}_i\}$ is a subset of $\text{span}(\{\mathbf{w}_1, \ldots, \mathbf{w}_i\})$, and hence of $\text{span}(\{\mathbf{w}_1, \ldots, \mathbf{w}_{i+1}\})$. But by definition, $\mathbf{v}_{i+1}$ is a linear combination of $\mathbf{w}_{i+1}$ and $\mathbf{v}_1, \ldots, \mathbf{v}_i$, all of which are in $\text{span}(\{\mathbf{w}_1, \ldots, \mathbf{w}_{i+1}\})$. Hence, $\mathbf{v}_{i+1} \in \text{span}(\{\mathbf{w}_1, \ldots, \mathbf{w}_{i+1}\})$.

To prove (2′), we assume that $\mathbf{v}_{i+1} = \mathbf{0}$ and produce a contradiction. Now, from the definition of $\mathbf{v}_{i+1}$, if $\mathbf{v}_{i+1} = \mathbf{0}$ we have

$$\mathbf{w}_{i+1} = \left(\frac{\mathbf{w}_{i+1} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 + \left(\frac{\mathbf{w}_{i+1} \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 + \cdots + \left(\frac{\mathbf{w}_{i+1} \cdot \mathbf{v}_i}{\mathbf{v}_i \cdot \mathbf{v}_i}\right)\mathbf{v}_i.$$

But then $\mathbf{w}_{i+1} \in \text{span}(\{\mathbf{v}_1, \ldots, \mathbf{v}_i\}) \subseteq \text{span}(\{\mathbf{w}_1, \ldots, \mathbf{w}_i\})$, from the inductive hypothesis. This result contradicts the fact that $B$ is a linearly independent set. Therefore, $\mathbf{v}_{i+1} \neq \mathbf{0}$.

Finally, we need to prove (3′). By the inductive hypothesis, $\{\mathbf{v}_1, \ldots, \mathbf{v}_i\}$ is an orthogonal set. Hence, we only need to show that $\mathbf{v}_{i+1}$ is orthogonal to each of $\mathbf{v}_1, \ldots, \mathbf{v}_i$. Now,

$$\mathbf{v}_{i+1} = \mathbf{w}_{i+1} - \left(\frac{\mathbf{w}_{i+1} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{w}_{i+1} \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 - \cdots - \left(\frac{\mathbf{w}_{i+1} \cdot \mathbf{v}_i}{\mathbf{v}_i \cdot \mathbf{v}_i}\right)\mathbf{v}_i.$$

Notice that

$$\begin{aligned}
\mathbf{v}_{i+1} \cdot \mathbf{v}_1 &= \mathbf{w}_{i+1} \cdot \mathbf{v}_1 - \left(\frac{\mathbf{w}_{i+1} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)(\mathbf{v}_1 \cdot \mathbf{v}_1) - \left(\frac{\mathbf{w}_{i+1} \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)(\mathbf{v}_2 \cdot \mathbf{v}_1) \\
&\quad - \cdots - \left(\frac{\mathbf{w}_{i+1} \cdot \mathbf{v}_i}{\mathbf{v}_i \cdot \mathbf{v}_i}\right)(\mathbf{v}_i \cdot \mathbf{v}_1) \\
&= \mathbf{w}_{i+1} \cdot \mathbf{v}_1 - \left(\frac{\mathbf{w}_{i+1} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)(\mathbf{v}_1 \cdot \mathbf{v}_1) - \left(\frac{\mathbf{w}_{i+1} \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)(0) \\
&\quad - \cdots - \left(\frac{\mathbf{w}_{i+1} \cdot \mathbf{v}_i}{\mathbf{v}_i \cdot \mathbf{v}_i}\right)(0) \qquad \text{inductive hypothesis} \\
&= \mathbf{w}_{i+1} \cdot \mathbf{v}_1 - \mathbf{w}_{i+1} \cdot \mathbf{v}_1 = 0.
\end{aligned}$$

Similar arguments show that $\mathbf{v}_{i+1} \cdot \mathbf{v}_2 = \mathbf{v}_{i+1} \cdot \mathbf{v}_3 = \cdots = \mathbf{v}_{i+1} \cdot \mathbf{v}_i = 0$. Hence, $\{\mathbf{v}_1, \ldots, \mathbf{v}_{i+1}\}$ is an orthogonal set. This finishes the Inductive Step, completing the proof of the theorem. □

Once we have an orthogonal basis for a subspace $\mathcal{W}$ of $\mathbb{R}^n$, we can easily convert it to an orthonormal basis for $\mathcal{W}$ by normalizing each vector. Also, a little thought will convince you that if any of the newly created vectors $\mathbf{v}_i$ in the Gram-Schmidt Process is replaced with a nonzero scalar multiple of itself, the proof of Theorem 6.4 still holds. Hence, in applying the Gram-Schmidt Process, we can often replace the $\mathbf{v}_i$'s we create with appropriate multiples to avoid fractions. The next example illustrates these techniques.

**Example 4**

You can verify that $B = \{[2,1,0,-1], [1,0,2,-1], [0,-2,1,0]\}$ is a linearly independent set in $\mathbb{R}^4$. Let $\mathcal{W} = \text{span}(B)$. Now, $B$ is not an orthogonal basis for $\mathcal{W}$, but we will apply the Gram-Schmidt Process to replace $B$ with an orthogonal basis. Let $\mathbf{w}_1 = [2,1,0,-1]$, $\mathbf{w}_2 = [1,0,2,-1]$, and $\mathbf{w}_3 = [0,-2,1,0]$. Beginning the Gram-Schmidt Process, we obtain $\mathbf{v}_1 = \mathbf{w}_1 = [2,1,0,-1]$ and

$$\begin{aligned}
\mathbf{v}_2 &= \mathbf{w}_2 - \left(\frac{\mathbf{w}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 \\
&= [1,0,2,-1] - \left(\frac{[1,0,2,-1] \cdot [2,1,0,-1]}{[2,1,0,-1] \cdot [2,1,0,-1]}\right)[2,1,0,-1] \\
&= [1,0,2,-1] - \left(\frac{3}{6}\right)[2,1,0,-1] = \left[0, -\frac{1}{2}, 2, -\frac{1}{2}\right].
\end{aligned}$$

To avoid fractions, we replace this vector with an appropriate scalar multiple. Multiplying by 2, we get $\mathbf{v}_2 = [0,-1,4,-1]$. Notice that $\mathbf{v}_2$ is orthogonal to $\mathbf{v}_1$. Finally,

$$\begin{aligned}
\mathbf{v}_3 &= \mathbf{w}_3 - \left(\frac{\mathbf{w}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{w}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 \\
&= [0,-2,1,0] - \left(\frac{[0,-2,1,0] \cdot [2,1,0,-1]}{[2,1,0,-1] \cdot [2,1,0,-1]}\right)[2,1,0,-1] - \left(\frac{[0,-2,1,0] \cdot [0,-1,4,-1]}{[0,-1,4,-1] \cdot [0,-1,4,-1]}\right)[0,-1,4,-1] \\
&= [0,-2,1,0] - \left(\frac{-2}{6}\right)[2,1,0,-1] - \left(\frac{6}{18}\right)[0,-1,4,-1] = \left[\frac{2}{3}, -\frac{4}{3}, -\frac{1}{3}, 0\right].
\end{aligned}$$

To avoid fractions, we multiply this vector by 3, yielding $\mathbf{v}_3 = [2,-4,-1,0]$. Notice that $\mathbf{v}_3$ is orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$. Hence,

$$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{[2,1,0,-1], [0,-1,4,-1], [2,-4,-1,0]\}$$

is an orthogonal basis for $\mathcal{W}$. To find an orthonormal basis for $\mathcal{W}$, we normalize $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ to obtain

$$\left\{\left[\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, 0, -\frac{1}{\sqrt{6}}\right], \left[0, -\frac{1}{3\sqrt{2}}, \frac{4}{3\sqrt{2}}, -\frac{1}{3\sqrt{2}}\right], \left[\frac{2}{\sqrt{21}}, -\frac{4}{\sqrt{21},}, -\frac{1}{\sqrt{21}}, 0\right]\right\}.$$

Suppose $T = \{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ is an orthogonal set of nonzero vectors in a subspace $\mathcal{W}$ of $\mathbb{R}^n$. By Theorem 6.1, $T$ is linearly independent. Hence, by Theorem 4.15, we can enlarge $T$ to an ordered basis $(\mathbf{w}_1, \ldots, \mathbf{w}_k, \mathbf{w}_{k+1}, \ldots, \mathbf{w}_l)$ for $\mathcal{W}$. Applying the Gram-Schmidt Process to this enlarged basis gives an ordered orthogonal basis $B = (\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_l)$ for $\mathcal{W}$. However, because $(\mathbf{w}_1, \ldots, \mathbf{w}_k)$ is already orthogonal, the first $k$ vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_k$, created by the Gram-Schmidt Process will be equal to $\mathbf{w}_1, \ldots, \mathbf{w}_k$, respectively (why?). Hence, $B$ is an ordered orthogonal basis for $\mathcal{W}$ that contains $T$. Similarly, if the original set $T = \{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ is *orthonormal*, $T$ can be enlarged to an *orthonormal* basis for $\mathcal{W}$ (why?). These remarks prove the following:

**Theorem 6.5** *Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$. Then any orthogonal set of nonzero vectors in $\mathcal{W}$ is contained in (can be enlarged to) an orthogonal basis for $\mathcal{W}$. Similarly, any orthonormal set of vectors in $\mathcal{W}$ is contained in an orthonormal basis for $\mathcal{W}$.*

**Example 5**

We will find an orthogonal basis $B$ for $\mathbb{R}^4$ that contains the orthogonal set $T = \{[2, 1, 0, -1], [0, -1, 4, -1], [2, -4, -1, 0]\}$ from Example 4. To enlarge $T$ to a basis for $\mathbb{R}^4$, we row reduce

$$\begin{bmatrix} 2 & 0 & 2 & 1 & 0 & 0 & 0 \\ 1 & -1 & -4 & 0 & 1 & 0 & 0 \\ 0 & 4 & -1 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \text{ to obtain } \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{1}{18} & -\frac{2}{9} & -\frac{17}{18} \\ 0 & 1 & 0 & 0 & -\frac{1}{18} & \frac{2}{9} & -\frac{1}{18} \\ 0 & 0 & 1 & 0 & -\frac{2}{9} & -\frac{1}{9} & -\frac{2}{9} \\ 0 & 0 & 0 & 1 & \frac{1}{3} & \frac{2}{3} & \frac{7}{3} \end{bmatrix}.$$

Hence, the Enlarging Method from Section 4.6 shows that $\{[2, 1, 0, -1], [0, -1, 4, -1], [2, -4, -1, 0], [1, 0, 0, 0]\}$ is a basis for $\mathbb{R}^4$. Now, we use the Gram-Schmidt Process to convert this basis to an orthogonal basis for $\mathbb{R}^4$.

Let $\mathbf{w}_1 = [2, 1, 0, -1]$, $\mathbf{w}_2 = [0, -1, 4, -1]$, $\mathbf{w}_3 = [2, -4, -1, 0]$, and $\mathbf{w}_4 = [1, 0, 0, 0]$. The first few steps of the Gram-Schmidt Process give $\mathbf{v}_1 = \mathbf{w}_1$, $\mathbf{v}_2 = \mathbf{w}_2$, and $\mathbf{v}_3 = \mathbf{w}_3$ (why?). Finally,

$$\begin{aligned} \mathbf{v}_4 &= \mathbf{w}_4 - \left(\frac{\mathbf{w}_4 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{w}_4 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 - \left(\frac{\mathbf{w}_4 \cdot \mathbf{v}_3}{\mathbf{v}_3 \cdot \mathbf{v}_3}\right)\mathbf{v}_3 \\ &= [1, 0, 0, 0] - \left(\frac{[1, 0, 0, 0] \cdot [2, 1, 0, -1]}{[2, 1, 0, -1] \cdot [2, 1, 0, -1]}\right)[2, 1, 0, -1] \\ &\quad - \left(\frac{[1, 0, 0, 0] \cdot [0, -1, 4, -1]}{[0, -1, 4, -1] \cdot [0, -1, 4, -1]}\right)[0, -1, 4, -1] \\ &\quad - \left(\frac{[1, 0, 0, 0] \cdot [2, -4, -1, 0]}{[2, -4, -1, 0] \cdot [2, -4, -1, 0]}\right)[2, -4, -1, 0] \\ &= [1, 0, 0, 0] - \frac{1}{3}[2, 1, 0, -1] - \frac{2}{21}[2, -4, -1, 0] = \left[\frac{1}{7}, \frac{1}{21}, \frac{2}{21}, \frac{1}{3}\right]. \end{aligned}$$

To avoid fractions, we multiply this vector by 21 to obtain $\mathbf{v}_4 = [3, 1, 2, 7]$. Notice that $\mathbf{v}_4$ is orthogonal to $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Hence, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is an orthogonal basis for $\mathbb{R}^4$ containing $T$.

Two useful results concerning orthonormal sets of vectors are Parseval's Equality and Bessel's Inequality, which are stated in Exercises 9 and 10.

## Orthogonal Matrices

**Definition** A nonsingular (square) matrix $\mathbf{A}$ is **orthogonal** if and only if $\mathbf{A}^T = \mathbf{A}^{-1}$.

The next theorem lists some fundamental properties of orthogonal matrices.

**Theorem 6.6** *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are orthogonal matrices of the same size, then*
(1) $|\mathbf{A}| = \pm 1$,
(2) $\mathbf{A}^T = \mathbf{A}^{-1}$ *is orthogonal, and*
(3) $\mathbf{AB}$ *is orthogonal.*

Part (1) of Theorem 6.6 is obviously true because if $\mathbf{A}$ is orthogonal, then $\left|\mathbf{A}^T\right| = \left|\mathbf{A}^{-1}\right| \Longrightarrow |\mathbf{A}| = 1/|\mathbf{A}| \Longrightarrow |\mathbf{A}|^2 = 1 \Longrightarrow |\mathbf{A}| = \pm 1$. (Beware! The converse is not true—if $|\mathbf{A}| = \pm 1$, then $\mathbf{A}$ is not necessarily orthogonal.) The proofs of parts (2) and (3) are straightforward, and you are asked to provide them in Exercise 11.

The next theorem characterizes all orthogonal matrices.

**Theorem 6.7** *Let* $\mathbf{A}$ *be an* $n \times n$ *matrix. Then* $\mathbf{A}$ *is orthogonal*
(1) *if and only if the rows of* $\mathbf{A}$ *form an orthonormal basis for* $\mathbb{R}^n$
(2) *if and only if the columns of* $\mathbf{A}$ *form an orthonormal basis for* $\mathbb{R}^n$.

Theorem 6.7 suggests that it is probably more appropriate to refer to orthogonal matrices as "orthonormal matrices." Unfortunately, the term *orthogonal matrix* has become traditional usage in linear algebra.

*Proof.* (Abridged) We prove half of part (1) and leave the rest as Exercise 17.

Suppose that $\mathbf{A}$ is an orthogonal $n \times n$ matrix. Then we have $\mathbf{AA}^T = \mathbf{I}_n$ (why?). Hence, for $1 \le i, j \le n$ with $i \ne j$, we have [$i$th row of $\mathbf{A}$] $\cdot$ [$j$th column of $\mathbf{A}^T$] $= 0$. Therefore, [$i$th row of $\mathbf{A}$] $\cdot$ [$j$th row of $\mathbf{A}$] $= 0$, which shows that distinct rows of $\mathbf{A}$ are orthogonal. Again, because $\mathbf{AA}^T = \mathbf{I}_n$, for each $i$, $1 \le i \le n$, we have [$i$th row of $\mathbf{A}$] $\cdot$ [$i$th column of $\mathbf{A}^T$] $= 1$. But then [$i$th row of $\mathbf{A}$] $\cdot$ [$i$th row of $\mathbf{A}$] $= 1$, which shows that each row of $\mathbf{A}$ is a unit vector. Thus, the $n$ rows of $\mathbf{A}$ form an orthonormal set, and hence, an orthonormal basis for $\mathbb{R}^n$. □

$\mathbf{I}_n$ is obviously an orthogonal matrix, for any $n \ge 1$. In the next example, we show how Theorem 6.7 can be used to find other orthogonal matrices.

## Example 6

Consider the orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for $\mathbb{R}^3$ from Example 2, where

$$\mathbf{v}_1 = \left[\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right], \mathbf{v}_2 = \left[-\frac{1}{3\sqrt{2}}, \frac{4}{3\sqrt{2}}, -\frac{1}{3\sqrt{2}}\right], \text{ and } \left[\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right].$$

By parts (1) and (2) of Theorem 6.7, respectively,

$$\mathbf{A} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{3\sqrt{2}} & \frac{4}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} \quad \text{and} \quad \mathbf{A}^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} & \frac{2}{3} \\ 0 & \frac{4}{3\sqrt{2}} & \frac{1}{3} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} & \frac{2}{3} \end{bmatrix}$$

are both orthogonal matrices. You can verify that both $\mathbf{A}$ and $\mathbf{A}^T$ are orthogonal by checking that $\mathbf{AA}^T = \mathbf{I}_3$.

One important example of orthogonal matrices is given in the next theorem.

**Theorem 6.8** *Let* $B$ *and* $C$ *be ordered orthonormal bases for* $\mathbb{R}^n$*. Then the transition matrix from* $B$ *to* $C$ *is an orthogonal matrix.*

In Exercise 20 you are asked to prove a partial converse as well as a generalization of Theorem 6.8.

*Proof.* Let $S$ be the standard basis for $\mathbb{R}^n$. The matrix $\mathbf{P}$ whose columns are the vectors in $B$ is the transition matrix from $B$ to $S$. Similarly, the matrix $\mathbf{Q}$, whose columns are the vectors in $C$, is the transition matrix from $C$ to $S$. Both $\mathbf{P}$ and $\mathbf{Q}$ are orthogonal matrices by part (2) of Theorem 6.7. But then $\mathbf{Q}^{-1}$ is also orthogonal. Now, by Theorems 4.18 and 4.19, $\mathbf{Q}^{-1}\mathbf{P}$ is the transition matrix from $B$ to $C$ (see Fig. 6.1), and $\mathbf{Q}^{-1}\mathbf{P}$ is orthogonal by part (3) of Theorem 6.6. □

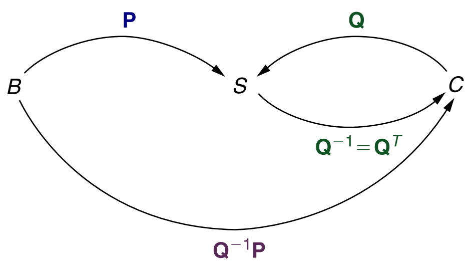


**FIGURE 6.1** Visualizing $\mathbf{Q}^{-1}\mathbf{P}$ as the transition matrix from $B$ to $C$

**Example 7**

Consider the following ordered orthonormal bases for $\mathbb{R}^2$:

$$B = \left( \left[ \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right], \left[ \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right] \right) \text{ and } C = \left( \left[ \frac{\sqrt{3}}{2}, \frac{1}{2} \right], \left[ -\frac{1}{2}, \frac{\sqrt{3}}{2} \right] \right).$$

By Theorem 6.8, the transition matrix from $B$ to $C$ is orthogonal. To verify this, we can use Theorem 6.3 to obtain

$$\left[ \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right]_C = \left[ \frac{\sqrt{6}+\sqrt{2}}{4}, \frac{\sqrt{6}-\sqrt{2}}{4} \right] \quad \text{and}$$

$$\left[ \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right]_C = \left[ \frac{\sqrt{6}-\sqrt{2}}{4}, \frac{-\sqrt{6}-\sqrt{2}}{4} \right].$$

Hence, the transition matrix from $B$ to $C$ is

$$\mathbf{A} = \frac{1}{4} \begin{bmatrix} \sqrt{6}+\sqrt{2} & \sqrt{6}-\sqrt{2} \\ \sqrt{6}-\sqrt{2} & -\sqrt{6}-\sqrt{2} \end{bmatrix}.$$

Because $\mathbf{A}\mathbf{A}^T = \mathbf{I}_2$ (verify!), $\mathbf{A}$ is an orthogonal matrix.

The final theorem of this section can be used to prove that multiplying two $n$-vectors by an orthogonal matrix does not change the angle between them (see Exercise 18).

**Theorem 6.9** *Let* $\mathbf{A}$ *be an* $n \times n$ *orthogonal matrix, and let* $\mathbf{v}$ *and* $\mathbf{w}$ *be vectors in* $\mathbb{R}^n$. *Then* $\mathbf{v} \cdot \mathbf{w} = \mathbf{A}\mathbf{v} \cdot \mathbf{A}\mathbf{w}$.

*Proof.* Notice that the dot product $\mathbf{x} \cdot \mathbf{y}$ of two column vectors $\mathbf{x}$ and $\mathbf{y}$ can be written in matrix multiplication form as $\mathbf{x}^T\mathbf{y}$. Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, and let $\mathbf{A}$ be an $n \times n$ orthogonal matrix. Then

$$\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T\mathbf{w} = \mathbf{v}^T\mathbf{I}_n\mathbf{w} = \mathbf{v}^T\mathbf{A}^T\mathbf{A}\mathbf{w} = (\mathbf{A}\mathbf{v})^T\mathbf{A}\mathbf{w} = \mathbf{A}\mathbf{v} \cdot \mathbf{A}\mathbf{w}.$$

□

## New Vocabulary

Bessel's Inequality
Gram-Schmidt Process
ordered orthogonal basis
ordered orthonormal basis
orthogonal basis
orthogonal matrix
orthogonal set (of vectors)
orthonormal basis
orthonormal set (of vectors)
Parseval's Equality

## Highlights

- If $T = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then $T$ is linearly independent.
- If $T$ is an orthogonal set of $n$ nonzero vectors in $\mathbb{R}^n$, then $T$ is an orthogonal basis for $\mathbb{R}^n$.

- If $T$ is an orthonormal set of $n$ vectors in $\mathbb{R}^n$, then $T$ is an orthonormal basis for $\mathbb{R}^n$.
- An orthogonal set $T$ of nonzero vectors can be converted into an orthonormal set by normalizing each vector in $T$.
- If a vector $\mathbf{v}$ is contained in a subspace $\mathcal{W}$ of $\mathbb{R}^n$, and $B = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$ is an ordered orthogonal basis for $\mathcal{W}$, then $[\mathbf{v}]_B = \left[\frac{(\mathbf{v}\cdot\mathbf{v}_1)}{\|\mathbf{v}_1\|^2}, \frac{(\mathbf{v}\cdot\mathbf{v}_2)}{\|\mathbf{v}_2\|^2}, \ldots, \frac{(\mathbf{v}\cdot\mathbf{v}_k)}{\|\mathbf{v}_k\|^2}\right]$. If $B$ is orthonormal, then $[\mathbf{v}]_B = [\mathbf{v}\cdot\mathbf{v}_1, \mathbf{v}\cdot\mathbf{v}_2, \ldots, \mathbf{v}\cdot\mathbf{v}_k]$.
- If $T$ is a set of $k$ linearly independent vectors in $\mathbb{R}^n$, applying the Gram-Schmidt Process to $T$ results in an orthogonal basis of $k$ vectors for span($T$).
- If $\mathcal{W}$ is a nontrivial subspace of $\mathbb{R}^n$, then $\mathcal{W}$ has an orthogonal (and hence, an orthonormal) basis.
- Any orthogonal set $T$ of nonzero vectors in a subspace $\mathcal{W}$ of $\mathbb{R}^n$ can be enlarged to an orthogonal basis for $\mathcal{W}$.
- A (nonsingular) matrix $\mathbf{A}$ is orthogonal if and only if $\mathbf{A}^T = \mathbf{A}^{-1}$.
- If a matrix $\mathbf{A}$ is orthogonal, then $|\mathbf{A}| = \pm 1$.
- An $n \times n$ matrix $\mathbf{A}$ is orthogonal if and only if the rows of $\mathbf{A}$ form an orthonormal basis for $\mathbb{R}^n$.
- An $n \times n$ matrix $\mathbf{A}$ is orthogonal if and only if the columns of $\mathbf{A}$ form an orthonormal basis for $\mathbb{R}^n$.
- If $B$ and $C$ are ordered orthonormal bases for $\mathbb{R}^n$, then the transition matrix from $B$ to $C$ is an orthogonal matrix.
- If $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, and $\mathbf{A}$ is an $n \times n$ orthogonal matrix, then $\mathbf{v}\cdot\mathbf{w} = \mathbf{A}\mathbf{v}\cdot\mathbf{A}\mathbf{w}$. Also, the angle between $\mathbf{v}$ and $\mathbf{w}$ equals the angle between $\mathbf{A}\mathbf{v}$ and $\mathbf{A}\mathbf{w}$.

## Exercises for Section 6.1

1. Which of the following sets of vectors are orthogonal? Which are orthonormal?
   - ★ (a) $\{[3,-2],[4,6]\}$
   - (b) $\left\{\left[\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right], \left[\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right]\right\}$
   - ★ (c) $\left\{\left[\frac{3}{\sqrt{13}}, -\frac{2}{\sqrt{13}}\right], \left[\frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}}\right]\right\}$
   - (d) $\left\{\left[3, \frac{13}{3}, 14\right], [5,3,-2], [4,-6,1]\right\}$
   - (e) $\left\{\left[\frac{3}{7}, -\frac{2}{7}, \frac{6}{7}\right]\right\}$
   - ★ (f) $\{[2,-3,1,2], [-1,2,8,0], [6,-1,1,-8]\}$
   - (g) $\left\{\left[\frac{1}{4}, -\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{1}{4}\right], \left[\frac{1}{6}, \frac{1}{2}, -\frac{2}{3}, \frac{1}{2}, \frac{1}{6}\right]\right\}$
2. Which of the following matrices are orthogonal?
   - ★ (a) $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$
   - (b) $\begin{bmatrix} \frac{1}{3} & 0 & \frac{2}{\sqrt{5}} \\ -\frac{2}{3} & \frac{\sqrt{2}}{2} & \frac{1}{\sqrt{5}} \\ \frac{2}{3} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix}$
   - ★ (c) $\begin{bmatrix} 3 & 0 & 10 \\ -1 & 3 & 3 \\ 3 & 1 & -9 \end{bmatrix}$
   - (d) $\begin{bmatrix} \frac{2}{15} & \frac{1}{3} & -\frac{14}{15} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{11}{15} & \frac{2}{3} & \frac{2}{15} \end{bmatrix}$
   - ★ (e) $\begin{bmatrix} \frac{2}{3} & \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & -\frac{2}{3} \\ 0 & \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \end{bmatrix}$
3. In each case, verify that the given ordered basis $B$ is orthonormal. Then, for the given $\mathbf{v}$, find $[\mathbf{v}]_B$, using the method of Theorem 6.3.
   - ★ (a) $\mathbf{v} = [-2, 3]$, $B = \left(\left[-\frac{\sqrt{3}}{2}, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{\sqrt{3}}{2}\right]\right)$
   - (b) $\mathbf{v} = [5, -2, 3]$, $B = \left(\left[\frac{3}{7}, \frac{2}{7}, -\frac{6}{7}\right], \left[\frac{6}{7}, -\frac{3}{7}, \frac{2}{7}\right], \left[\frac{2}{7}, \frac{6}{7}, \frac{3}{7}\right]\right)$
   - ★ (c) $\mathbf{v} = [8, 4, -3, 5]$, $B = \left(\left[\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right], \left[\frac{3}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, -\frac{1}{2\sqrt{3}}, -\frac{1}{2\sqrt{3}}\right], \left[0, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right], \left[0, 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]\right)$
4. Each of the following represents a basis for a subspace of $\mathbb{R}^n$, for some $n$. Use the Gram-Schmidt Process to find an orthogonal basis for the subspace.
   - ★ (a) $\{[5,-1,2], [2,-1,-4]\}$ in $\mathbb{R}^3$
   - (b) $\{[1,3,-1,2], [2,-5,10,-11]\}$ in $\mathbb{R}^4$
   - ★ (c) $\{[2,1,0,-1], [1,1,1,-1], [1,-2,1,1]\}$ in $\mathbb{R}^4$
   - (d) $\{[2,4,1,-3], [9,11,10,16], [5,-11,11,-1]\}$ in $\mathbb{R}^4$
   - (e) $\{[1,-1,2,-2], [1,5,0,8], [-10,2,-4,-15], [-13,3,4,-34]\}$ in $\mathbb{R}^4$

**5.** Enlarge each of the following orthogonal sets to an orthogonal basis for $\mathbb{R}^n$. (Avoid fractions by using appropriate scalar multiples.)

★ **(a)** $\{[2, 2, -3]\}$

**(b)** $\{[5, -2, 1]\}$

★ **(c)** $\{[1, -3, 1], [2, 5, 13]\}$

**(d)** $\{[4, 7, -1], [2, -1, 1]\}$

★ **(e)** $\{[2, 1, -2, 1]\}$

**(f)** $\{[1, 1, 1, 2], [2, -1, 1, -1]\}$

**6.** Let $\mathcal{W} = \{[a, b, c, d, e] \mid a + b + c + d + e = 0\}$, a subspace of $\mathbb{R}^5$. Let $T = \{[-2, -1, 4, -2, 1], [4, -3, 0, -2, 1]\}$, an orthogonal subset of $\mathcal{W}$. Enlarge $T$ to an orthogonal basis for $\mathcal{W}$. (Hint: Use the fact that $B = \{[1, -1, 0, 0, 0], [0, 1, -1, 0, 0], [0, 0, 1, -1, 0], [0, 0, 0, 1, -1]\}$ is a basis for $\mathcal{W}$.)

**7.** It can be shown (see Exercise 13 in Section 6.3) that the linear operator represented by a $3 \times 3$ orthogonal matrix with determinant 1 (with respect to the standard basis) always represents a rotation about some axis in $\mathbb{R}^3$ and that the axis of rotation is parallel to an eigenvector corresponding to the eigenvalue $\lambda = 1$. Verify that each of the following matrices is orthogonal with determinant 1, and thereby represents a rotation about an axis in $\mathbb{R}^3$. Solve in each case for a vector in the direction of the axis of rotation. (In part (f), round the coordinates of the axis of rotation to three places after the decimal point.)

★ **(a)** $\frac{1}{11}\begin{bmatrix} 2 & 6 & -9 \\ -9 & 6 & 2 \\ 6 & 7 & 6 \end{bmatrix}$

**(b)** $\frac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ 2 & 2 & 1 \\ -2 & 1 & 2 \end{bmatrix}$

★ **(c)** $\frac{1}{7}\begin{bmatrix} 6 & 2 & 3 \\ 3 & -6 & -2 \\ 2 & 3 & -6 \end{bmatrix}$

**(d)** $\frac{1}{17}\begin{bmatrix} 12 & -9 & 8 \\ 1 & 12 & 12 \\ -12 & -8 & 9 \end{bmatrix}$

**(e)** $\frac{1}{15}\begin{bmatrix} -11 & 10 & 2 \\ 10 & 10 & 5 \\ 2 & 5 & -14 \end{bmatrix}$

**(f)** $\frac{1}{\sqrt{66}}\begin{bmatrix} 4 & 7 & -1 \\ 2\sqrt{11} & -\sqrt{11} & \sqrt{11} \\ \sqrt{6} & -\sqrt{6} & -3\sqrt{6} \end{bmatrix}$

**8.** This exercise relates orthogonal sets and scalar multiplication.

**(a)** Show that if $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an orthogonal set in $\mathbb{R}^n$ and $c_1, \ldots, c_k$ are nonzero scalars, then $\{c_1\mathbf{v}_1, \ldots, c_k\mathbf{v}_k\}$ is also an orthogonal set.

★ **(b)** Is part (a) still true if *orthogonal* is replaced by *orthonormal* everywhere?

**9.** Suppose that $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is an orthonormal basis for $\mathbb{R}^n$.

**(a)** If $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, show that

$$\mathbf{v} \cdot \mathbf{w} = (\mathbf{v} \cdot \mathbf{u}_1)(\mathbf{w} \cdot \mathbf{u}_1) + (\mathbf{v} \cdot \mathbf{u}_2)(\mathbf{w} \cdot \mathbf{u}_2) + \cdots + (\mathbf{v} \cdot \mathbf{u}_n)(\mathbf{w} \cdot \mathbf{u}_n).$$

**(b)** If $\mathbf{v} \in \mathbb{R}^n$, use part (a) to prove **Parseval's Equality**,

$$\|\mathbf{v}\|^2 = (\mathbf{v} \cdot \mathbf{u}_1)^2 + (\mathbf{v} \cdot \mathbf{u}_2)^2 + \cdots + (\mathbf{v} \cdot \mathbf{u}_n)^2.$$

**10.** Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ be an orthonormal set of vectors in $\mathbb{R}^n$. For any vector $\mathbf{v} \in \mathbb{R}^n$, prove **Bessel's Inequality**,

$$(\mathbf{v} \cdot \mathbf{u}_1)^2 + \cdots + (\mathbf{v} \cdot \mathbf{u}_k)^2 \leq \|\mathbf{v}\|^2.$$

(Hint: Let $\mathcal{W}$ be the subspace spanned by $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$. Enlarge $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ to an orthonormal basis for $\mathbb{R}^n$. Then use Theorem 6.3.) (Bessel's Inequality is a generalization of Parseval's Equality, which appears in Exercise 9.)

▶ **11.** This exercise asks for proofs for parts of Theorem 6.6.

**(a)** Prove part (2) of Theorem 6.6.

**(b)** Prove part (3) of Theorem 6.6.

**12.** Let $\mathbf{A}$ be an $n \times n$ matrix with $\mathbf{A}^2 = \mathbf{I}_n$. Prove that $\mathbf{A}$ is symmetric if and only if $\mathbf{A}$ is orthogonal. (Note: The matrix in part (e) of Exercise 7 is an example of a symmetric orthogonal matrix.)

**13.** Show that if $n$ is odd and $\mathbf{A}$ is an orthogonal $n \times n$ matrix, then $\mathbf{A}$ is not skew-symmetric. (Hint: Suppose $\mathbf{A}$ is both orthogonal and skew-symmetric. Show that $\mathbf{A}^2 = -\mathbf{I}_n$, and then use determinants.)

**14.** If $\mathbf{A}$ is an $n \times n$ orthogonal matrix with $|\mathbf{A}| = -1$, show that $\mathbf{A} + \mathbf{I}_n$ has no inverse. (Hint: Show that $\mathbf{A} + \mathbf{I}_n = \mathbf{A}(\mathbf{A} + \mathbf{I}_n)^T$, and then use determinants.)

**15.** Suppose that $\mathbf{A}$ is a $3 \times 3$ upper triangular orthogonal matrix. Show that $\mathbf{A}$ is diagonal and that all main diagonal entries of $\mathbf{A}$ equal $\pm 1$. (Note: This result is true for any $n \times n$ upper triangular orthogonal matrix.)

**16.** This exercise relates unit vectors and orthogonal matrices.

   **(a)** If $\mathbf{u}$ is any unit vector in $\mathbb{R}^n$, explain why there exists an $n \times n$ orthogonal matrix with $\mathbf{u}$ as its first row. (Hint: Consider Theorem 6.5.)

   ★ **(b)** Find an orthogonal matrix whose first row is $\frac{1}{\sqrt{6}}[1, 2, 1]$.

▶ **17.** Finish the proof of Theorem 6.7.

**18.** Suppose that $\mathbf{A}$ is an $n \times n$ orthogonal matrix.

   **(a)** Prove that for every $\mathbf{v} \in \mathbb{R}^n$, $\|\mathbf{v}\| = \|\mathbf{Av}\|$.

   **(b)** Prove that for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, the angle between $\mathbf{v}$ and $\mathbf{w}$ equals the angle between $\mathbf{Av}$ and $\mathbf{Aw}$.

**19.** Let $B$ be an ordered orthonormal basis for a $k$-dimensional subspace $\mathcal{V}$ of $\mathbb{R}^n$. Prove that for all $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}$, $\mathbf{v}_1 \cdot \mathbf{v}_2 = [\mathbf{v}_1]_B \cdot [\mathbf{v}_2]_B$, where the first dot product takes place in $\mathbb{R}^n$ and the second takes place in $\mathbb{R}^k$. (Hint: Let $B = (\mathbf{b}_1, \ldots, \mathbf{b}_k)$, and express $\mathbf{v}_1$ and $\mathbf{v}_2$ as linear combinations of the vectors in $B$. Substitute these linear combinations in the left side of $\mathbf{v}_1 \cdot \mathbf{v}_2 = [\mathbf{v}_1]_B \cdot [\mathbf{v}_2]_B$ and simplify. Then use the same linear combinations to express $\mathbf{v}_1$ and $\mathbf{v}_2$ in $B$-coordinates to calculate the right side.)

**20.** Prove each of the following statements related to Theorem 6.8. (Hint: Use the result of Exercise 19 in proving parts (b) and (c).)

   **(a)** Let $B$ be an orthonormal basis for $\mathbb{R}^n$, $C$ be a basis for $\mathbb{R}^n$, and $\mathbf{P}$ be the transition matrix from $B$ to $C$. If $\mathbf{P}$ is an orthogonal matrix, then $C$ is an orthonormal basis for $\mathbb{R}^n$.

   **(b)** Let $\mathcal{V}$ be a subspace of $\mathbb{R}^n$, and let $B$ and $C$ be orthonormal bases for $\mathcal{V}$. Then the transition matrix from $B$ to $C$ is an orthogonal matrix.

   **(c)** Let $\mathcal{V}$ be a subspace of $\mathbb{R}^n$, $B$ be an orthonormal basis for $\mathcal{V}$, $C$ be a basis for $\mathcal{V}$, and $\mathbf{P}$ be the transition matrix from $B$ to $C$. If $\mathbf{P}$ is an orthogonal matrix, then $C$ is an orthonormal basis for $\mathcal{V}$.

**21.** If $\mathbf{A}$ is an $m \times n$ matrix and the columns of $\mathbf{A}$ form an orthonormal set in $\mathbb{R}^m$, prove that $\mathbf{A}^T\mathbf{A} = \mathbf{I}_n$.

★ **22.** True or False:

   **(a)** Any subset of $\mathbb{R}^n$ containing $\mathbf{0}$ is automatically an orthogonal set of vectors.

   **(b)** The standard basis in $\mathbb{R}^n$ is an orthonormal set of vectors.

   **(c)** If $B = (\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n)$ is an ordered orthonormal basis for $\mathbb{R}^n$, and $\mathbf{v} \in \mathbb{R}^n$, then $[\mathbf{v}]_B = [\mathbf{v} \cdot \mathbf{u}_1, \mathbf{v} \cdot \mathbf{u}_2, \ldots, \mathbf{v} \cdot \mathbf{u}_n]$.

   **(d)** The Gram-Schmidt Process can be used to enlarge any linearly independent set $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ in $\mathbb{R}^n$ to an orthogonal basis $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k, \mathbf{w}_{k+1}, \ldots, \mathbf{w}_n\}$ for $\mathbb{R}^n$.

   **(e)** If $\mathcal{W}$ is a nontrivial subspace of $\mathbb{R}^n$, then an orthogonal basis for $\mathcal{W}$ exists.

   **(f)** If $\mathbf{A}$ is a square matrix, and $\mathbf{A}^T\mathbf{A} = \mathbf{I}_n$, then $\mathbf{A}$ is orthogonal.

   **(g)** If $\mathbf{A}$ and $\mathbf{B}$ are orthogonal $n \times n$ matrices, then $\mathbf{BA}$ is orthogonal and $|\mathbf{BA}| = \pm 1$.

   **(h)** If either the rows or columns of $\mathbf{A}$ form an orthogonal basis for $\mathbb{R}^n$, then $\mathbf{A}$ is orthogonal.

   **(i)** If $\mathbf{A}$ is an orthogonal matrix and $R$ is a Type (III) row operation, then $R(\mathbf{A})$ is an orthogonal matrix.

   **(j)** If $\mathbf{P}$ is the transition matrix from $B$ to $C$, where $B$ and $C$ are ordered orthonormal bases for $\mathbb{R}^n$, then $\mathbf{P}$ is orthogonal.

## 6.2 Orthogonal Complements

For each subspace $\mathcal{W}$ of $\mathbb{R}^n$, there is a corresponding subspace of $\mathbb{R}^n$ consisting of the vectors orthogonal to all vectors in $\mathcal{W}$, called the orthogonal complement of $\mathcal{W}$. In this section, we study many elementary properties of orthogonal complements and investigate the orthogonal projection of a vector onto a subspace of $\mathbb{R}^n$.

### Orthogonal Complements

**Definition** Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$. The **orthogonal complement**, $\mathcal{W}^\perp$, of $\mathcal{W}$ in $\mathbb{R}^n$ is the set of all vectors $\mathbf{x} \in \mathbb{R}^n$ with the property that $\mathbf{x} \cdot \mathbf{w} = 0$, for all $\mathbf{w} \in \mathcal{W}$. That is, $\mathcal{W}^\perp$ contains those vectors of $\mathbb{R}^n$ orthogonal to every vector in $\mathcal{W}$.

The proof of the next theorem is left as Exercise 17.

**Theorem 6.10** *Suppose $\mathcal{W}$ is a subspace of $\mathbb{R}^n$ and $S$ is any spanning set for $\mathcal{W}$. Then $\mathbf{v} \in \mathcal{W}^\perp$ if and only if $\mathbf{v}$ is orthogonal to every vector in $S$.*

**Example 1**

Consider the subspace $\mathcal{W} = \{[a, b, 0] \mid a, b \in \mathbb{R}\}$ of $\mathbb{R}^3$. Now, $\mathcal{W}$ is spanned by $\{[1, 0, 0], [0, 1, 0]\}$. By Theorem 6.10, a vector $[x, y, z]$ is in $\mathcal{W}^\perp$, the orthogonal complement of $\mathcal{W}$, if and only if it is orthogonal to both $[1, 0, 0]$ and $[0, 1, 0]$ (why?)—that is, if and only if $x = y = 0$. Hence, $\mathcal{W}^\perp = \{[0, 0, z] \mid z \in \mathbb{R}\}$. Notice that $\mathcal{W}^\perp$ is a subspace of $\mathbb{R}^3$ of dimension 1 and that $\dim(\mathcal{W}) + \dim\left(\mathcal{W}^\perp\right) = \dim\left(\mathbb{R}^3\right)$.

**Example 2**

Consider the subspace $\mathcal{W} = \{a[-3, 2, 4] \mid a \in \mathbb{R}\}$ of $\mathbb{R}^3$. Since $\{[-3, 2, 4]\}$ spans $\mathcal{W}$, Theorem 6.10 tells us that the orthogonal complement $\mathcal{W}^\perp$ of $\mathcal{W}$ is the set of all vectors $[x, y, z]$ in $\mathbb{R}^3$ such that $[x, y, z] \cdot [-3, 2, 4] = 0$. That is, $\mathcal{W}^\perp$ is precisely the set of all vectors $[x, y, z]$ lying in the plane $-3x + 2y + 4z = 0$. Notice that $\mathcal{W}^\perp$ is a subspace of $\mathbb{R}^3$ of dimension 2 and that $\dim(\mathcal{W}) + \dim\left(\mathcal{W}^\perp\right) = \dim(\mathbb{R}^3)$.

**Example 3**

The orthogonal complement of $\mathbb{R}^n$ itself is just the trivial subspace $\{\mathbf{0}\}$, since $\mathbf{0}$ is the only vector orthogonal to all of $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n \in \mathbb{R}^n$ (why?).

Conversely, the orthogonal complement of the trivial subspace in $\mathbb{R}^n$ is all of $\mathbb{R}^n$ because every vector in $\mathbb{R}^n$ is orthogonal to the zero vector.

Hence, $\{\mathbf{0}\}$ and $\mathbb{R}^n$ itself are orthogonal complements of each other in $\mathbb{R}^n$. Notice that the dimensions of these two subspaces add up to $\dim(\mathbb{R}^n)$.

## Properties of Orthogonal Complements

Examples 1, 2, and 3 suggest that the orthogonal complement $\mathcal{W}^\perp$ of a subspace $\mathcal{W}$ is a subspace of $\mathbb{R}^n$. This result is part of the next theorem.

**Theorem 6.11** *Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$. Then $\mathcal{W}^\perp$ is a subspace of $\mathbb{R}^n$, and $\mathcal{W} \cap \mathcal{W}^\perp = \{\mathbf{0}\}$.*

*Proof.* $\mathcal{W}^\perp$ is nonempty because $\mathbf{0} \in \mathcal{W}^\perp$ (why?). Thus, to show that $\mathcal{W}^\perp$ is a subspace, we need only verify the closure properties for $\mathcal{W}^\perp$.

Suppose $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{W}^\perp$. We want to show $\mathbf{x}_1 + \mathbf{x}_2 \in \mathcal{W}^\perp$. However, for all $\mathbf{w} \in \mathcal{W}$, $(\mathbf{x}_1 + \mathbf{x}_2) \cdot \mathbf{w} = (\mathbf{x}_1 \cdot \mathbf{w}) + (\mathbf{x}_2 \cdot \mathbf{w}) = 0 + 0 = 0$, since $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{W}^\perp$. Hence, $\mathbf{x}_1 + \mathbf{x}_2 \in \mathcal{W}^\perp$. Next, suppose that $\mathbf{x} \in \mathcal{W}^\perp$ and $c \in \mathbb{R}$. We want to show that $c\mathbf{x} \in \mathcal{W}^\perp$. However, for all $\mathbf{w} \in \mathcal{W}$, $(c\mathbf{x}) \cdot \mathbf{w} = c(\mathbf{x} \cdot \mathbf{w}) = c(0) = 0$, since $\mathbf{x} \in \mathcal{W}^\perp$. Hence, $c\mathbf{x} \in \mathcal{W}^\perp$. Thus, $\mathcal{W}^\perp$ is a subspace of $\mathbb{R}^n$.

Finally, suppose $\mathbf{w} \in \mathcal{W} \cap \mathcal{W}^\perp$. Then $\mathbf{w} \in \mathcal{W}$ and $\mathbf{w} \in \mathcal{W}^\perp$, so $\mathbf{w}$ is orthogonal to itself. Hence, $\mathbf{w} \cdot \mathbf{w} = 0$, and so $\mathbf{w} = \mathbf{0}$. □

The next theorem shows how we can obtain an orthogonal basis for $\mathcal{W}^\perp$.

**Theorem 6.12** *Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be an orthogonal basis for $\mathcal{W}$ contained in an orthogonal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$ for $\mathbb{R}^n$. Then $\{\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$ is an orthogonal basis for $\mathcal{W}^\perp$.*

*Proof.* Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be an orthogonal basis for $\mathbb{R}^n$, with $\mathcal{W} = \text{span}(\{\mathbf{v}_1, \ldots, \mathbf{v}_k\})$. Let $\mathcal{X} = \text{span}(\{\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\})$. Since $\{\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$ is linearly independent (why?), it is a basis for $\mathcal{W}^\perp$ if $\mathcal{X} = \mathcal{W}^\perp$. We will show that $\mathcal{X} \subseteq \mathcal{W}^\perp$ and $\mathcal{W}^\perp \subseteq \mathcal{X}$.

To show $\mathcal{X} \subseteq \mathcal{W}^\perp$, we must prove that any vector $\mathbf{x}$ of the form $d_{k+1}\mathbf{v}_{k+1} + \cdots + d_n\mathbf{v}_n$ (for some scalars $d_{k+1}, \ldots, d_n$) is orthogonal to every vector $\mathbf{w} \in \mathcal{W}$. Now, if $\mathbf{w} \in \mathcal{W}$, then $\mathbf{w} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$, for some scalars $c_1, \ldots, c_k$. Hence,

$$\mathbf{x} \cdot \mathbf{w} = (d_{k+1}\mathbf{v}_{k+1} + \cdots + d_n\mathbf{v}_n) \cdot (c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k),$$

which equals zero when expanded because each vector in $\{\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$ is orthogonal to every vector in $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$. Hence, $\mathbf{x} \in \mathcal{W}^\perp$, and so $\mathcal{X} \subseteq \mathcal{W}^\perp$.

To show $\mathcal{W}^\perp \subseteq \mathcal{X}$, we must show that any vector $\mathbf{x}$ in $\mathcal{W}^\perp$ is also in span($\{\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$). Let $\mathbf{x} \in \mathcal{W}^\perp$. Since $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is an orthogonal basis for $\mathbb{R}^n$, Theorem 6.3 tells us that

$$\mathbf{x} = \frac{(\mathbf{x}\cdot\mathbf{v}_1)}{(\mathbf{v}_1\cdot\mathbf{v}_1)}\mathbf{v}_1 + \cdots + \frac{(\mathbf{x}\cdot\mathbf{v}_k)}{(\mathbf{v}_k\cdot\mathbf{v}_k)}\mathbf{v}_k + \frac{(\mathbf{x}\cdot\mathbf{v}_{k+1})}{(\mathbf{v}_{k+1}\cdot\mathbf{v}_{k+1})}\mathbf{v}_{k+1} + \cdots + \frac{(\mathbf{x}\cdot\mathbf{v}_n)}{(\mathbf{v}_n\cdot\mathbf{v}_n)}\mathbf{v}_n.$$

However, since each of $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is in $\mathcal{W}$, we know that $\mathbf{x}\cdot\mathbf{v}_1 = \cdots = \mathbf{x}\cdot\mathbf{v}_k = 0$. Hence,

$$\mathbf{x} = \frac{(\mathbf{x}\cdot\mathbf{v}_{k+1})}{(\mathbf{v}_{k+1}\cdot\mathbf{v}_{k+1})}\mathbf{v}_{k+1} + \cdots + \frac{(\mathbf{x}\cdot\mathbf{v}_n)}{(\mathbf{v}_n\cdot\mathbf{v}_n)}\mathbf{v}_n,$$

and so $\mathbf{x} \in$ span($\{\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$). Thus, $\mathcal{W}^\perp \subseteq \mathcal{X}$. □

**Example 4**

Consider the subspace $\mathcal{W} = \text{span}(\{[2,-1,0,1],[-1,3,1,-1]\})$ of $\mathbb{R}^4$. We want to find an orthogonal basis for $\mathcal{W}^\perp$. We start by finding an orthogonal basis for $\mathcal{W}$.

Let $\mathbf{w}_1 = [2,-1,0,1]$ and $\mathbf{w}_2 = [-1,3,1,-1]$. Performing the Gram-Schmidt Process yields $\mathbf{v}_1 = \mathbf{w}_1 = [2,-1,0,1]$ and $\mathbf{v}_2 = \mathbf{w}_2 - ((\mathbf{w}_2\cdot\mathbf{v}_1)/(\mathbf{v}_1\cdot\mathbf{v}_1))\mathbf{v}_1 = [1,2,1,0]$. Hence, $\{\mathbf{v}_1, \mathbf{v}_2\} = \{[2,-1,0,1],[1,2,1,0]\}$ is an orthogonal basis for $\mathcal{W}$.

We now expand this basis for $\mathcal{W}$ to a basis for all of $\mathbb{R}^4$ using the Enlarging Method of Section 4.6. Row reducing

$$\begin{bmatrix} 2 & 1 & 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{yields} \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{bmatrix}.$$

Thus, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}_3, \mathbf{w}_4\}$ is a basis for $\mathbb{R}^4$, where $\mathbf{w}_3 = [1,0,0,0]$ and $\mathbf{w}_4 = [0,1,0,0]$. Applying the Gram-Schmidt Process to $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}_3, \mathbf{w}_4\}$, we replace $\mathbf{w}_3$ and $\mathbf{w}_4$, respectively, with $\mathbf{v}_3 = [1,0,-1,-2]$ and $\mathbf{v}_4 = [0,1,-2,1]$ (verify!). Then

$$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} = \{[2,-1,0,1],[1,2,1,0],[1,0,-1,-2],[0,1,-2,1]\}$$

is an orthogonal basis for $\mathbb{R}^4$. Since $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal basis for $\mathcal{W}$, Theorem 6.12 tells us that $\{\mathbf{v}_3, \mathbf{v}_4\} = \{[1,0,-1,-2],[0,1,-2,1]\}$ is an orthogonal basis for $\mathcal{W}^\perp$.

The following is an important corollary of Theorem 6.12, which was illustrated in Examples 1, 2, and 3:

**Corollary 6.13** *Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$. Then* $\dim(\mathcal{W}) + \dim\left(\mathcal{W}^\perp\right) = n = \dim(\mathbb{R}^n)$.

*Proof.* Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$ of dimension $k$. By Theorem 6.4, $\mathcal{W}$ has an orthogonal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$. By Theorem 6.5, we can expand this basis for $\mathcal{W}$ to an orthogonal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$ for all of $\mathbb{R}^n$. Then, by Theorem 6.12, $\{\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$ is a basis for $\mathcal{W}^\perp$, and so $\dim\left(\mathcal{W}^\perp\right) = n - k$. Hence, $\dim(\mathcal{W}) + \dim(\mathcal{W}^\perp) = n$. □

**Example 5**

If $\mathcal{W}$ is a one-dimensional subspace of $\mathbb{R}^n$, then Corollary 6.13 asserts that $\dim\left(\mathcal{W}^\perp\right) = n - 1$. For example, in $\mathbb{R}^2$, the one-dimensional subspace $\mathcal{W} = \text{span}(\{[a,b]\})$, where $[a,b] \neq [0,0]$, has a one-dimensional orthogonal complement. In fact, $\mathcal{W}^\perp = \text{span}(\{[b,-a]\})$ (see Fig. 6.2(a)). That is, $\mathcal{W}^\perp$ is the set of all vectors on the line through the origin perpendicular to $[a,b]$.

In $\mathbb{R}^3$, the one-dimensional subspace $\mathcal{W} = \text{span}(\{[a,b,c]\})$, where $[a,b,c] \neq [0,0,0]$, has a two-dimensional orthogonal complement. A little thought will convince you that $\mathcal{W}^\perp$ is the plane through the origin perpendicular to $[a,b,c]$; that is, the plane $ax + by + cz = 0$ (see Fig. 6.2(b)).

If $\mathcal{W}$ is a subspace of $\mathbb{R}^n$, Corollary 6.13 indicates that the dimensions of $\mathcal{W}$ and $\mathcal{W}^\perp$ add up to $n$. For this reason, many students get the mistaken impression that every vector in $\mathbb{R}^n$ lies either in $\mathcal{W}$ or in $\mathcal{W}^\perp$. But $\mathcal{W}$ and $\mathcal{W}^\perp$ are not "setwise" complements of each other; a more accurate depiction is given in Fig. 6.3. For example, recall the subspace $\mathcal{W} = \{[a,b,0] \mid a,b \in \mathbb{R}\}$ of Example 1. We showed that $\mathcal{W}^\perp = \{[0,0,z] \mid z \in \mathbb{R}\}$. Yet $[1,1,1]$ is in neither $\mathcal{W}$ nor $\mathcal{W}^\perp$, even though $\dim(\mathcal{W}) + \dim\left(\mathcal{W}^\perp\right) = \dim\left(\mathbb{R}^3\right)$. In this case, $\mathcal{W}$ is the $xy$-plane and $\mathcal{W}^\perp$ is the $z$-axis.

The next corollary asserts that each subspace $\mathcal{W}$ of $\mathbb{R}^n$ is, in fact, the orthogonal complement of $\mathcal{W}^\perp$. Hence, $\mathcal{W}$ and $\mathcal{W}^\perp$ are orthogonal complements of each other. The proof is left as Exercise 18.

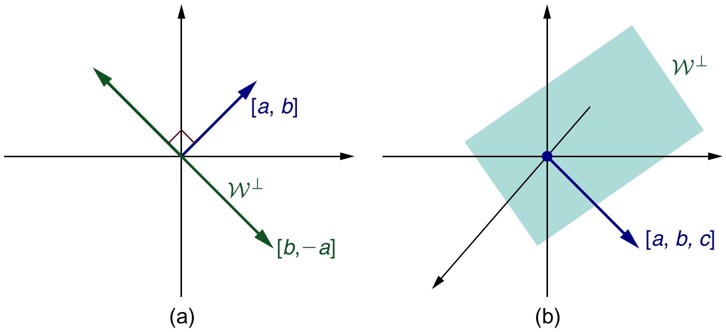


**FIGURE 6.2** (a) The orthogonal complement of $\mathcal{W} = \text{span}(\{[a,b]\})$ in $\mathbb{R}^2$, a line through the origin perpendicular to $[a,b]$, when $[a,b] \neq [0,0]$; (b) the orthogonal complement of $\mathcal{W} = \text{span}(\{[a,b,c]\})$ in $\mathbb{R}^3$, a plane through the origin perpendicular to $[a,b,c]$, when $[a,b,c] \neq [0,0,0]$

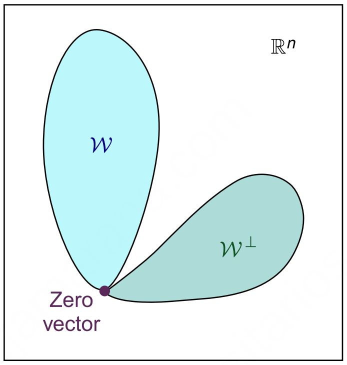


**FIGURE 6.3** Symbolic depiction of $\mathcal{W}$ and $\mathcal{W}^{\perp}$

**Corollary 6.14** *Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$. Then $\left(\mathcal{W}^{\perp}\right)^{\perp} = \mathcal{W}$.*

## Orthogonal Projection Onto a Subspace

Next, we present the Projection Theorem, a generalization of Theorem 1.11. Recall from Theorem 1.11 that every nonzero vector in $\mathbb{R}^n$ can be decomposed into the sum of two vectors, one parallel to a given vector **a** and another orthogonal to **a**.

**Theorem 6.15 (Projection Theorem)** *Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$. Then every vector $\mathbf{v} \in \mathbb{R}^n$ can be expressed in a unique way as $\mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1 \in \mathcal{W}$ and $\mathbf{w}_2 \in \mathcal{W}^{\perp}$.*

*Proof.* Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$, and let $\mathbf{v} \in \mathbb{R}^n$. We first show that $\mathbf{v}$ can be expressed as $\mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1 \in \mathcal{W}$, $\mathbf{w}_2 \in \mathcal{W}^{\perp}$. Then we will show that there is a unique pair $\mathbf{w}_1, \mathbf{w}_2$ for each $\mathbf{v}$.

Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ be an orthonormal basis for $\mathcal{W}$. Expand $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ to an orthonormal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ for $\mathbb{R}^n$. Then by Theorem 6.3, $\mathbf{v} = (\mathbf{v} \cdot \mathbf{u}_1)\mathbf{u}_1 + \cdots + (\mathbf{v} \cdot \mathbf{u}_n)\mathbf{u}_n$. Let $\mathbf{w}_1 = (\mathbf{v} \cdot \mathbf{u}_1)\mathbf{u}_1 + \cdots + (\mathbf{v} \cdot \mathbf{u}_k)\mathbf{u}_k$ and $\mathbf{w}_2 = (\mathbf{v} \cdot \mathbf{u}_{k+1})\mathbf{u}_{k+1} + \cdots + (\mathbf{v} \cdot \mathbf{u}_n)\mathbf{u}_n$. Clearly, $\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2$. Also, Theorem 6.12 implies that $\mathbf{w}_1 \in \mathcal{W}$ and $\mathbf{w}_2$ is in $\mathcal{W}^{\perp}$.

Finally, we want to show uniqueness of decomposition. Suppose that $\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2$ and $\mathbf{v} = \mathbf{w}_1' + \mathbf{w}_2'$, where $\mathbf{w}_1, \mathbf{w}_1' \in \mathcal{W}$ and $\mathbf{w}_2, \mathbf{w}_2' \in \mathcal{W}^{\perp}$. We want to show that $\mathbf{w}_1 = \mathbf{w}_1'$ and $\mathbf{w}_2 = \mathbf{w}_2'$. Now, $\mathbf{w}_1 - \mathbf{w}_1' = \mathbf{w}_2' - \mathbf{w}_2$ (why?). Also, $\mathbf{w}_1 - \mathbf{w}_1' \in \mathcal{W}$, but $\mathbf{w}_2' - \mathbf{w}_2 \in \mathcal{W}^{\perp}$. Thus, $\mathbf{w}_1 - \mathbf{w}_1' = \mathbf{w}_2' - \mathbf{w}_2 \in \mathcal{W} \cap \mathcal{W}^{\perp}$. By Theorem 6.11, $\mathbf{w}_1 - \mathbf{w}_1' = \mathbf{w}_2' - \mathbf{w}_2 = \mathbf{0}$. Hence, $\mathbf{w}_1 = \mathbf{w}_1'$ and $\mathbf{w}_2 = \mathbf{w}_2'$. □

We give a special name to the vector $\mathbf{w}_1$ in the proof of Theorem 6.15.

**Definition** Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$ with orthonormal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$, and let $\mathbf{v} \in \mathbb{R}^n$. Then the **orthogonal projection of v onto** $\mathcal{W}$ is the vector

$$\mathbf{proj}_{\mathcal{W}}\mathbf{v} = (\mathbf{v}\cdot\mathbf{u}_1)\mathbf{u}_1 + \cdots + (\mathbf{v}\cdot\mathbf{u}_k)\mathbf{u}_k.$$

If $\mathcal{W}$ is the trivial subspace of $\mathbb{R}^n$, then $\mathbf{proj}_{\mathcal{W}}\mathbf{v} = \mathbf{0}$.

Notice that the choice of orthonormal basis for $\mathcal{W}$ in this definition does not matter. This is because if $\mathbf{v}$ is any vector in $\mathbb{R}^n$, Theorem 6.15 asserts there is a unique expression $\mathbf{w}_1 + \mathbf{w}_2$ for $\mathbf{v}$ with $\mathbf{w}_1 \in \mathcal{W}$, $\mathbf{w}_2 \in \mathcal{W}^\perp$, and we see from the proof of the theorem that $\mathbf{w}_1 = \mathbf{proj}_{\mathcal{W}}\mathbf{v}$. Hence, if $\{\mathbf{z}_1, \ldots, \mathbf{z}_k\}$ is any other orthonormal basis for $\mathcal{W}$, then $\mathbf{proj}_{\mathcal{W}}\mathbf{v}$ is equal to $(\mathbf{v}\cdot\mathbf{z}_1)\mathbf{z}_1 + \cdots + (\mathbf{v}\cdot\mathbf{z}_k)\mathbf{z}_k$ as well. This fact is illustrated in the next example.

## Example 6

Consider the orthonormal subset

$$B = \{\mathbf{u}_1, \mathbf{u}_2\} = \left\{\left[\frac{8}{9}, -\frac{1}{9}, -\frac{4}{9}\right], \left[\frac{4}{9}, \frac{4}{9}, \frac{7}{9}\right]\right\}$$

of $\mathbb{R}^3$, and let $\mathcal{W} = \text{span}(B)$. Notice that $B$ is an orthonormal basis for $\mathcal{W}$.

Also consider the orthogonal set $S = \{[4, 1, 1], [4, -5, -11]\}$. Now since

$$[4, 1, 1] = 3\left[\frac{8}{9}, -\frac{1}{9}, -\frac{4}{9}\right] + 3\left[\frac{4}{9}, \frac{4}{9}, \frac{7}{9}\right]$$
$$\text{and}\ \ [4, -5, -11] = 9\left[\frac{8}{9}, -\frac{1}{9}, -\frac{4}{9}\right] - 9\left[\frac{4}{9}, \frac{4}{9}, \frac{7}{9}\right],$$

$S$ is an orthogonal subset of $\mathcal{W}$. Since $|S| = \dim(\mathcal{W})$, $S$ is also an orthogonal basis for $\mathcal{W}$. Hence, after normalizing the vectors in $S$, we obtain the following second orthonormal basis for $\mathcal{W}$:

$$C = \{\mathbf{z}_1, \mathbf{z}_2\} = \left\{\left[\frac{4}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}\right], \left[\frac{4}{9\sqrt{2}}, -\frac{5}{9\sqrt{2}}, -\frac{11}{9\sqrt{2}}\right]\right\}.$$

Let $\mathbf{v} = [1, 2, 3]$. We will verify that the same vector for $\mathbf{proj}_{\mathcal{W}}\mathbf{v}$ is obtained whether $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ or $C = \{\mathbf{z}_1, \mathbf{z}_2\}$ is used as the orthonormal basis for $\mathcal{W}$. Now, using $B$ yields

$$(\mathbf{v}\cdot\mathbf{u}_1)\,\mathbf{u}_1 + (\mathbf{v}\cdot\mathbf{u}_2)\,\mathbf{u}_2 = -\frac{2}{3}\left[\frac{8}{9}, -\frac{1}{9}, -\frac{4}{9}\right] + \frac{11}{3}\left[\frac{4}{9}, \frac{4}{9}, \frac{7}{9}\right] = \left[\frac{28}{27}, \frac{46}{27}, \frac{85}{27}\right].$$

Similarly, using $C$ gives

$$(\mathbf{v}\cdot\mathbf{z}_1)\,\mathbf{z}_1 + (\mathbf{v}\cdot\mathbf{z}_2)\,\mathbf{z}_2 = \frac{3}{\sqrt{2}}\left[\frac{4}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}\right] + \left(-\frac{13}{3\sqrt{2}}\right)\left[\frac{4}{9\sqrt{2}}, -\frac{5}{9\sqrt{2}}, -\frac{11}{9\sqrt{2}}\right]$$
$$= \left[\frac{28}{27}, \frac{46}{27}, \frac{85}{27}\right].$$

Hence, with either orthonormal basis we obtain $\mathbf{proj}_{\mathcal{W}}\mathbf{v} = \left[\frac{28}{27}, \frac{46}{27}, \frac{85}{27}\right]$.

The proof of Theorem 6.15 illustrates the following:

**Corollary 6.16** *If $\mathcal{W}$ is a subspace of $\mathbb{R}^n$ and $\mathbf{v} \in \mathbb{R}^n$, then there are unique vectors $\mathbf{w}_1$ and $\mathbf{w}_2$, with $\mathbf{w}_1 \in \mathcal{W}$ and $\mathbf{w}_2 \in \mathcal{W}^\perp$ such that $\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2$. Moreover, $\mathbf{w}_1 = \mathbf{proj}_{\mathcal{W}}\mathbf{v}$ and $\mathbf{w}_2 = \mathbf{v} - \mathbf{proj}_{\mathcal{W}}\mathbf{v} = \mathbf{proj}_{\mathcal{W}^\perp}\mathbf{v}$.*

The vector $\mathbf{w}_1$ is the generalization of the projection vector $\mathbf{proj_a b}$ from Section 1.2 (see Exercise 16).

## Example 7

Let $\mathcal{W}$ be the subspace of $\mathbb{R}^3$ whose vectors (beginning at the origin) lie in the plane $\mathcal{L}$ with equation $2x + y + z = 0$. Let $\mathbf{v} = [-6, 10, 5]$. (Notice that $\mathbf{v} \notin \mathcal{W}$.) We will find $\mathbf{proj}_{\mathcal{W}}\mathbf{v}$.

First, notice that $[1, 0, -2]$ and $[0, 1, -1]$ are two linearly independent vectors in $\mathcal{W}$. (To find the first vector, choose $x = 1$, $y = 0$, and for the other, let $x = 0$ and $y = 1$.) Using the Gram-Schmidt Process on these vectors, we obtain the orthogonal basis $\{[1, 0, -2], [-2, 5, -1]\}$

for $\mathcal{W}$ (verify!). After normalizing, we have the orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ for $\mathcal{W}$, where

$$\mathbf{u}_1 = \left[\frac{1}{\sqrt{5}}, 0, -\frac{2}{\sqrt{5}}\right] \quad \text{and} \quad \mathbf{u}_2 = \left[-\frac{2}{\sqrt{30}}, \frac{5}{\sqrt{30}}, -\frac{1}{\sqrt{30}}\right].$$

Now,

$$\begin{aligned}\mathbf{proj}_{\mathcal{W}}\mathbf{v} &= (\mathbf{v}\cdot\mathbf{u}_1)\,\mathbf{u}_1 + (\mathbf{v}\cdot\mathbf{u}_2)\,\mathbf{u}_2\\ &= -\frac{16}{\sqrt{5}}\left[\frac{1}{\sqrt{5}}, 0, -\frac{2}{\sqrt{5}}\right] + \frac{57}{\sqrt{30}}\left[-\frac{2}{\sqrt{30}}, \frac{5}{\sqrt{30}}, -\frac{1}{\sqrt{30}}\right]\\ &= \left[-\frac{16}{5}, 0, \frac{32}{5}\right] + \left[-\frac{114}{30}, \frac{285}{30}, -\frac{57}{30}\right]\\ &= \left[-7, \frac{19}{2}, \frac{9}{2}\right].\end{aligned}$$

Notice that this vector is in $\mathcal{W}$. Finally, $\mathbf{v} - \mathbf{proj}_{\mathcal{W}}\mathbf{v} = \left[1, \frac{1}{2}, \frac{1}{2}\right]$, which is indeed in $\mathcal{W}^{\perp}$ because it is orthogonal to both $\mathbf{u}_1$ and $\mathbf{u}_2$ (verify!). Hence, we have decomposed $\mathbf{v} = [-6, 10, 5]$ as the sum of two vectors $\left[-7, \frac{19}{2}, \frac{9}{2}\right]$ and $\left[1, \frac{1}{2}, \frac{1}{2}\right]$, where the first is in $\mathcal{W}$ and the second is in $\mathcal{W}^{\perp}$.

We can think of the orthogonal projection vector $\mathbf{proj}_{\mathcal{W}}\mathbf{v}$ in Example 7 as the "shadow" that $\mathbf{v}$ casts on the plane $\mathcal{L}$ as light falls directly onto $\mathcal{L}$ from a light source above and parallel to $\mathcal{L}$. This concept is illustrated in Fig. 6.4.

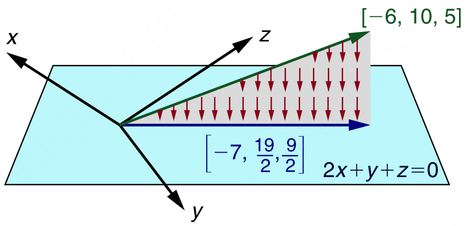


**FIGURE 6.4** The orthogonal projection vector $\left[-7, \frac{19}{2}, \frac{9}{2}\right]$ of $\mathbf{v} = [-6, 10, 5]$ onto the plane $2x + y + z = 0$, pictured as a shadow cast by $\mathbf{v}$ from a light source above and parallel to the plane

There are two special cases of Corollary 6.16. First, if $\mathbf{v} \in \mathcal{W}$, then $\mathbf{proj}_{\mathcal{W}}\mathbf{v}$ simply equals $\mathbf{v}$ itself. Also, if $\mathbf{v} \in \mathcal{W}^{\perp}$, then $\mathbf{proj}_{\mathcal{W}}\mathbf{v}$ equals $\mathbf{0}$. These results are left as Exercise 13.

The next theorem assures us that orthogonal projection onto a subspace of $\mathbb{R}^n$ is a linear operator on $\mathbb{R}^n$. The proof is left as Exercise 19.

**Theorem 6.17** *Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$. Then the mapping $L: \mathbb{R}^n \to \mathbb{R}^n$ given by $L(\mathbf{v}) = \mathbf{proj}_{\mathcal{W}}\mathbf{v}$ is a linear operator with* $\ker(L) = \mathcal{W}^{\perp}$.

## Application: Orthogonal Projections and Reflections in $\mathbb{R}^3$

From Theorem 6.17, an orthogonal projection onto a plane through the origin in $\mathbb{R}^3$ is a linear operator on $\mathbb{R}^3$. We can use eigenvectors and the Generalized Diagonalization Method to find the matrix for such an operator with respect to the standard basis.

### Example 8

Let $L: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the orthogonal projection onto the plane $\mathcal{W} = \{[x, y, z] \,|\, 4x - 7y + 4z = 0\}$. To find the matrix for $L$ with respect to the standard basis, we first find bases for $\mathcal{W}$ and $\mathcal{W}^{\perp}$, which as we will see, are actually bases for the eigenspaces of $L$.

Since $[4, -7, 4] \cdot [x, y, z] = 0$ for every vector in $\mathcal{W}$, $\mathbf{v}_1 = [4, -7, 4] \in \mathcal{W}^{\perp}$. Since $\dim(\mathcal{W}) = 2$, we have $\dim(\mathcal{W}^{\perp}) = 1$ by Corollary 6.13 and so $\{\mathbf{v}_1\}$ is a basis for $\mathcal{W}^{\perp}$. Notice that $\mathcal{W}^{\perp} = \ker(L)$ (by Theorem 6.17), and so $\mathcal{W}^{\perp} =$ the eigenspace $E_0$ for $L$. Hence, $\{\mathbf{v}_1\}$ is actually a basis for $E_0$.

Next, notice that the plane $\mathcal{W} = \{[x, y, z] \,|\, 4x - 7y + 4z = 0\}$ can be expressed as $\{[x, y, \frac{1}{4}(-4x + 7y)]\} = \{x[1, 0, -1] + y[0, 1, \frac{7}{4}]\}$. Let $\mathbf{v}_2 = [1, 0, -1]$ and $\mathbf{v}_3 = [0, 1, \frac{7}{4}]$. Then $\{\mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent subset of $\mathcal{W}$. Hence, $\{\mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathcal{W}$, since $\dim(\mathcal{W}) = 2$.

But since every vector in the plane $\mathcal{W}$ is mapped to itself by $L$, $\mathcal{W}=$ the eigenspace $E_1$ for $L$. Thus, $\{\mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $E_1$. The union $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of the bases for $E_0$ and $E_1$ is a linearly independent set of three vectors for $\mathbb{R}^3$ by Theorem 5.27, and so $L$ is diagonalizable.

Now, by the Generalized Diagonalization Method of Section 5.6, if $\mathbf{A}$ is the matrix for $L$ with respect to the standard basis, then $\mathbf{P}^{-1}\mathbf{AP}=\mathbf{D}$, where $\mathbf{P}$ is the transition matrix whose columns are the eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, and $\mathbf{D}$ is the diagonal matrix with the eigenvalues 0, 1, and 1 on the main diagonal. Hence, we compute $\mathbf{P}^{-1}$, and use $\mathbf{A}=\mathbf{PDP}^{-1}$ to obtain

$$\mathbf{A}=\begin{bmatrix} 4 & 1 & 0 \\ -7 & 0 & 1 \\ 4 & -1 & \frac{7}{4} \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{4}{81} & -\frac{7}{81} & \frac{4}{81} \\ \frac{65}{81} & \frac{28}{81} & -\frac{16}{81} \\ \frac{28}{81} & \frac{32}{81} & \frac{28}{81} \end{bmatrix}=\frac{1}{81}\begin{bmatrix} 65 & 28 & -16 \\ 28 & 32 & 28 \\ -16 & 28 & 65 \end{bmatrix},$$

which is the matrix for $L$ with respect to the standard basis.

The technique used in Example 8 can be generalized as follows:

Let $\mathcal{W}$ be a plane in $\mathbb{R}^3$ through the origin, with $\mathbf{v}_1=[a,b,c]$ orthogonal to $\mathcal{W}$. Let $\mathbf{v}_2$ and $\mathbf{v}_3$ be any linearly independent pair of vectors in $\mathcal{W}$. Then the **matrix A for the orthogonal projection onto $\mathcal{W}$ with respect to the standard basis** is $\mathbf{A}=\mathbf{PDP}^{-1}$, where $\mathbf{P}$ is the transition matrix whose columns are $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, in any order, and $\mathbf{D}$ is the diagonal matrix with the eigenvalues 0, 1, and 1 in a corresponding order on the main diagonal. (That is, the column containing eigenvalue 0 in $\mathbf{D}$ corresponds to the column in $\mathbf{P}$ containing $\mathbf{v}_1$.)

Similarly, we can reverse the process to determine whether a given $3\times 3$ matrix $\mathbf{A}$ represents an orthogonal projection onto a plane through the origin. Such a matrix must diagonalize to the diagonal matrix $\mathbf{D}$ having eigenvalues 0, 1, and 1 on the main diagonal, and the transition matrix $\mathbf{P}$ such that $\mathbf{A}=\mathbf{PDP}^{-1}$ must have the property that the column of $\mathbf{P}$ corresponding to the eigenvalue 0 be orthogonal to the other two columns of $\mathbf{P}$.

### Example 9

The matrix

$$\mathbf{A}=\begin{bmatrix} 18 & -6 & -30 \\ -25 & 10 & 45 \\ 17 & -6 & -29 \end{bmatrix}$$

has eigenvalues 0, 1, and $-2$ (verify!). Since there is an eigenvalue other than 0 or 1, $\mathbf{A}$ can not represent an orthogonal projection onto a plane through the origin.

Similarly, you can verify that

$$\mathbf{A}_1=\begin{bmatrix} -3 & 1 & -1 \\ 16 & -3 & 4 \\ 28 & -7 & 8 \end{bmatrix} \quad \text{diagonalizes to} \quad \mathbf{D}_1=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now, $\mathbf{D}_1$ clearly has the proper form. However, the transition matrix $\mathbf{P}_1$ used in the diagonalization is found to be

$$\mathbf{P}_1=\begin{bmatrix} -1 & 0 & 1 \\ 4 & -1 & -6 \\ 7 & -1 & -10 \end{bmatrix}.$$

Since the first column of $\mathbf{P}_1$ (corresponding to eigenvalue 0) is not orthogonal to the other two columns of $\mathbf{P}_1$, $\mathbf{A}_1$ does not represent an orthogonal projection onto a plane through the origin.

In contrast, the matrix

$$\mathbf{A}_2=\frac{1}{14}\begin{bmatrix} 5 & -3 & -6 \\ -3 & 13 & -2 \\ -6 & -2 & 10 \end{bmatrix} \quad \text{diagonalizes to} \quad \mathbf{D}_2=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with transition matrix

$$\mathbf{P}_2=\begin{bmatrix} 3 & -4 & 1 \\ 1 & 2 & -1 \\ 2 & 5 & -1 \end{bmatrix}.$$

Now, $\mathbf{D}_2$ has the correct form, as does $\mathbf{P}_2$, since the first column of $\mathbf{P}_2$ is orthogonal to both other columns. Hence, $\mathbf{A}_2$ represents an orthogonal projection onto a plane through the origin in $\mathbb{R}^3$. In fact, it is the orthogonal projection onto the plane $3x + y + 2z = 0$, that is, all $[x, y, z]$ orthogonal to the first column of $\mathbf{P}_2$.

We can analyze linear operators that are orthogonal reflections through a plane through the origin in $\mathbb{R}^3$ in a manner similar to the techniques we used for orthogonal projections.[1] However, the vector $\mathbf{v}_1$ orthogonal to the plane now corresponds to the eigenvalue $\lambda_1 = -1$ (instead of $\lambda_1 = 0$), since $\mathbf{v}_1$ reflects through the plane into $-\mathbf{v}_1$.

### Example 10

Consider the orthogonal reflection $R$ through the plane $\{[x, y, z] \,|\, 5x - y + 3z = 0\} = \{[x, y, \frac{1}{3}(-5x + y)]\} = \{x[1, 0, -\frac{5}{3}] + y[0, 1, \frac{1}{3}]\}$. The matrix for $R$ with respect to the standard basis for $\mathbb{R}^3$ is $\mathbf{A} = \mathbf{PDP}^{-1}$, where $\mathbf{D}$ has the eigenvalues $-1$, 1, and 1 on the main diagonal, and where the first column of the transition matrix $\mathbf{P}$ is orthogonal to the plane, and the other two columns of $\mathbf{P}$ are linearly independent vectors in the plane. Hence,

$$\mathbf{A} = \mathbf{PDP}^{-1} = \begin{bmatrix} 5 & 1 & 0 \\ -1 & 0 & 1 \\ 3 & -\frac{5}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{7} & -\frac{1}{35} & \frac{3}{35} \\ \frac{2}{7} & \frac{1}{7} & -\frac{3}{7} \\ \frac{1}{7} & \frac{34}{35} & \frac{3}{35} \end{bmatrix} = \begin{bmatrix} -\frac{3}{7} & \frac{2}{7} & -\frac{6}{7} \\ \frac{2}{7} & \frac{33}{35} & \frac{6}{35} \\ -\frac{6}{7} & \frac{6}{35} & \frac{17}{35} \end{bmatrix}.$$

The technique used in Example 10 can be generalized as follows:

Let $\mathcal{W}$ be a plane in $\mathbb{R}^3$ through the origin, with $\mathbf{v}_1 = [a, b, c]$ orthogonal to $\mathcal{W}$. Let $\mathbf{v}_2$ and $\mathbf{v}_3$ be any linearly independent pair of vectors in $\mathcal{W}$. Then the **matrix A for the orthogonal reflection through $\mathcal{W}$ with respect to the standard basis** is $\mathbf{A} = \mathbf{PDP}^{-1}$, where $\mathbf{P}$ is the transition matrix whose columns are $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, in any order, and $\mathbf{D}$ is the diagonal matrix with the eigenvalues $-1$, 1, and 1 in a corresponding order on the main diagonal. (That is, the column containing eigenvalue $-1$ in $\mathbf{D}$ corresponds to the column in $\mathbf{P}$ containing $\mathbf{v}_1$.)

Similarly, we can reverse the process to determine whether a given $3 \times 3$ matrix $\mathbf{A}$ represents an orthogonal reflection through a plane through the origin. Such a matrix must diagonalize to the diagonal matrix $\mathbf{D}$ having eigenvalues $-1$, 1, and 1, respectively, on the main diagonal, and the transition matrix $\mathbf{P}$ such that $\mathbf{A} = \mathbf{PDP}^{-1}$ must have the property that the column of $\mathbf{P}$ corresponding to the eigenvalue $-1$ be orthogonal to the other two columns of $\mathbf{P}$.

## Application: Distance From a Point to a Subspace

**Definition** Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$, and assume all vectors in $\mathcal{W}$ have initial point at the origin. Let $P$ be any point in $n$-dimensional space. Then the **minimum distance** from $P$ to $\mathcal{W}$ is the shortest distance between $P$ and the terminal point of any vector in $\mathcal{W}$.

The next theorem gives a formula for the minimum distance, and its proof is left as Exercise 22.

**Theorem 6.18** *Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$, and let $P$ be a point in $n$-dimensional space. If $\mathbf{v}$ is the vector from the origin to $P$, then the minimum distance from $P$ to $\mathcal{W}$ is $\|\mathbf{v} - \mathbf{proj}_{\mathcal{W}}\mathbf{v}\|$.*

Notice that if $S$ is the terminal point of $\mathbf{proj}_{\mathcal{W}}\mathbf{v}$, then $\|\mathbf{v} - \mathbf{proj}_{\mathcal{W}}\mathbf{v}\|$ represents the distance from $P$ to $S$, as illustrated in Fig. 6.5. Therefore, Theorem 6.18 can be interpreted as saying that no other vector in $\mathcal{W}$ is closer to $\mathbf{v}$ than $\mathbf{proj}_{\mathcal{W}}\mathbf{v}$; that is, the norm of the difference between $\mathbf{v}$ and $\mathbf{proj}_{\mathcal{W}}\mathbf{v}$ is less than or equal to the norm of the difference between $\mathbf{v}$ and any other vector in $\mathcal{W}$. In fact, it can be shown that if $\mathbf{w}$ is a vector in $\mathcal{W}$ equally close to $\mathbf{v}$, then $\mathbf{w}$ must equal $\mathbf{proj}_{\mathcal{W}}\mathbf{v}$.[2]

[1] All of the reflection operators we have studied earlier in this text are, in fact, *orthogonal* reflections.
[2] This statement, in a slightly different form, is proved as part of Theorem 8.13 in Section 8.10.

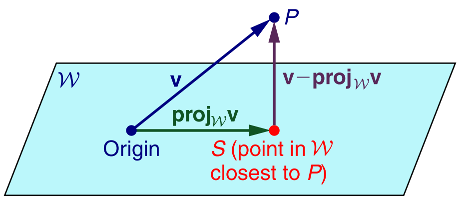


FIGURE 6.5 The minimum distance from $P$ to $\mathcal{W}$, $\|\mathbf{v}-\mathbf{proj}_{\mathcal{W}}\mathbf{v}\|$

**Example 11**

Consider the subspace $\mathcal{W}$ of $\mathbb{R}^3$ from Example 7, whose vectors lie in the plane $2x+y+z=0$. In that example, for $\mathbf{v}=[-6,10,5]$, we calculated that $\mathbf{v}-\mathbf{proj}_{\mathcal{W}}\mathbf{v}=\left[1,\frac{1}{2},\frac{1}{2}\right]$. Hence, the minimum distance from $P=(-6,10,5)$ to $\mathcal{W}$ is $\|\mathbf{v}-\mathbf{proj}_{\mathcal{W}}\mathbf{v}\|=\sqrt{1^2+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}$ $=\sqrt{\frac{3}{2}}\approx 1.2247$.

♦ **Application:** You have now covered the prerequisites for Section 8.10, "Least-Squares Solutions for Inconsistent Systems."

## New Vocabulary

minimum distance from a point to a subspace
orthogonal complement (of a subspace)
orthogonal projection (of a vector onto a subspace)
orthogonal reflection (of a vector through a plane)
Projection Theorem

## Highlights

- If $\{\mathbf{v}_1,\ldots,\mathbf{v}_k\}$ spans a subspace $\mathcal{W}$ of $\mathbb{R}^n$, then $\mathcal{W}^\perp$, the orthogonal complement of $\mathcal{W}$, is the subspace of $\mathbb{R}^n$ consisting precisely of the vectors that are orthogonal to all of $\mathbf{v}_1,\ldots,\mathbf{v}_k$.
- If $\mathcal{W}$ is a subspace of $\mathbb{R}^n$, then $\mathcal{W}\cap\mathcal{W}^\perp=\{\mathbf{0}\}$ and $\left(\mathcal{W}^\perp\right)^\perp=\mathcal{W}$.
- If $B=\{\mathbf{v}_1,\ldots,\mathbf{v}_k\}$ is an orthogonal basis for a subspace $\mathcal{W}$ of $\mathbb{R}^n$, and if $B$ is enlarged to an orthogonal basis $\{\mathbf{v}_1,\ldots,\mathbf{v}_k,\mathbf{v}_{k+1},\ldots,\mathbf{v}_n\}$ for $\mathbb{R}^n$, then $\{\mathbf{v}_{k+1},\ldots,\mathbf{v}_n\}$ is an orthogonal basis for $\mathcal{W}^\perp$.
- If $\mathcal{W}$ is a subspace of $\mathbb{R}^n$, then $\dim(\mathcal{W})+\dim\left(\mathcal{W}^\perp\right)=n=\dim(\mathbb{R}^n)$.
- In $\mathbb{R}^2$, if $[a,b]\neq[0,0]$, and $\mathcal{W}=\text{span}(\{[a,b]\})$, then $\mathcal{W}^\perp=\text{span}(\{[b,-a]\})$.
- In $\mathbb{R}^3$, if $[a,b,c]\neq[0,0,0]$, and $\mathcal{W}=\text{span}(\{[a,b,c]\})$, then $\mathcal{W}^\perp$ consists of the plane $ax+by+cz=0$ through the origin perpendicular to $[a,b,c]$.
- If $\mathcal{W}$ is a subspace of $\mathbb{R}^n$ having orthonormal basis $\{\mathbf{u}_1,\ldots,\mathbf{u}_k\}$, the orthogonal projection of a vector $\mathbf{v}$ onto $\mathcal{W}$ is $\mathbf{proj}_{\mathcal{W}}\mathbf{v}=(\mathbf{v}\cdot\mathbf{u}_1)\mathbf{u}_1+\cdots+(\mathbf{v}\cdot\mathbf{u}_k)\mathbf{u}_k$. The result obtained for $\mathbf{proj}_{\mathcal{W}}\mathbf{v}$ is the same regardless of the particular orthonormal basis chosen for $\mathcal{W}$.
- If $\mathcal{W}$ is a subspace of $\mathbb{R}^n$, then every vector $\mathbf{v}\in\mathbb{R}^n$ can be expressed as $\mathbf{w}_1+\mathbf{w}_2$, where $\mathbf{w}_1=\mathbf{proj}_{\mathcal{W}}\mathbf{v}\in\mathcal{W}$ and $\mathbf{w}_2=\mathbf{v}-\mathbf{proj}_{\mathcal{W}}\mathbf{v}=\mathbf{proj}_{\mathcal{W}^\perp}\mathbf{v}\in\mathcal{W}^\perp$.
- If $\mathcal{W}$ is a subspace of $\mathbb{R}^n$, then $L\colon\mathbb{R}^n\to\mathbb{R}^n$ given by $L(\mathbf{v})=\mathbf{proj}_{\mathcal{W}}\mathbf{v}$ is a linear operator, and $\ker(L)=\mathcal{W}^\perp$.
- The matrix $\mathbf{A}$ for any orthogonal projection onto a plane through the origin in $\mathbb{R}^3$ is diagonalizable. Also, $\mathbf{A}=\mathbf{PDP}^{-1}$ where $\mathbf{D}$ is a diagonal matrix having eigenvalues 0, 1, 1 on the main diagonal, and where the column of the transition matrix $\mathbf{P}$ corresponding to eigenvalue 0 is orthogonal to the columns of $\mathbf{P}$ corresponding to the eigenvalue 1.
- The matrix $\mathbf{A}$ for any orthogonal reflection through a plane through the origin in $\mathbb{R}^3$ is diagonalizable. Also, $\mathbf{A}=\mathbf{PDP}^{-1}$ where $\mathbf{D}$ is a diagonal matrix having eigenvalues $-1$, 1, 1 on the main diagonal, and where the column of the transition matrix $\mathbf{P}$ corresponding to eigenvalue $-1$ is orthogonal to the columns of $\mathbf{P}$ corresponding to the eigenvalue 1.
- The minimum distance from a point $P$ to a subspace $\mathcal{W}$ of $\mathbb{R}^n$ is $\|\mathbf{v}-\mathbf{proj}_{\mathcal{W}}\mathbf{v}\|$, where $\mathbf{v}$ is the vector from the origin to $P$.

# Exercises for Section 6.2

1. For each of the following subspaces $\mathcal{W}$ of $\mathbb{R}^n$, find a basis for $\mathcal{W}^\perp$, and verify Corollary 6.13:
   ★ **(a)** In $\mathbb{R}^2$, $\mathcal{W}=\text{span}(\{[3,-2]\})$
   **(b)** In $\mathbb{R}^3$, $\mathcal{W}=$ the plane $4x-7y+5z=0$

★ **(c)** In $\mathbb{R}^3$, $\mathcal{W} = \text{span}(\{[1, 4, -2], [2, 1, -1]\})$
**(d)** In $\mathbb{R}^3$, $\mathcal{W} = \text{span}(\{[5, 1, 3]\})$
★ **(e)** In $\mathbb{R}^3$, $\mathcal{W} =$ the plane $-2x + 5y - z = 0$
★ **(f)** In $\mathbb{R}^4$, $\mathcal{W} = \text{span}(\{[1, -1, 0, 2], [0, 1, 2, -1]\})$
**(g)** In $\mathbb{R}^4$, $\mathcal{W} = \{[x, y, z, w] \mid 4x + 5y - 2z + 3w = 0\}$
**(h)** In $\mathbb{R}^5$, $\mathcal{W} =$ the solution set for the system $\mathbf{AX} = \mathbf{0}$, with $\mathbf{A} = \begin{bmatrix} 1 & -2 & 4 & -8 & 5 \\ 2 & -1 & 5 & -7 & 4 \\ 3 & -3 & 9 & -10 & -1 \end{bmatrix}$

**2.** For each of the following subspaces $\mathcal{W}$ of $\mathbb{R}^n$ and for the given $\mathbf{v} \in \mathbb{R}^n$, find $\mathbf{proj}_{\mathcal{W}}\mathbf{v}$, and decompose $\mathbf{v}$ into $\mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1 \in \mathcal{W}$ and $\mathbf{w}_2 \in \mathcal{W}^{\perp}$. (Hint: You may need to find an orthonormal basis for $\mathcal{W}$ first.)
★ **(a)** In $\mathbb{R}^3$, $\mathcal{W} = \text{span}(\{[1, -2, -1], [3, -1, 0]\})$, $\mathbf{v} = [-1, 3, 2]$
★ **(b)** In $\mathbb{R}^3$, $\mathcal{W} =$ the plane $2x - 2y + z = 0$, $\mathbf{v} = [1, -4, 3]$
**(c)** In $\mathbb{R}^3$, $\mathcal{W} = \text{span}(\{[-1, 3, 2], [2, 2, -3]\})$, $\mathbf{v} = [24, 17, 7]$
**(d)** In $\mathbb{R}^4$, $\mathcal{W} = \text{span}(\{[2, 1, -3, 2], [2, 4, -5, 2]\})$, $\mathbf{v} = [2, 6, 7, 1]$

**3.** Let $\mathbf{v} = [a, b, c]$. If $\mathcal{W}$ is the $xy$-plane, verify that $\mathbf{proj}_{\mathcal{W}}\mathbf{v} = [a, b, 0]$.

**4.** In each of the following, find the minimum distance between the given point $P$ and the given subspace $\mathcal{W}$ of $\mathbb{R}^n$:
★ **(a)** $P = (-2, 3, 1)$, $\mathcal{W} = \text{span}(\{[-1, 4, 4], [2, -1, 0]\})$ in $\mathbb{R}^3$
**(b)** $P = (7, -1, 19)$, $\mathcal{W} = \text{span}(\{[4, -2, 3]\})$ in $\mathbb{R}^3$
**(c)** $P = (4, -13, 10, -3)$, $\mathcal{W} = \text{span}(\{[-1, 2, -1, 2], [2, -1, 2, -1]\})$ in $\mathbb{R}^4$
★ **(d)** $P = (-1, 4, -2, 2)$, $\mathcal{W} = \{[x, y, z, w] \mid 2x - 3z + 2w = 0\}$ in $\mathbb{R}^4$

**5.** In each part, let $L$ be the linear operator on $\mathbb{R}^3$ with the given matrix representation with respect to the standard basis. Determine whether $L$ is
**(i)** An orthogonal projection onto a plane through the origin
**(ii)** An orthogonal reflection through a plane through the origin
**(iii)** Neither
Also, if $L$ is of type (i) or (ii), state the equation of the plane.

★ **(a)** $\frac{1}{11}\begin{bmatrix} 2 & -3 & -3 \\ -3 & 10 & -1 \\ -3 & -1 & 10 \end{bmatrix}$

**(b)** $\frac{1}{3}\begin{bmatrix} -1 & -2 & -2 \\ -2 & -1 & 2 \\ -2 & 2 & -1 \end{bmatrix}$

**(c)** $\frac{1}{15}\begin{bmatrix} 14 & 2 & -5 \\ 2 & 11 & 10 \\ -5 & 10 & -10 \end{bmatrix}$

★ **(d)** $\frac{1}{15}\begin{bmatrix} 7 & -2 & -14 \\ -4 & 14 & -7 \\ -12 & -3 & -6 \end{bmatrix}$

★ **6.** Let $L: \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal projection onto the plane $2x - y + 2z = 0$. Use eigenvalues and eigenvectors to find the matrix representation of $L$ with respect to the standard basis.

**7.** Let $L: \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal reflection through the plane $6x - 4y + 5z = 0$. Use eigenvalues and eigenvectors to find the matrix representation of $L$ with respect to the standard basis.

**8.** Let $L: \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal projection onto the plane $2x + y + z = 0$ from Example 7.
**(a)** Use eigenvalues and eigenvectors to find the matrix representation of $L$ with respect to the standard basis.
**(b)** Use the matrix in part (a) to confirm the computation in Example 7 that $L([-6, 10, 5]) = [-7, \frac{19}{2}, \frac{9}{2}]$.

**9.** Find the characteristic polynomial for each of the given linear operators. (Hint: This requires almost no computation.)
★ **(a)** $L: \mathbb{R}^3 \to \mathbb{R}^3$, where $L$ is the orthogonal projection onto the plane $4x - 3y + 2z = 0$
**(b)** $L: \mathbb{R}^3 \to \mathbb{R}^3$, where $L$ is the orthogonal projection onto the line through the origin spanned by $[5, 2, -7]$
★ **(c)** $L: \mathbb{R}^3 \to \mathbb{R}^3$, where $L$ is the orthogonal reflection through the plane $3x + 5y - z = 0$
**(d)** $L: \mathbb{R}^3 \to \mathbb{R}^3$, where $L$ is the orthogonal reflection through the line through the origin spanned by $[7, -3, 2]$

**10.** In each of the following, find the matrix representation of the operator $L: \mathbb{R}^n \to \mathbb{R}^n$ given by $L(\mathbf{v}) = \mathbf{proj}_{\mathcal{W}}\mathbf{v}$, with respect to the standard basis for $\mathbb{R}^n$:
★ **(a)** In $\mathbb{R}^3$, $\mathcal{W} = \text{span}(\{[2, -1, 1], [1, 0, -3]\})$
**(b)** In $\mathbb{R}^3$, $\mathcal{W} =$ the plane $4x - y + 2z = 0$
★ **(c)** In $\mathbb{R}^4$, $\mathcal{W} = \text{span}(\{[1, 2, 1, 0], [-1, 0, -2, 1]\})$
**(d)** In $\mathbb{R}^4$, $\mathcal{W} = \text{span}(\{[-3, -1, 1, 2], [5, 3, -1, -2]\})$

**11.** Prove that if $\mathcal{W}_1$ and $\mathcal{W}_2$ are subspaces of $\mathbb{R}^n$ with $\mathcal{W}_1^\perp = \mathcal{W}_2^\perp$, then $\mathcal{W}_1 = \mathcal{W}_2$.

**12.** Prove that if $\mathcal{W}_1$ and $\mathcal{W}_2$ are subspaces of $\mathbb{R}^n$ with $\mathcal{W}_1 \subseteq \mathcal{W}_2$, then $\mathcal{W}_2^\perp \subseteq \mathcal{W}_1^\perp$.

**13.** Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$.

(a) Show that if $\mathbf{v} \in \mathcal{W}$, then $\mathbf{proj}_{\mathcal{W}}\mathbf{v} = \mathbf{v}$.

(b) Show that if $\mathbf{v} \in \mathcal{W}^\perp$, then $\mathbf{proj}_{\mathcal{W}}\mathbf{v} = \mathbf{0}$.

**14.** Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$. Suppose that $\mathbf{v}$ is a nonzero vector with initial point at the origin and terminal point $P$. Prove that $\mathbf{v} \in \mathcal{W}^\perp$ if and only if the minimum distance between $P$ and $\mathcal{W}$ is $\|\mathbf{v}\|$.

**15.** We can represent matrices in $\mathcal{M}_{nn}$ as $n^2$-vectors by using their coordinatization with respect to the standard basis in $\mathcal{M}_{nn}$. Use this technique to prove that the orthogonal complement of the subspace $\mathcal{V}$ of symmetric matrices in $\mathcal{M}_{nn}$ is the subspace $\mathcal{W}$ of $n \times n$ skew-symmetric matrices. (Hint: First show that $\mathcal{W} \subseteq \mathcal{V}^\perp$. Then prove equality by showing that $\dim(\mathcal{W}) = n^2 - \dim(\mathcal{V})$.)

**16.** Show that if $\mathcal{W}$ is a one-dimensional subspace of $\mathbb{R}^n$ spanned by $\mathbf{a}$ and if $\mathbf{b} \in \mathbb{R}^n$, then the value of $\mathbf{proj}_{\mathcal{W}}\mathbf{b}$ agrees with the definition for $\mathbf{proj_a b}$ in Section 1.2.

▶ **17.** Prove Theorem 6.10.

**18.** Prove Corollary 6.14. (Hint: First show that $\mathcal{W} \subseteq \left(\mathcal{W}^\perp\right)^\perp$. Then use Corollary 6.13 to show that $\dim(\mathcal{W}) = \dim(\left(\mathcal{W}^\perp\right)^\perp)$, and apply Theorem 4.13.)

▶ **19.** Prove Theorem 6.17. (Hint: To prove $\ker(L) = \mathcal{W}^\perp$, first show that $\text{range}(L) = \mathcal{W}$. Hence, $\dim(\ker(L)) = n - \dim(\mathcal{W}) = \dim\left(\mathcal{W}^\perp\right)$ (why?). Finally, show $\mathcal{W}^\perp \subseteq \ker(L)$, and apply Theorem 4.13.)

**20.** Let $L\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with matrix $\mathbf{A}$ (with respect to the standard basis). Show that $\ker(L)$ is the orthogonal complement of the row space of $\mathbf{A}$.

**21.** Let $L\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Consider the mapping $T\colon (\ker(L))^\perp \to \mathbb{R}^m$ given by $T(\mathbf{v}) = L(\mathbf{v})$, for all $\mathbf{v} \in (\ker(L))^\perp$. ($T$ is the **restriction** of $L$ to $(\ker(L))^\perp$.) Prove that $T$ is one-to-one.

▶ **22.** Prove Theorem 6.18. (Hint: Suppose that $T$ is any point in $\mathcal{W}$ and $\mathbf{w}$ is the vector from the origin to $T$. We need to show that $\|\mathbf{v} - \mathbf{w}\| \geq \left\|\mathbf{v} - \mathbf{proj}_{\mathcal{W}}\mathbf{v}\right\|$; that is, the distance from $P$ to $T$ is at least as large as the distance from $P$ to the terminal point of $\mathbf{proj}_{\mathcal{W}}\mathbf{v}$. Let $\mathbf{a} = \mathbf{v} - \mathbf{proj}_{\mathcal{W}}\mathbf{v}$ and $\mathbf{b} = (\mathbf{proj}_{\mathcal{W}}\mathbf{v}) - \mathbf{w}$. Show that $\mathbf{a} \in \mathcal{W}^\perp$, $\mathbf{b} \in \mathcal{W}$, and $\|\mathbf{v} - \mathbf{w}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2$.)

**23.** Let $\mathcal{L}$ be a subspace of $\mathbb{R}^n$, and let $\mathcal{W}$ be a subspace of $\mathcal{L}$. We define the **orthogonal complement of $\mathcal{W}$ in $\mathcal{L}$** to be the set of all vectors in $\mathcal{L}$ that are orthogonal to every vector in $\mathcal{W}$.

(a) Prove that the orthogonal complement of $\mathcal{W}$ in $\mathcal{L}$ is a subspace of $\mathcal{L}$.

(b) Prove that the dimensions of $\mathcal{W}$ and its orthogonal complement in $\mathcal{L}$ add up to the dimension of $\mathcal{L}$. (Hint: Let $B$ be an orthonormal basis for $\mathcal{W}$. First enlarge $B$ to an orthonormal basis for $\mathcal{L}$, and then enlarge this basis to an orthonormal basis for $\mathbb{R}^n$.)

**24.** Let $\mathbf{A}$ be an $m \times n$ matrix and let $L_1 : \mathbb{R}^n \to \mathbb{R}^m$ and $L_2 : \mathbb{R}^m \to \mathbb{R}^n$ be given by $L_1(\mathbf{x}) = \mathbf{Ax}$, and $L_2(\mathbf{y}) = \mathbf{A}^T\mathbf{y}$.

(a) Prove that for all $\mathbf{v} \in \mathbb{R}^m$ and $\mathbf{w} \in \mathbb{R}^n$, $\mathbf{v} \cdot L_1(\mathbf{w}) = L_2(\mathbf{v}) \cdot \mathbf{w}$ (or, equivalently, $\mathbf{v} \cdot (\mathbf{Aw}) = (\mathbf{A}^T\mathbf{v}) \cdot \mathbf{w}$).

(b) Prove that $\ker(L_2) \subseteq (\text{range}(L_1))^\perp$. (Hint: Use part (a).)

(c) Prove that $\ker(L_2) = (\text{range}(L_1))^\perp$. (Hint: Use part (b) and the Dimension Theorem.)

(d) Show that $(\ker(L_1))^\perp$ equals the row space of $\mathbf{A}$. (Hint: Row space of $\mathbf{A}$ = column space of $\mathbf{A}^T = \text{range}(L_2)$.)

★ **25.** True or False:

(a) If $\mathcal{W}$ is a subspace of $\mathbb{R}^n$, then $\mathcal{W}^\perp = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \cdot \mathbf{w} = 0 \text{ for all } \mathbf{w} \in \mathcal{W}\}$.

(b) If $\mathcal{W}$ is a subspace of $\mathbb{R}^n$ and every vector in a basis for $\mathcal{W}$ is orthogonal to $\mathbf{v}$, then $\mathbf{v} \in \mathcal{W}^\perp$.

(c) If $\mathcal{W}$ is a subspace of $\mathbb{R}^n$, then $\mathcal{W} \cap \mathcal{W}^\perp = \{\ \}$.

(d) If $\mathcal{W}$ is a subspace of $\mathbb{R}^7$, and $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_7\}$ is a basis for $\mathbb{R}^7$ and $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4\}$ is a basis for $\mathcal{W}$, then $\{\mathbf{b}_5, \mathbf{b}_6, \mathbf{b}_7\}$ is a basis for $\mathcal{W}^\perp$.

(e) If $\mathcal{W}$ is a subspace of $\mathbb{R}^5$, then $\dim(\mathcal{W}^\perp) = 5 - \dim(\mathcal{W})$.

(f) If $\mathcal{W}$ is a subspace of $\mathbb{R}^n$, then every vector $\mathbf{v} \in \mathbb{R}^n$ lies in $\mathcal{W}$ or $\mathcal{W}^\perp$.

(g) The orthogonal complement of the orthogonal complement of a subspace $\mathcal{W}$ of $\mathbb{R}^n$ is $\mathcal{W}$ itself.

(h) The orthogonal complement of a plane through the origin in $\mathbb{R}^3$ is a line through the origin perpendicular to the plane.

(i) The mapping $L\colon \mathbb{R}^n \to \mathbb{R}^n$ given by $L(\mathbf{v}) = \mathbf{proj}_{\mathcal{W}}\mathbf{v}$, where $\mathcal{W}$ is a given subspace of $\mathbb{R}^n$, has $\mathcal{W}^\perp$ as its kernel.

(j) The matrix for an orthogonal projection onto a plane through the origin in $\mathbb{R}^3$ diagonalizes to a matrix with $-1, 1, 1$ on the main diagonal.

(k) If $\mathcal{W}$ is a subspace of $\mathbb{R}^n$, and $\mathbf{v} \in \mathbb{R}^n$, then the minimum distance from $\mathbf{v}$ to $\mathcal{W}$ is $\left\|\mathbf{proj}_{\mathcal{W}^\perp}\mathbf{v}\right\|$.

(l) If $\mathbf{v} \in \mathbb{R}^n$, and $\mathcal{W}$ is a subspace of $\mathbb{R}^n$, then $\mathbf{v} = \mathbf{proj}_{\mathcal{W}}\mathbf{v} + \mathbf{proj}_{\mathcal{W}^\perp}\mathbf{v}$.

## 6.3 Orthogonal Diagonalization

In this section, we determine which linear operators on $\mathbb{R}^n$ have an orthonormal basis $B$ of eigenvectors. Such operators are said to be orthogonally diagonalizable. For this type of operator, the transition matrix $\mathbf{P}$ from $B$-coordinates to standard coordinates is an orthogonal matrix. Such a change of basis preserves much of the geometric structure of $\mathbb{R}^n$, including lengths of vectors and the angles between them. Essentially, then, an orthogonally diagonalizable operator is one for which we can find a diagonal form while keeping certain important geometric properties of the operator.

We begin by defining symmetric operators and studying their properties. Then we show that these operators are precisely the ones that are orthogonally diagonalizable. Also, we present a method for orthogonally diagonalizing an operator analogous to the Generalized Diagonalization Method in Section 5.6. Finally, we introduce the Spectral Theorem which provides a sum-of-products form for a symmetric matrix.

### Symmetric Operators

**Definition** Let $\mathcal{V}$ be a subspace of $\mathbb{R}^n$. A linear operator $L\colon \mathcal{V} \to \mathcal{V}$ is a **symmetric operator** on $\mathcal{V}$ if and only if $L(\mathbf{v}_1) \cdot \mathbf{v}_2 = \mathbf{v}_1 \cdot L(\mathbf{v}_2)$, for every $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}$.

**Example 1**

The operator $L$ on $\mathbb{R}^3$ given by $L([a,b,c]) = [b,a,-c]$ is symmetric since

$$\begin{aligned} & L([a,b,c]) \cdot [d,e,f] = [b,a,-c] \cdot [d,e,f] = bd + ae - cf \\ \text{and} \quad & [a,b,c] \cdot L([d,e,f]) = [a,b,c] \cdot [e,d,-f] = ae + bd - cf. \end{aligned}$$

You can verify that the matrix representation for the operator $L$ in Example 1 with respect to the standard basis is

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

a symmetric matrix. The next theorem asserts that an operator on a subspace $\mathcal{V}$ of $\mathbb{R}^n$ is symmetric if and only if its matrix representation with respect to any orthonormal basis for $\mathcal{V}$ is symmetric.

**Theorem 6.19** *Let $\mathcal{V}$ be a nontrivial subspace of $\mathbb{R}^n$, $L$ be a linear operator on $\mathcal{V}$, $B$ be an ordered orthonormal basis for $\mathcal{V}$, and $\mathbf{A}_{BB}$ be the matrix for $L$ with respect to $B$. Then $L$ is a symmetric operator if and only if $\mathbf{A}_{BB}$ is a symmetric matrix.*

Theorem 6.19 gives a quick way of recognizing symmetric operators just by looking at their matrix representations. Such operators occur frequently in applications. (For example, see Section 8.11, "Quadratic Forms.") The proof of Theorem 6.19 is long, and so we have placed it in Appendix A for the interested reader.

### Orthogonally Diagonalizable Operators

We know that a linear operator $L$ on a finite dimensional vector space $\mathcal{V}$ can be diagonalized if we can find a basis for $\mathcal{V}$ consisting of eigenvectors for $L$. We now examine the special case where the basis of eigenvectors is *orthonormal*.

**Definition** Let $\mathcal{V}$ be a nontrivial subspace of $\mathbb{R}^n$, and let $L\colon \mathcal{V} \to \mathcal{V}$ be a linear operator. Then $L$ is an **orthogonally diagonalizable operator** if and only if there is an ordered orthonormal basis $B$ for $\mathcal{V}$ such that the matrix for $L$ with respect to $B$ is a diagonal matrix.

A square matrix $\mathbf{A}$ is **orthogonally diagonalizable** if and only if there is an orthogonal matrix $\mathbf{P}$ such that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is a diagonal matrix.

These two definitions are related by the next theorem.

**Theorem 6.20** *Let $L$ be a linear operator on a nontrivial subspace $\mathcal{V}$ of $\mathbb{R}^n$, and let $B$ be an ordered orthonormal basis for $\mathcal{V}$. Then $L$ is orthogonally diagonalizable if and only if the matrix for $L$ with respect to $B$ is orthogonally diagonalizable.*

*Proof.* Suppose $L$ is a linear operator on a nontrivial $k$-dimensional subspace $\mathcal{V}$ of $\mathbb{R}^n$, and let $B$ be an ordered orthonormal basis for $\mathcal{V}$.

If $L$ is orthogonally diagonalizable, then there is an ordered orthonormal basis $C$ for $\mathcal{V}$ such that the matrix for $L$ with respect to $C$ is a diagonal matrix $\mathbf{D}$. By part (b) of Exercise 20 in Section 6.1 (a generalization of Theorem 6.8), the transition matrix $\mathbf{P}$ from $C$ to $B$ is an orthogonal matrix. Then, if $\mathbf{A}$ is the matrix for $L$ with respect to $B$, $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$, by the remarks just before Example 6 in Section 5.2, and so $\mathbf{A}$ is an orthogonally diagonalizable matrix.

Conversely, suppose that the matrix $\mathbf{A}$ for $L$ with respect to $B$ is orthogonally diagonalizable. Then there is an orthogonal matrix $\mathbf{Q}$ such that $\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}$ is diagonal. Let $C = (\mathbf{v}_1, \ldots, \mathbf{v}_k)$, where $\mathbf{v}_i$ is the vector in $\mathcal{V}$ such that $[\mathbf{v}_i]_B = i$th column of $\mathbf{Q}$. Then $C$ is a basis for $\mathcal{V}$ (it has $k$ linearly independent vectors) and, by definition, $\mathbf{Q}$ is the transition matrix from $C$ to $B$. Then, because $\mathbf{Q}$ is an orthogonal matrix, part (c) of Exercise 20 in Section 6.1 implies that $C$ is an orthonormal basis for $\mathcal{V}$. But the matrix for $L$ with respect to $C$ is the diagonal matrix $\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}$, thus proving that $L$ is orthogonally diagonalizable. □

**Example 2**

Let $L: \mathbb{R}^4 \to \mathbb{R}^4$ be the operator given by $L(\mathbf{v}) = \mathbf{A}\mathbf{v}$, where

$$\mathbf{A} = \frac{1}{7}\begin{bmatrix} 15 & -21 & -3 & -5 \\ -21 & 35 & -7 & 0 \\ -3 & -7 & 23 & 15 \\ -5 & 0 & 15 & 39 \end{bmatrix}.$$

Also consider the *orthogonal* matrix

$$\mathbf{P} = \frac{1}{\sqrt{14}}\begin{bmatrix} 3 & 1 & -2 & 0 \\ 2 & 0 & 3 & -1 \\ 1 & -3 & 0 & 2 \\ 0 & 2 & 1 & 3 \end{bmatrix}.$$

You can verify that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{P}^T\mathbf{A}\mathbf{P} = \mathbf{D}$, where

$$\mathbf{D} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

is a diagonal matrix. Then, by definition, the matrix $\mathbf{A}$ is orthogonally diagonalizable. Therefore, by Theorem 6.20, $L$ is an orthogonally diagonalizable operator. In fact, the columns of $\mathbf{P}$ form the following ordered orthonormal basis for $\mathbb{R}^4$:

$$\left(\frac{1}{\sqrt{14}}[3, 2, 1, 0], \frac{1}{\sqrt{14}}[1, 0, -3, 2], \frac{1}{\sqrt{14}}[-2, 3, 0, 1], \frac{1}{\sqrt{14}}[0, -1, 2, 3]\right).$$

Notice that the matrix for $L$ with respect to this basis is the diagonal matrix $\mathbf{D}$.

## A Symmetric Operator Always Has an Eigenvalue

The following lemma is needed for the proof of Theorem 6.22, the main theorem of this section:

**Lemma 6.21** *Let $L$ be a symmetric operator on a nontrivial subspace $\mathcal{V}$ of $\mathbb{R}^n$. Then $L$ has at least one eigenvalue.*

Lemma 6.21 is a direct corollary of Theorem 7.11 in Section 7.4, which has a short, easy proof, but uses complex vectors and matrices.[3] An alternate but more complicated proof of Lemma 6.21 that avoids the use of complex numbers is outlined in Exercise 14.

**Example 3**

The operator $L([a,b,c]) = [b,a,-c]$ on $\mathbb{R}^3$ is symmetric, as shown in Example 1. Lemma 6.21 then states that $L$ has at least one eigenvalue. In fact, $L$ has two eigenvalues, which are $\lambda_1 = 1$ and $\lambda_2 = -1$. The eigenspaces $E_{\lambda_1}$ and $E_{\lambda_2}$ have bases $\{[1,1,0]\}$ and $\{[1,-1,0],[0,0,1]\}$, respectively.

## Equivalence of Symmetric and Orthogonally Diagonalizable Operators

We are now ready to show that symmetric operators and orthogonally diagonalizable operators are really the same.

**Theorem 6.22** *Let $\mathcal{V}$ be a nontrivial subspace of $\mathbb{R}^n$, and let $L$ be a linear operator on $\mathcal{V}$. Then $L$ is orthogonally diagonalizable if and only if $L$ is symmetric.*

*Proof.* Suppose that $L$ is a linear operator on a nontrivial subspace $\mathcal{V}$ of $\mathbb{R}^n$.

First, we show that if $L$ is orthogonally diagonalizable, then $L$ is symmetric. Suppose $L$ is orthogonally diagonalizable. Then, by definition, there is an ordered orthonormal basis $B$ for $\mathcal{V}$ such that the matrix representation $\mathbf{A}$ for $L$ with respect to $B$ is diagonal. Since every diagonal matrix is also symmetric, $L$ is a symmetric operator by Theorem 6.19.

To finish the proof, we must show that if $L$ is a symmetric operator, then $L$ is orthogonally diagonalizable. Suppose $L$ is symmetric. If $L$ has an ordered orthonormal basis $B$ consisting entirely of eigenvectors of $L$, then, clearly, the matrix for $L$ with respect to $B$ is a diagonal matrix (having the eigenvalues corresponding to the eigenvectors in $B$ along its main diagonal), and then $L$ is orthogonally diagonalizable. Therefore, our goal is to find an orthonormal basis of eigenvectors for $L$. We give a proof by induction on $\dim(\mathcal{V})$.

**Base Step:** Assume that $\dim(\mathcal{V}) = 1$. Normalize any nonzero vector in $\mathcal{V}$ to obtain a unit vector $\mathbf{u} \in \mathcal{V}$. Then, $\{\mathbf{u}\}$ is an orthonormal basis for $\mathcal{V}$. Since $L(\mathbf{u}) \in \mathcal{V}$ and $\{\mathbf{u}\}$ is a basis for $\mathcal{V}$, we must have $L(\mathbf{u}) = \lambda\mathbf{u}$, for some real number $\lambda$, and so $\lambda$ is an eigenvalue for $L$. Hence, $\{\mathbf{u}\}$ is an orthonormal basis of eigenvectors for $\mathcal{V}$, thus completing the Base Step.

**Inductive Step:** The inductive hypothesis is as follows:

If $\mathcal{W}$ is a subspace of $\mathbb{R}^n$ with dimension $k$, and $T$ is any symmetric operator on $\mathcal{W}$, then $\mathcal{W}$ has an orthonormal basis of eigenvectors for $T$.

We must prove the following:

If $\mathcal{V}$ is a subspace of $\mathbb{R}^n$ with dimension $k+1$, and $L$ is a symmetric operator on $\mathcal{V}$, then $\mathcal{V}$ has an orthonormal basis of eigenvectors for $L$.

Now, $L$ has at least one eigenvalue $\lambda$, by Lemma 6.21. Take any eigenvector for $L$ corresponding to $\lambda$ and normalize it to create a unit eigenvector $\mathbf{v}$. Let $\mathcal{Y} = \operatorname{span}(\{\mathbf{v}\})$. Now, we want to enlarge $\{\mathbf{v}\}$ to an orthonormal basis of eigenvectors for $L$ in $\mathcal{V}$.

Our goal is to find a subspace $\mathcal{W}$ of $\mathcal{V}$ of dimension $k$ that is orthogonal to $\mathcal{Y}$, together with a symmetric operator on $\mathcal{W}$. We can then invoke the inductive hypothesis to find the remaining orthonormal basis vectors for $L$.

Since $\dim(\mathcal{V}) = k+1$, we can use the Gram-Schmidt Process to find vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ such that $\{\mathbf{v}, \mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an orthonormal basis for $\mathcal{V}$ containing $\mathbf{v}$. Since $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are orthogonal to $\mathbf{v}$, we have $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subseteq \mathcal{Y}^\perp \cap \mathcal{V}$, the orthogonal complement of $\mathcal{Y}$ in $\mathcal{V}$ (see Exercise 23 in Section 6.2). Let $\mathcal{W} = \mathcal{Y}^\perp \cap \mathcal{V}$. Since $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a linearly independent subset of $\mathcal{W}$, $\dim(\mathcal{W}) \geq k$. But $\mathbf{v} \notin \mathcal{W}$ implies $\dim(\mathcal{W}) < \dim(\mathcal{V}) = k+1$, and so $\dim(\mathcal{W}) = k$.

Next, we claim that for every $\mathbf{w} \in \mathcal{W}$, we have $L(\mathbf{w}) \in \mathcal{W}$. For,

$$\begin{aligned}
\mathbf{v} \cdot L(\mathbf{w}) &= L(\mathbf{v}) \cdot \mathbf{w} && \text{since } L \text{ is symmetric} \\
&= (\lambda\mathbf{v}) \cdot \mathbf{w} && \text{since } \lambda \text{ is an eigenvalue for } L \\
&= \lambda(\mathbf{v} \cdot \mathbf{w}) = \lambda(0) = 0,
\end{aligned}$$

[3] The proof of Theorem 7.11 is independent of Section 6.3.

which shows that $L(\mathbf{w})$ is orthogonal to $\mathbf{v}$ and hence is in $\mathcal{W}$. Therefore, we can define a linear operator $T: \mathcal{W} \to \mathcal{W}$ by $T(\mathbf{w}) = L(\mathbf{w})$. ($T$ is the **restriction** of $L$ to $\mathcal{W}$.) Now, $T$ is a symmetric operator on $\mathcal{W}$, since, for every $\mathbf{w}_1, \mathbf{w}_2 \in \mathcal{W}$,

$$\begin{aligned} T(\mathbf{w}_1)\cdot\mathbf{w}_2 &= L(\mathbf{w}_1)\cdot\mathbf{w}_2 && \text{definition of } T \\ &= \mathbf{w}_1\cdot L(\mathbf{w}_2) && \text{since } L \text{ is symmetric} \\ &= \mathbf{w}_1\cdot T(\mathbf{w}_2). && \text{definition of } T \end{aligned}$$

Since $\dim(\mathcal{W}) = k$, the inductive hypothesis implies that $\mathcal{W}$ has an orthonormal basis $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ of eigenvectors for $T$. Then, by definition of $T$, $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is also a set of eigenvectors for $L$, all of which are orthogonal to $\mathbf{v}$ (since they are in $\mathcal{W}$). Hence, $B = \{\mathbf{v}, \mathbf{u}_1, \dots, \mathbf{u}_k\}$ is an orthonormal basis for $\mathcal{V}$ of eigenvectors for $L$, and we have finished the proof of the Inductive Step. □

The following corollary now follows immediately from Theorems 6.19, 6.20, and 6.22.

**Corollary 6.23** *If* $\mathbf{A}$ *is a square matrix, then* $\mathbf{A}$ *is symmetric if and only if* $\mathbf{A}$ *is orthogonally diagonalizable.*

*Proof.* Let $\mathbf{A}$ be an $n \times n$ matrix. Consider the linear operator $L$ on $\mathbb{R}^n$ having matrix $\mathbf{A}$ with respect to the standard basis (which is orthonormal). By Theorem 6.19, $\mathbf{A}$ is symmetric if and only if $L$ is a symmetric operator. By Theorem 6.22, $L$ is symmetric if and only if $L$ is orthogonally diagonalizable. But, by Theorem 6.20, $L$ is orthogonally diagonalizable if and only if $\mathbf{A}$ is orthogonally diagonalizable. □

## Method for Orthogonally Diagonalizing a Linear Operator

We now present a method for orthogonally diagonalizing a symmetric operator, based on Theorem 6.22. You should compare this to the method for diagonalizing a linear operator given in Section 5.6. Notice that the following method assumes that eigenvectors of a symmetric operator corresponding to distinct eigenvalues are orthogonal. The proof of this is left as Exercise 11.

**Method for Orthogonally Diagonalizing a Symmetric Operator (Orthogonal Diagonalization Method)**
Let $L: \mathcal{V} \to \mathcal{V}$ be a symmetric operator on a nontrivial subspace $\mathcal{V}$ of $\mathbb{R}^n$, with $\dim(\mathcal{V}) = k$.

**Step 1:** Find an ordered orthonormal basis $C$ for $\mathcal{V}$ (if $\mathcal{V} = \mathbb{R}^n$, we can use the standard basis), and calculate the matrix representation $\mathbf{A}$ for $L$ with respect to $C$ (which should be a $k \times k$ symmetric matrix).
**Step 2:** (a) Apply the Diagonalization Method of Section 3.4 to $\mathbf{A}$ in order to obtain all of the eigenvalues $\lambda_1, \dots, \lambda_k$ of $\mathbf{A}$, and a basis in $\mathbb{R}^k$ for each eigenspace $E_{\lambda_i}$ of $\mathbf{A}$ (by solving an appropriate homogeneous system if necessary).
(b) Perform the Gram-Schmidt Process on the basis for each $E_{\lambda_i}$ from Step 2(a), and then normalize to get an orthonormal basis for each $E_{\lambda_i}$.
(c) Let $Z = (\mathbf{z}_1, \dots, \mathbf{z}_k)$ be an ordered basis for $\mathbb{R}^k$ consisting of the union of the orthonormal bases for the $E_{\lambda_i}$.
**Step 3:** Reverse the $C$-coordinatization isomorphism on the vectors in $Z$ to obtain an ordered orthonormal basis $B = (\mathbf{v}_1, \dots, \mathbf{v}_k)$ for $\mathcal{V}$; that is, $[\mathbf{v}_i]_C = \mathbf{z}_i$.

The matrix representation for $L$ with respect to $B$ is the diagonal matrix $\mathbf{D}$, where $d_{ii}$ is the eigenvalue for $L$ corresponding to $\mathbf{v}_i$. In most practical situations, the transition matrix $\mathbf{P}$ from $B$- to $C$-coordinates is useful. $\mathbf{P}$ is the $k \times k$ matrix whose columns are $[\mathbf{v}_1]_C, \dots, [\mathbf{v}_k]_C$—that is, the vectors $\mathbf{z}_1, \dots, \mathbf{z}_k$ in $Z$. Note that $\mathbf{P}$ is an orthogonal matrix, and $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{P}^T\mathbf{A}\mathbf{P}$.

The following example illustrates this method, while revealing how the orthonormal basis for the operator given in Example 2 is obtained.

### Example 4

Consider the operator $L: \mathbb{R}^4 \to \mathbb{R}^4$ given by $L(\mathbf{v}) = \mathbf{A}\mathbf{v}$, where

$$\mathbf{A} = \frac{1}{7}\begin{bmatrix} 15 & -21 & -3 & -5 \\ -21 & 35 & -7 & 0 \\ -3 & -7 & 23 & 15 \\ -5 & 0 & 15 & 39 \end{bmatrix}.$$

$L$ is clearly symmetric, since its matrix $\mathbf{A}$ with respect to the standard basis $C$ for $\mathbb{R}^4$ is symmetric. We find an orthonormal basis $B$ such that the matrix for $L$ with respect to $B$ is diagonal.

**Step 1:** We have already seen that $\mathbf{A}$ is the matrix for $L$ with respect to the standard basis $C$ for $\mathbb{R}^4$.

**Step 2:** (a) A lengthy calculation yields

$$p_{\mathbf{A}}(x) = x^4 - 16x^3 + 77x^2 - 98x = x(x-2)(x-7)^2,$$

giving eigenvalues $\lambda_1 = 0$, $\lambda_2 = 2$, and $\lambda_3 = 7$. Solving the appropriate homogeneous systems to find bases for the eigenspaces produces:

$$\text{Basis for } E_{\lambda_1} = \{[3,2,1,0]\}$$
$$\text{Basis for } E_{\lambda_2} = \{[1,0,-3,2]\}$$
$$\text{Basis for } E_{\lambda_3} = \{[-2,3,0,1],[3,-5,1,0]\}.$$

(b) There is no need to perform the Gram-Schmidt Process on the bases for $E_{\lambda_1}$ and $E_{\lambda_2}$, since each of these eigenspaces is one-dimensional. Normalizing the basis vectors yields:

$$\text{Orthonormal basis for } E_{\lambda_1} = \left\{\frac{1}{\sqrt{14}}[3,2,1,0]\right\}$$
$$\text{Orthonormal basis for } E_{\lambda_2} = \left\{\frac{1}{\sqrt{14}}[1,0,-3,2]\right\}.$$

Let us label the vectors in these bases as $\mathbf{z}_1$, $\mathbf{z}_2$, respectively. However, we must perform the Gram-Schmidt Process on the basis for $E_{\lambda_3}$. Take $[-2,3,0,1]$, the first basis vector for $E_{\lambda_3}$, as the first vector in the Gram-Schmidt Process, and calculate the second vector in the Gram-Schmidt Process using the second basis vector $[3,-5,1,0]$ as follows:

$$[3,-5,1,0] - \left(\frac{[3,-5,1,0]\cdot[-2,3,0,1]}{[-2,3,0,1]\cdot[-2,3,0,1]}\right)[-2,3,0,1] = \left[0,-\frac{1}{2},1,\frac{3}{2}\right].$$

Finally, normalizing these two vectors, we obtain:

$$\text{Orthonormal basis for } E_{\lambda_3} = \left\{\frac{1}{\sqrt{14}}[-2,3,0,1],\ \frac{1}{\sqrt{14}}[0,-1,2,3]\right\}.$$

Let us label the vectors in this basis as $\mathbf{z}_3$, $\mathbf{z}_4$, respectively.

(c) We let

$$Z = (\mathbf{z}_1,\mathbf{z}_2,\mathbf{z}_3,\mathbf{z}_4) = \left(\frac{1}{\sqrt{14}}[3,2,1,0],\ \frac{1}{\sqrt{14}}[1,0,-3,2],\ \frac{1}{\sqrt{14}}[-2,3,0,1],\ \frac{1}{\sqrt{14}}[0,-1,2,3]\right)$$

be the union of the orthonormal bases for $E_{\lambda_1}$, $E_{\lambda_2}$, and $E_{\lambda_3}$.

**Step 3:** Since $C$ is the standard basis for $\mathbb{R}^4$, the $C$-coordinatization isomorphism is the identity mapping, so $\mathbf{v}_1 = \mathbf{z}_1$, $\mathbf{v}_2 = \mathbf{z}_2$, $\mathbf{v}_3 = \mathbf{z}_3$, and $\mathbf{v}_4 = \mathbf{z}_4$ here, and $B = (\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4)$ is an ordered orthonormal basis for $\mathbb{R}^4$. The matrix representation $\mathbf{D}$ of $L$ with respect to $B$ is

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}.$$

The transition matrix $\mathbf{P}$ from $B$ to $C$ is the *orthogonal* matrix

$$\mathbf{P} = \frac{1}{\sqrt{14}}\begin{bmatrix} 3 & 1 & -2 & 0 \\ 2 & 0 & 3 & -1 \\ 1 & -3 & 0 & 2 \\ 0 & 2 & 1 & 3 \end{bmatrix}.$$

You can verify that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{P}^T\mathbf{A}\mathbf{P} = \mathbf{D}$.

## The Spectral Theorem

Theorem 6.22 and Corollary 6.23 imply that it is possible to express each orthogonally diagonalizable (= symmetric) matrix as a linear combination of matrix products involving unit eigenvectors. This result is commonly referred to as the Spectral Theorem.

**Theorem 6.24 (Spectral Theorem)** *If* $\mathbf{A}$ *is an* $n \times n$ *orthogonally diagonalizable (= symmetric) matrix, then*

$$\mathbf{A} = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \lambda_2 \mathbf{u}_2 \mathbf{u}_2^T + \cdots + \lambda_n \mathbf{u}_n \mathbf{u}_n^T,$$

*where* $\lambda_1, \ldots, \lambda_n$ *are the (not necessarily distinct) eigenvalues of* $\mathbf{A}$, *and* $\mathbf{u}_1, \ldots, \mathbf{u}_n$ *are corresponding column eigenvectors that form an orthonormal basis for* $\mathbb{R}^n$.

*Proof.* Let $\mathbf{A}$ be an $n \times n$ orthogonally diagonalizable matrix, with $\lambda_1, \ldots, \lambda_n$ as the (not necessarily distinct) eigenvalues of $\mathbf{A}$, and $\mathbf{u}_1, \ldots, \mathbf{u}_n$ as corresponding column eigenvectors forming an orthonormal basis for $\mathbb{R}^n$. Consider the linear operator $L_1(\mathbf{v}) = \mathbf{A}\mathbf{v}$ on $\mathbb{R}^n$. Since each $\mathbf{u}_i$ is an eigenvector for $\mathbf{A}$, we have $L_1(\mathbf{u}_i) = \mathbf{A}\mathbf{u}_i = \lambda_i \mathbf{u}_i$, for $1 \le i \le n$. Next, from the right-hand side of the equation in the theorem, consider the linear operator $L_2(\mathbf{v}) = (\lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \cdots + \lambda_i \mathbf{u}_i \mathbf{u}_i^T + \cdots + \lambda_n \mathbf{u}_n \mathbf{u}_n^T)\mathbf{v}$ on $\mathbb{R}^n$. Notice that, for $1 \le i \le n$, $L_2(\mathbf{u}_i) = (\lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \cdots + \lambda_i \mathbf{u}_i \mathbf{u}_i^T + \cdots + \lambda_n \mathbf{u}_n \mathbf{u}_n^T)\mathbf{u}_i = (\lambda_1 \mathbf{u}_1 \mathbf{u}_1^T \mathbf{u}_i) + \cdots + (\lambda_i \mathbf{u}_i \mathbf{u}_i^T \mathbf{u}_i) + \cdots + (\lambda_n \mathbf{u}_n \mathbf{u}_n^T \mathbf{u}_i)$, which reduces to $\lambda_i \mathbf{u}_i$ (since $\mathbf{u}_j^T \mathbf{u}_i = 0$, for $j \ne i$, and $\mathbf{u}_i^T \mathbf{u}_i = 1$). Thus, $L_1$ and $L_2$ have the same effect on the basis $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ for $\mathbb{R}^n$, and must be identical linear operators by Theorem 5.4. Therefore, their matrices with respect to the standard basis for $\mathbb{R}^n$ must be equal. □

## Example 5

Consider the matrix $\mathbf{A}$ in Example 4. In that example, we found the eigenvalues for $\mathbf{A}$ to be $\lambda_1 = 0$, $\lambda_2 = 2$, and $\lambda_3 = \lambda_4 = 7$ (since the eigenspace for this eigenvalue is two-dimensional). The corresponding unit eigenvectors are $\mathbf{u}_1 = \frac{1}{\sqrt{14}}[3, 2, 1, 0]$, $\mathbf{u}_2 = \frac{1}{\sqrt{14}}[1, 0, -3, 2]$, $\mathbf{u}_3 = \frac{1}{\sqrt{14}}[-2, 3, 0, 1]$, and $\mathbf{u}_4 = \frac{1}{\sqrt{14}}[0, -1, 2, 3]$. Notice that

$$\lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \lambda_2 \mathbf{u}_2 \mathbf{u}_2^T + \lambda_3 \mathbf{u}_3 \mathbf{u}_3^T + \lambda_4 \mathbf{u}_4 \mathbf{u}_4^T = \left( \begin{array}{l} 0\left(\frac{1}{\sqrt{14}}\right)\begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}\left(\frac{1}{\sqrt{14}}\right)\begin{bmatrix} 3 & 2 & 1 & 0 \end{bmatrix} \\ + 2\left(\frac{1}{\sqrt{14}}\right)\begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}\left(\frac{1}{\sqrt{14}}\right)\begin{bmatrix} 1 & 0 & -3 & 2 \end{bmatrix} \\ + 7\left(\frac{1}{\sqrt{14}}\right)\begin{bmatrix} -2 \\ 3 \\ 0 \\ 1 \end{bmatrix}\left(\frac{1}{\sqrt{14}}\right)\begin{bmatrix} -2 & 3 & 0 & 1 \end{bmatrix} \\ + 7\left(\frac{1}{\sqrt{14}}\right)\begin{bmatrix} 0 \\ -1 \\ 2 \\ 3 \end{bmatrix}\left(\frac{1}{\sqrt{14}}\right)\begin{bmatrix} 0 & -1 & 2 & 3 \end{bmatrix} \end{array} \right)$$

$$= \frac{1}{14}\left( \begin{array}{l} 0\begin{bmatrix} 9 & 6 & 3 & 0 \\ 6 & 4 & 2 & 0 \\ 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \\ -3 & 0 & 9 & -6 \\ 2 & 0 & -6 & 4 \end{bmatrix} \\ + 7\begin{bmatrix} 4 & -6 & 0 & -2 \\ -6 & 9 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ -2 & 3 & 0 & 1 \end{bmatrix} + 7\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & -3 \\ 0 & -2 & 4 & 6 \\ 0 & -3 & 6 & 9 \end{bmatrix} \end{array} \right)$$

$$= \frac{1}{14}\left( \begin{bmatrix} 30 & -42 & -6 & -10 \\ -42 & 70 & -14 & 21 \\ -6 & -14 & 46 & 30 \\ -10 & 0 & 30 & 78 \end{bmatrix} \right) = \frac{1}{7}\begin{bmatrix} 15 & -21 & -3 & -5 \\ -21 & 35 & -7 & 0 \\ -3 & -7 & 23 & 15 \\ -5 & 0 & 15 & 39 \end{bmatrix} = \mathbf{A}.$$

The Spectral Theorem has a close analog in the Outer Product form of the Singular Value Decomposition of a matrix (Theorem 9.7 in Section 9.5).

We conclude by examining a symmetric operator whose domain is a proper subspace of $\mathbb{R}^n$.

## Example 6

Consider the operators $L_1$, $L_2$, and $L_3$ on $\mathbb{R}^3$ given by

$L_1$: orthogonal projection onto the plane $x+y+z=0$
$L_2$: orthogonal projection onto the plane $x+y-z=0$
$L_3$: orthogonal projection onto the $xy$-plane (that is, $z=0$).

Let $L: \mathbb{R}^3 \to \mathbb{R}^3$ be given by $L = L_3 \circ L_2 \circ L_1$, and let $\mathcal{V}$ be the $xy$-plane in $\mathbb{R}^3$. Then, since $\text{range}(L_3) = \mathcal{V}$, we see that $\text{range}(L) \subseteq \mathcal{V}$. Thus, restricting the domain of $L$ to $\mathcal{V}$, we can think of $L$ as a linear operator on $\mathcal{V}$. We will show that $L$ is a symmetric operator on $\mathcal{V}$ and orthogonally diagonalize $L$.

**Step 1:** Choose $C = ([1,0,0],[0,1,0])$ as an ordered orthonormal basis for $\mathcal{V}$. We need to calculate the matrix representation $\mathbf{A}$ of $L$ with respect to $C$. Using the orthonormal basis $\left\{\frac{1}{\sqrt{2}}[1,-1,0], \frac{1}{\sqrt{6}}[1,1,-2]\right\}$ for the plane $x+y+z=0$, the orthonormal basis $\left\{\frac{1}{\sqrt{2}}[1,-1,0], \frac{1}{\sqrt{6}}[1,1,2]\right\}$ for the plane $x+y-z=0$, and the orthonormal basis $C$ for the $xy$-plane, we can use the method of Example 7 in Section 6.2 to compute the required orthogonal projections.

$$L([1,0,0]) = L_3(L_2(L_1([1,0,0]))) = L_3\left(L_2\left(\left[\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right]\right)\right) = L_3\left(\left[\frac{4}{9}, -\frac{5}{9}, -\frac{1}{9}\right]\right) = \frac{1}{9}[4,-5,0]$$

$$\text{and} \quad L([0,1,0]) = L_3(L_2(L_1([0,1,0]))) = L_3\left(L_2\left(\left[-\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}\right]\right)\right) = L_3\left(\left[-\frac{5}{9}, \frac{4}{9}, -\frac{1}{9}\right]\right) = \frac{1}{9}[-5,4,0].$$

Expressing these vectors in $C$-coordinates, we see that the matrix representation of $L$ with respect to $C$ is $\mathbf{A} = \frac{1}{9}\begin{bmatrix} 4 & -5 \\ -5 & 4 \end{bmatrix}$, a symmetric matrix. Thus, by Theorem 6.19, $L$ is a symmetric operator on $\mathcal{V}$.[4] Hence, $L$ is, indeed, orthogonally diagonalizable.

**Step 2:** (a) The characteristic polynomial for $\mathbf{A}$ is $p_{\mathbf{A}}(x) = x^2 - \frac{8}{9}x - \frac{1}{9} = (x-1)\left(x+\frac{1}{9}\right)$, giving eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -\frac{1}{9}$. Solving the appropriate homogeneous systems to find bases for these eigenspaces yields:

$$\text{Basis for } E_{\lambda_1} = \{[1,-1]\}, \qquad \text{Basis for } E_{\lambda_2} = \{[1,1]\}.$$

Notice that we expressed the bases in $C$-coordinates.

(b) Since the eigenspaces are one-dimensional, there is no need to perform the Gram-Schmidt Process on the bases for $E_{\lambda_1}$ and $E_{\lambda_2}$. Normalizing the basis vectors produces:

$$\text{Orthonormal basis for } E_{\lambda_1} = \left\{\frac{1}{\sqrt{2}}[1,-1]\right\}$$
$$\text{Orthonormal basis for } E_{\lambda_2} = \left\{\frac{1}{\sqrt{2}}[1,1]\right\}.$$

Let us denote these vectors as $\mathbf{z}_1$, $\mathbf{z}_2$, respectively.

(c) Let $Z = (\mathbf{z}_1, \mathbf{z}_2)$ be the union of the (ordered) orthonormal bases for $E_{\lambda_1}$ and $E_{\lambda_2}$.

**Step 3:** Reversing the $C$-coordinatization isomorphism on $Z$, we obtain $\mathbf{v}_1 = \frac{1}{\sqrt{2}}[1,-1,0]$, and $\mathbf{v}_2 = \frac{1}{\sqrt{2}}[1,1,0]$, respectively. Thus, an ordered orthonormal basis in $\mathbb{R}^3$ for $\mathcal{V}$ is $B = (\mathbf{v}_1, \mathbf{v}_2)$. The matrix $\mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{9} \end{bmatrix}$ is the matrix representation for $L$ with respect to $B$. The transition matrix $\mathbf{P} = \left(\frac{1}{\sqrt{2}}\right)\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ from $B$ to $C$ is the orthogonal matrix whose columns are the vectors in $B$ expressed in $C$-coordinates. You can verify that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{P}^T\mathbf{AP} = \mathbf{D}$.

Finally, notice that the Spectral Theorem holds for the matrix $\mathbf{A}$ since

$$\lambda_1\mathbf{u}_1\mathbf{u}_1^T + \lambda_2\mathbf{u}_2\mathbf{u}_2^T = 1\left(\frac{1}{\sqrt{2}}\right)\begin{bmatrix} 1 \\ -1 \end{bmatrix}\left(\frac{1}{\sqrt{2}}\right)[1,-1] + \left(-\frac{1}{9}\right)\left(\frac{1}{\sqrt{2}}\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix}\left(\frac{1}{\sqrt{2}}\right)[1,1]$$
$$= \frac{9}{18}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \frac{1}{18}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 4 & -5 \\ -5 & 4 \end{bmatrix} = \mathbf{A}.$$

- **Supplemental Material**: You have now covered the prerequisites for Section 7.4, "Orthogonality in $\mathbb{C}^n$," and Section 7.5, "Inner Product Spaces."
- **Application**: You have now covered the prerequisites for Section 8.11, "Quadratic Forms."

[4] You can easily verify that $L$ is *not* a symmetric operator on all of $\mathbb{R}^3$, even though it is symmetric on the subspace $\mathcal{V}$.

## New Vocabulary

Orthogonal Diagonalization Method
orthogonally diagonalizable matrix
orthogonally diagonalizable operator
Spectral Theorem
symmetric operator

## Highlights

- A linear operator $L$ on a subspace $\mathcal{V}$ of $\mathbb{R}^n$ is symmetric if and only if, for every $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}$, we have $L(\mathbf{v}_1) \cdot \mathbf{v}_2 = \mathbf{v}_1 \cdot L(\mathbf{v}_2)$.
- Let $L$ be a linear operator on a nontrivial subspace $\mathcal{V}$ of $\mathbb{R}^n$, let $B$ be an ordered orthonormal basis for $\mathcal{V}$, and let $\mathbf{A}$ be the matrix for $L$ with respect to $B$. Then $L$ is symmetric if and only if $\mathbf{A}$ is a symmetric matrix.
- A matrix $\mathbf{A}$ is orthogonally diagonalizable if and only if there is some orthogonal matrix $\mathbf{P}$ such that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is a diagonal matrix.
- Let $L$ be a linear operator on a nontrivial subspace $\mathcal{V}$ of $\mathbb{R}^n$. Then $L$ is orthogonally diagonalizable if and only if there is an ordered orthonormal basis $B$ for $\mathcal{V}$ such that the matrix for $L$ with respect to $B$ is a diagonal matrix.
- Let $L$ be a linear operator on a nontrivial subspace $\mathcal{V}$ of $\mathbb{R}^n$, and let $B$ be an ordered orthonormal basis for $\mathcal{V}$. Then $L$ is orthogonally diagonalizable if and only if the matrix for $L$ with respect to $B$ is orthogonally diagonalizable.
- A linear operator $L$ on a nontrivial subspace of $\mathbb{R}^n$ is orthogonally diagonalizable if and only if $L$ is symmetric.
- A square matrix $\mathbf{A}$ is symmetric if and only if $\mathbf{A}$ is orthogonally diagonalizable.
- If $L$ is a symmetric linear operator on a nontrivial subspace $\mathcal{V}$ of $\mathbb{R}^n$, with matrix $\mathbf{A}$ with respect to an ordered orthonormal basis for $\mathcal{V}$, then the Orthogonal Diagonalization Method produces an orthogonal matrix $\mathbf{P}$ such that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{P}^T\mathbf{A}\mathbf{P}$ is diagonal.
- Spectral Theorem: If $\mathbf{A}$ is an $n \times n$ orthogonally diagonalizable ($=$ symmetric) matrix, then $\mathbf{A} = \lambda_1\mathbf{u}_1\mathbf{u}_1^T + \lambda_2\mathbf{u}_2\mathbf{u}_2^T + \cdots + \lambda_n\mathbf{u}_n\mathbf{u}_n^T$, where $\lambda_1, \ldots, \lambda_n$ are the (not necessarily distinct) eigenvalues of $\mathbf{A}$, and $\mathbf{u}_1, \ldots, \mathbf{u}_n$ are corresponding column eigenvectors that form an orthonormal basis for $\mathbb{R}^n$.

## Exercises for Section 6.3

**Note**: Use a calculator or computer (when needed) in solving for eigenvalues and eigenvectors and performing the Gram-Schmidt Process.

1. Determine which of the following linear operators are symmetric. Explain why each is, or is not, symmetric.

★ **(a)** $L: \mathbb{R}^2 \to \mathbb{R}^2$ given by $L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3x+2y \\ 2x+5y \end{bmatrix}$

**(b)** $L: \mathbb{R}^3 \to \mathbb{R}^3$ given as the shear in the $z$-direction with factor 2. (See Table 5.1 in Section 5.2.)

**(c)** $L: \mathbb{R}^3 \to \mathbb{R}^3$, where $L([1,-3,6]) = [23,-10,24]$, $L([4,5,-2]) = [-3,19,9]$, and $L([2,1,7]) = [24,10,35]$

★ **(d)** $L: \mathbb{R}^3 \to \mathbb{R}^3$ given by the orthogonal projection onto the plane $ax + by + cz = 0$

★ **(e)** $L: \mathbb{R}^3 \to \mathbb{R}^3$ given by a counterclockwise rotation through an angle of $\frac{\pi}{3}$ radians about the line through the origin in the direction $[1,1,-1]$

**(f)** $L: \mathbb{R}^3 \to \mathbb{R}^3$ given by $L = L_2 \circ L_1$, where $L_1$ is the orthogonal reflection through the $xy$-plane, and $L_2$ is the orthogonal projection onto the plane $x + 2y - 2z = 0$

★ **(g)** $L: \mathbb{R}^4 \to \mathbb{R}^4$ given by $L = L_1^{-1} \circ L_2 \circ L_1$ where $L_1: \mathbb{R}^4 \to \mathcal{M}_{22}$ is given by $L_1([a,b,c,d]) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and $L_2: \mathcal{M}_{22} \to \mathcal{M}_{22}$ is given by $L_2(\mathbf{K}) = \begin{bmatrix} 4 & 3 \\ 3 & 9 \end{bmatrix}\mathbf{K}$

2. In each part, find a symmetric matrix having the given eigenvalues and the given bases for their associated eigenspaces.

★ **(a)** $\lambda_1 = 1, \lambda_2 = -1, E_{\lambda_1} = \text{span}\left(\left\{\frac{1}{5}[3,4]\right\}\right), E_{\lambda_2} = \text{span}\left(\left\{\frac{1}{5}[4,-3]\right\}\right)$

**(b)** $\lambda_1 = 0, \lambda_2 = -1, \lambda_3 = 3,$

$$E_{\lambda_1} = \text{span}\left(\left\{\frac{1}{11}[2,6,-9]\right\}\right),\ E_{\lambda_2} = \text{span}\left(\left\{\frac{1}{11}[6,7,6]\right\}\right),\ E_{\lambda_3} = \text{span}\left(\left\{\frac{1}{11}[-9,6,2]\right\}\right)$$

**(c)** $\lambda_1 = -1, \lambda_2 = 1, E_{\lambda_1} = \text{span}(\{[6,3,2],[2,0,1]\}), E_{\lambda_2} = \text{span}(\{[-6,4,12]\})$

★ **(d)** $\lambda_1 = -1, \lambda_2 = 1,$

$$E_{\lambda_1} = \text{span}(\{[12, 3, 4, 0], [12, -1, 7, 12]\}),\ E_{\lambda_2} = \text{span}(\{[-3, 12, 0, 4], [-2, 24, -12, 11]\})$$

**3.** In each part of this exercise, the matrix $\mathbf{A}$ with respect to the standard basis for a symmetric linear operator on $\mathbb{R}^n$ is given. Orthogonally diagonalize each operator by following Steps 2 and 3 of the method given in the text. Your answers should include the ordered orthonormal basis $B$, the orthogonal matrix $\mathbf{P}$, and the diagonal matrix $\mathbf{D}$. Check your work by verifying that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$, and that the Spectral Theorem holds for $\mathbf{A}$. (Hint: In (e), $p_{\mathbf{A}}(x) = (x-2)^2(x+3)(x-5)$.)

★ **(a)** $\mathbf{A} = \begin{bmatrix} 144 & -60 \\ -60 & 25 \end{bmatrix}$

**(b)** $\mathbf{A} = \frac{1}{25}\begin{bmatrix} 39 & 48 \\ 48 & 11 \end{bmatrix}$

★ **(c)** $\mathbf{A} = \frac{1}{9}\begin{bmatrix} 17 & 8 & -4 \\ 8 & 17 & -4 \\ -4 & -4 & 11 \end{bmatrix}$

**(d)** $\mathbf{A} = \frac{1}{27}\begin{bmatrix} -13 & -40 & -16 \\ -40 & 176 & -124 \\ -16 & -124 & -1 \end{bmatrix}$

★ **(e)** $\mathbf{A} = \frac{1}{14}\begin{bmatrix} 23 & 0 & 15 & -10 \\ 0 & 31 & -6 & -9 \\ 15 & -6 & -5 & 48 \\ -10 & -9 & 48 & 35 \end{bmatrix}$

**(f)** $\mathbf{A} = \begin{bmatrix} 3 & 4 & 12 \\ 4 & -12 & 3 \\ 12 & 3 & -4 \end{bmatrix}$

★ **(g)** $\mathbf{A} = \begin{bmatrix} 11 & 2 & -10 \\ 2 & 14 & 5 \\ -10 & 5 & -10 \end{bmatrix}$

**4.** In each part of this exercise, use the Orthogonal Diagonalization Method on the given symmetric linear operator $L$, defined on a subspace $\mathcal{V}$ of $\mathbb{R}^n$. Your answers should include the ordered orthonormal basis $C$ for $\mathcal{V}$, the matrix $\mathbf{A}$ for $L$ with respect to $C$, the ordered orthonormal basis $B$ for $\mathcal{V}$, the orthogonal matrix $\mathbf{P}$, and the diagonal matrix $\mathbf{D}$. Check your work by verifying that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$, and that the Spectral Theorem holds for $\mathbf{A}$.

★ **(a)** $L: \mathcal{V} \to \mathcal{V}$, where $\mathcal{V}$ is the plane $6x + 10y - 15z = 0$ in $\mathbb{R}^3$, $L([-10, 15, 6]) = [50, -18, 8]$, and $L([15, 6, 10]) = [-5, 36, 22]$

**(b)** $L: \mathcal{V} \to \mathcal{V}$, where $\mathcal{V}$ is the subspace of $\mathbb{R}^4$ spanned by $\{[1, -1, 1, 1], [-1, 1, 1, 1], [1, 1, 1, -1]\}$ and $L$ is given by

$$L\left(\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & -2 & 1 & 1 \\ -1 & 2 & 0 & 2 \\ 2 & 2 & 1 & 2 \\ 1 & 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$$

**5.** In each case, use orthogonal diagonalization to find a symmetric matrix $\mathbf{A}$ such that

★ **(a)** $\mathbf{A}^3 = \frac{1}{25}\begin{bmatrix} 119 & -108 \\ -108 & 56 \end{bmatrix}$.

**(b)** $\mathbf{A}^2 = \begin{bmatrix} 244 & 180 \\ 180 & 601 \end{bmatrix}$.

★ **(c)** $\mathbf{A}^2 = \begin{bmatrix} 17 & 16 & -16 \\ 16 & 41 & -32 \\ -16 & -32 & 41 \end{bmatrix}$.

**(d)** $\mathbf{A}^3 = \frac{1}{9}\begin{bmatrix} 4 & -4 & 0 & -2 \\ -4 & 12 & 16 & -14 \\ 0 & 16 & 32 & -32 \\ -2 & -14 & -32 & 33 \end{bmatrix}$.

★ **6.** Give an example of a $3 \times 3$ matrix that is diagonalizable but not orthogonally diagonalizable.

★ **7.** Find the diagonal matrix $\mathbf{D}$ to which $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ is similar by an orthogonal change of coordinates. (Hint: Think! The full method for orthogonal diagonalization is not needed.)

**8.** Let $L$ be a symmetric linear operator on a nontrivial subspace $\mathcal{V}$ of $\mathbb{R}^n$.

**(a)** If 1 is the only eigenvalue for $L$, prove that $L$ is the identity operator.

★ **(b)** What must be true about $L$ if zero is its only eigenvalue? Prove it.

**9.** Let $L_1$ and $L_2$ be symmetric operators on $\mathbb{R}^n$. Prove that $L_2 \circ L_1$ is symmetric if and only if $L_2 \circ L_1 = L_1 \circ L_2$.

**10.** Two $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ are said to be **orthogonally similar** if and only if there is an orthogonal matrix $\mathbf{P}$ such that $\mathbf{B} = \mathbf{P}^{-1}\mathbf{AP}$. Prove the following statements are equivalent for $n \times n$ symmetric matrices $\mathbf{A}$ and $\mathbf{B}$:

(i) **A** and **B** are similar.
(ii) **A** and **B** have the same characteristic polynomial.
(iii) **A** and **B** are orthogonally similar.

(Hint: Show that (i) ⇒ (ii) ⇒ (iii) ⇒ (i).)

11. Let $L$ be a symmetric operator on a subspace $\mathcal{V}$ of $\mathbb{R}^n$. Suppose that $\lambda_1$ and $\lambda_2$ are distinct eigenvalues for $L$ with corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Prove that $\mathbf{v}_1 \perp \mathbf{v}_2$. (Hint: Use the definition of a symmetric operator to show that $(\lambda_2 - \lambda_1)(\mathbf{v}_1 \cdot \mathbf{v}_2) = 0$.)

12. Let **A** be an $n \times n$ symmetric matrix. Prove that **A** is orthogonal if and only if all eigenvalues for **A** are either 1 or $-1$. (Hint: For one half of the proof, use Theorem 6.9. For the other half, orthogonally diagonalize to help calculate $\mathbf{A}^2 = \mathbf{A}\mathbf{A}^T$.)

13. Let **A** be an $n \times n$ orthogonal matrix with $n$ odd. We know from part (1) of Theorem 6.6 that $|\mathbf{A}| = \pm 1$.
   (a) If $|\mathbf{A}| = 1$, prove that $\mathbf{A} - \mathbf{I}_n$ is singular. (Hint: Show that $\mathbf{A} - \mathbf{I}_n = -\mathbf{A}(\mathbf{A} - \mathbf{I}_n)^T$, and then use determinants.)
   (b) If $|\mathbf{A}| = 1$, show that **A** has $\lambda = 1$ as an eigenvalue.
   (c) If $|\mathbf{A}| = 1$ and $n = 3$, show that there is an orthogonal matrix **Q** with $|\mathbf{Q}| = 1$ such that

$$\mathbf{Q}^T\mathbf{A}\mathbf{Q} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{for some value of } \theta.$$

   (Hint: Let **v** be a unit eigenvector for **A** corresponding to $\lambda = 1$. Expand the set $\{\mathbf{v}\}$ to an orthonormal basis for $\mathbb{R}^3$. Let **Q** be the matrix whose columns are these basis vectors, with **v** listed last and the first two columns ordered so that $|\mathbf{Q}| = 1$. Note that $\mathbf{Q}^T\mathbf{A}\mathbf{Q}$ is an orthogonal matrix. Show that the last column of $\mathbf{Q}^T\mathbf{A}\mathbf{Q}$ is $\mathbf{e}_3$, and then that the last row of $\mathbf{Q}^T\mathbf{A}\mathbf{Q}$ is also $\mathbf{e}_3$. Finally use the facts that the columns of $\mathbf{Q}^T\mathbf{A}\mathbf{Q}$ are orthonormal and $|\mathbf{Q}^T\mathbf{A}\mathbf{Q}| = 1$ to show that the remaining entries of $\mathbf{Q}^T\mathbf{A}\mathbf{Q}$ have the desired form.)
   (d) Use part (c) of this problem to prove the claim made in Exercise 7 of Section 6.1 that a linear operator on $\mathbb{R}^3$ represented by a $3 \times 3$ orthogonal matrix with determinant 1 (with respect to the standard basis) always represents a rotation about some axis in $\mathbb{R}^3$. (Hint: With **Q** and $\theta$ as in part (c), **A** will represent a rotation through the angle $\theta$ about the axis in the direction corresponding to the last column of **Q**. The rotation will be in the direction from the first column of **Q** toward the second column of **Q**.)
   ★ (e) Find the direction of the axis of rotation and the angle of rotation (to the nearest degree) corresponding to the orthogonal matrix in part (a) of Exercise 7 in Section 6.1. (Hint: Compute $\mathbf{Q}^T\mathbf{A}\mathbf{Q}$ as in part (c). Use the signs of both $\cos\theta$ and $\sin\theta$ to determine the quadrant in which the angle $\theta$ resides. Note that the rotation will be in the direction from the first column of **Q** toward the second column of **Q**. However, even though the angle between these column vectors is 90°, the angle of rotation could be higher than 180°.)
   (f) If $|\mathbf{A}| = -1$ and $n = 3$, prove that **A** is the product of an orthogonal reflection through a plane in $\mathbb{R}^3$ followed by a rotation about some axis in $\mathbb{R}^3$. (Hence, every $3 \times 3$ orthogonal matrix with determinant $-1$ can be thought of as the product of an orthogonal reflection and a rotation.) (Hint: Let **G** be the matrix with respect to the standard basis for any chosen orthogonal reflection through a plane in $\mathbb{R}^3$. Note that $|\mathbf{G}| = -1$, and $\mathbf{G}^2 = \mathbf{I}_3$. Thus, $\mathbf{A} = \mathbf{A}\mathbf{G}^2$. Let $\mathbf{C} = \mathbf{A}\mathbf{G}$, and note that **C** is orthogonal and $|\mathbf{C}| = 1$. Finally, use part (d) of this problem.)

▶ 14. This exercise outlines a proof for Lemma 6.21 that does not require the use of complex numbers. It uses the Fundamental Theorem of Algebra which states that every nonconstant polynomial in $x$ can be factored into a nonzero constant times a product of linear and irreducible quadratic factors. Also, by factoring out a constant and completing the square, each quadratic factor can be put into the form $(x-a)^2 + b^2$, where $a$ and $b$ are real numbers, with $b \neq 0$.
   Suppose **A** is an $n \times n$ symmetric matrix.
   (a) Let $p_{\mathbf{A}}(x) = x^n + (a_{k-1})x^{n-1} + \cdots + a_1x + a_0$. Explain why $p_{\mathbf{A}}(\mathbf{A}) = \mathbf{A}^k + (a_{k-1})\mathbf{A}^{k-1} + \cdots + a_1\mathbf{A} + a_0\mathbf{I}_n = \mathbf{O}_n$.
   (b) Suppose $q(x)$ is a polynomial that factors as

$$q(x) = c\,r_1(x)r_2(x)\cdots r_k(x),$$

   for some constant $c$ and some polynomials $r_1(x), \ldots, r_k(x)$. Use Theorem 1.16 and Exercise 22 in Section 1.5 to explain why

$$q(\mathbf{A}) = c\,r_1(\mathbf{A})r_2(\mathbf{A})\cdots r_k(\mathbf{A}).$$

**(c)** Use parts (a) and (b) and the Fundamental Theorem of Algebra to prove that there is a polynomial $t(x)$ which is either linear or of the form $(x-a)^2+b^2$, with $b \neq 0$, such that $|t(\mathbf{A})| = 0$.

**(d)** Prove that the polynomial $t(x)$ in part (c) cannot be of the form $(x-a)^2+b^2$, with $b \neq 0$. (Hint: Otherwise, there exists a nonzero vector $\mathbf{v}$ such that $\left((\mathbf{A}-a\mathbf{I})^2+b^2\mathbf{I}\right)\mathbf{v} = \mathbf{0}$. Note that $\mathbf{D} = (\mathbf{A}-a\mathbf{I})$ is symmetric, and $\mathbf{D}^2\mathbf{v} = -b^2\mathbf{v}$. Finally, use Theorem 6.19 and the definition of a symmetric operator to obtain a contradiction.)

**(e)** Use parts (c) and (d) to complete the proof of Lemma 6.21.

**15.** Prove the converse of the Spectral Theorem: If the column vectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$ form an orthonormal basis for $\mathbb{R}^n$, and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, then the $n \times n$ matrix $\mathbf{A} = \lambda_1\mathbf{u}_1\mathbf{u}_1^T + \lambda_2\mathbf{u}_2\mathbf{u}_2^T + \cdots + \lambda_n\mathbf{u}_n\mathbf{u}_n^T$ is symmetric. (Hint: Show that $\mathbf{A}$ is orthogonally diagonalizable.)

★ **16.** True or False:

**(a)** If $\mathcal{V}$ is a nontrivial subspace of $\mathbb{R}^n$, a linear operator $L$ on $\mathcal{V}$ with the property $\mathbf{v}_1 \cdot L(\mathbf{v}_2) = L(\mathbf{v}_1) \cdot \mathbf{v}_2$ for every $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}$ has at least one eigenvalue.

**(b)** A symmetric operator on a nontrivial subspace $\mathcal{V}$ of $\mathbb{R}^n$ has a symmetric matrix with respect to any ordered basis for $\mathcal{V}$.

**(c)** If a linear operator $L$ on a nontrivial subspace $\mathcal{V}$ of $\mathbb{R}^n$ is symmetric, then the matrix for $L$ with respect to any ordered orthonormal basis for $\mathcal{V}$ is symmetric.

**(d)** A linear operator $L$ on a nontrivial subspace $\mathcal{V}$ of $\mathbb{R}^n$ is symmetric if and only if the matrix for $L$ with respect to some ordered orthonormal basis for $\mathcal{V}$ is diagonal.

**(e)** Let $L$ be a symmetric linear operator on a nontrivial subspace of $\mathbb{R}^n$ having matrix $\mathbf{A}$ with respect to an ordered orthonormal basis. In using the Orthogonal Diagonalization Method on $L$, the transition matrix $\mathbf{P}$ and the diagonal matrix $\mathbf{D}$ obtained from this process have the property that $\mathbf{A} = \mathbf{PDP}^T$.

**(f)** The orthogonal matrix $\mathbf{P}$ in the equation $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$ for a symmetric matrix $\mathbf{A}$ and diagonal matrix $\mathbf{D}$ is the transition matrix from some ordered orthonormal basis to standard coordinates.

**(g)** In the statement of the Spectral Theorem, each product of the form $\mathbf{u}_i\mathbf{u}_i^T$ reduces to [1], since the $\mathbf{u}_i$'s form an orthonormal basis.

## Review Exercises for Chapter 6

**1.** In each case, verify that the given ordered basis $B$ is orthogonal. Then, for the given $\mathbf{v}$, find $[\mathbf{v}]_B$, using the method of Theorem 6.3.

★ **(a)** $\mathbf{v} = [5, 3, 14]$; $B = ([1, 3, -2], [-1, 1, 1], [5, 1, 4])$

**(b)** $\mathbf{v} = [1, 31, -3]$; $B = ([5, -3, 4], [-1, 1, 2], [-5, -7, 1])$

**2.** Each of the following represents a basis for a subspace of $\mathbb{R}^n$, for some $n$. Use the Gram-Schmidt Process to find an orthogonal basis for the subspace.

★ **(a)** $\{[1, -1, -1, 1],\ [5, 1, 1, 5]\}$ in $\mathbb{R}^4$

**(b)** $\{[1, 1, 2, 3, -1],\ [4, 5, 11, 16, -1],\ [7, -11, 10, 14, 10]\}$ in $\mathbb{R}^5$

★ **3.** Enlarge the orthogonal set $\{[6, 3, -6],\ [3, 6, 6]\}$ to an orthogonal basis for $\mathbb{R}^3$. (Avoid fractions by using appropriate scalar multiples.)

**4.** Consider the orthogonal set $S = \{[1, 1, 1, 1],\ [1, -1, 1, -1]\}$ in $\mathbb{R}^4$.

**(a)** Enlarge $S$ to an orthogonal basis for $\mathbb{R}^4$.

**(b)** Normalize the vectors in the basis you found in part (a) to create an orthonormal basis $B$ for $\mathbb{R}^4$.

**(c)** Find the transition matrix from standard coordinates to $B$-coordinates without using row reduction. (Hint: The transition matrix from $B$-coordinates to standard coordinates is an orthogonal matrix.)

**5.** Suppose $\mathbf{A}$ is an $n \times n$ matrix such that for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, $\mathbf{v} \cdot \mathbf{w} = \mathbf{Av} \cdot \mathbf{Aw}$. Prove that $\mathbf{A}$ is an orthogonal matrix. (Note: This is the converse to Theorem 6.9.) (Hint: Notice that $\mathbf{Av} \cdot \mathbf{Aw} = \mathbf{v} \cdot (\mathbf{A}^T\mathbf{Aw})$. Use this with the vectors $\mathbf{e}_i$ and $\mathbf{e}_j$ for $\mathbf{v}$ and $\mathbf{w}$ to show that $\mathbf{A}^T\mathbf{A} = \mathbf{I}_n$.)

**6.** For each of the following subspaces $\mathcal{W}$ of $\mathbb{R}^n$ and for the given $\mathbf{v} \in \mathbb{R}^n$, find $\mathbf{proj}_{\mathcal{W}}\mathbf{v}$, and decompose $\mathbf{v}$ into $\mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1 \in \mathcal{W}$ and $\mathbf{w}_2 \in \mathcal{W}^\perp$. (Hint: You may need to find an orthonormal basis for $\mathcal{W}$ first.)

★ **(a)** $\mathcal{W} = \text{span}(\{[8, 1, -4], [16, 11, -26]\})$, $\mathbf{v} = [2, 7, 26]$

**(b)** $\mathcal{W} = \text{span}(\{[-1, 1, 3, -1, -6], [-4, 4, 9, 5, -3], [-2, 5, 8, 1, -3]\})$, $\mathbf{v} = [89, -5, 69, 125, -162]$

**7.** In each part, find the minimum distance between the given point $P$ and the given subspace $\mathcal{W}$ of $\mathbb{R}^4$:

★ **(a)** $\mathcal{W} = \text{span}(\{[2, 9, -6, 0], [2, 5, -12, 12]\})$, $P = (1, 29, -29, -2)$

**(b)** $\mathcal{W} = \text{span}(\{[-3, 1, 2, 2], [10, 2, -5, -8]\})$, $P = (-7, 19, 2, 14)$

**8.** If $\mathcal{W} = \text{span}(\{[2, 4, 1, -1], [3, 9, 1, -1]\})$, find a basis for $\mathcal{W}^\perp$.

**9.** Let $L: \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal projection onto the plane $3x - y + 2z = 0$. Use eigenvalues and eigenvectors to find the matrix representation of $L$ with respect to the standard basis.

★ **10.** Let $L: \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal reflection through the plane $2x - 3y + z = 0$. Use eigenvalues and eigenvectors to find the matrix representation of $L$ with respect to the standard basis.

**11.** Find the characteristic polynomial for each of the given linear operators. (Hint: This requires almost no computation.)

**(a)** $L: \mathbb{R}^3 \to \mathbb{R}^3$, where $L$ is the orthogonal reflection through the plane $7x + 6y - 4z = 0$

★ **(b)** $L: \mathbb{R}^3 \to \mathbb{R}^3$, where $L$ is the orthogonal projection onto the line through the origin spanned by $[6, -1, 4]$

**(c)** $L: \mathbb{R}^3 \to \mathbb{R}^3$, where $L$ is the orthogonal projection onto the plane $2x - 9y + 5z = 0$

**12.** Determine which of the following linear operators are symmetric. Explain why each is, or is not, symmetric.

★ **(a)** $L: \mathbb{R}^4 \to \mathbb{R}^4$ given by $L = L_1^{-1} \circ L_2 \circ L_1$ where $L_1: \mathbb{R}^4 \to \mathcal{P}_3$ is given by $L_1([a, b, c, d]) = ax^3 + bx^2 + cx + d$, and $L_2: \mathcal{P}_3 \to \mathcal{P}_3$ is given by $L_2(\mathbf{p}(\mathbf{x})) = \mathbf{p}'(x)$

**(b)** $L: \mathbb{R}^3 \to \mathbb{R}^3$ such that $L([2, 1, 2]) = [10, 15, 16]$, $L([1, 2, 1]) = [8, 12, 14]$, and $L([1, 1, 2]) = [5, 13, 17]$

**(c)** $L: \mathbb{R}^4 \to \mathbb{R}^4$ given by $L = L_2 \circ L_1$ where $L_1: \mathbb{R}^4 \to \mathbb{R}^4$ and $L_2: \mathbb{R}^4 \to \mathbb{R}^4$ such that the matrices for $L_1$ and $L_2$ with respect to the standard basis, are, respectively, $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 3 & -1 & 1 \\ 1 & -1 & 4 & -2 \\ 2 & 1 & -2 & 1 \end{bmatrix}$ and $\begin{bmatrix} 3 & -1 & 2 & 1 \\ -1 & 2 & 1 & 5 \\ 2 & 1 & 2 & 4 \\ 1 & 5 & 4 & -6 \end{bmatrix}$

**(d)** $L: \mathbb{R}^9 \to \mathbb{R}^9$ given by $L = L_1^{-1} \circ L_2 \circ L_1$ where $L_1: \mathbb{R}^9 \to \mathcal{M}_{33}$ is given by $L_1([a, b, c, d, e, f, g, h, i]) = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, and $L_2: \mathcal{M}_{33} \to \mathcal{M}_{33}$ is given by $L_2(\mathbf{A}) = \mathbf{A}^T$

**13.** In each part of this exercise, the matrix $\mathbf{A}$ with respect to the standard basis for a symmetric linear operator on $\mathbb{R}^3$ is given. Orthogonally diagonalize each operator by following Steps 2 and 3 of the method given in Section 6.3. Your answers should include the ordered orthonormal basis $B$, the orthogonal matrix $\mathbf{P}$, and the diagonal matrix $\mathbf{D}$. Check your work by verifying that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$, and that the Spectral Theorem holds for $\mathbf{A}$.

★ **(a)** $\mathbf{A} = \frac{1}{30}\begin{bmatrix} -17 & 26 & 5 \\ 26 & 22 & 10 \\ 5 & 10 & 55 \end{bmatrix}$

**(b)** $\mathbf{A} = \begin{bmatrix} 13 & -2 & -3 \\ -2 & 10 & -6 \\ -3 & -6 & 5 \end{bmatrix}$

**14.** Give an example of a $4 \times 4$ matrix that is diagonalizable but not orthogonally diagonalizable.

**15.** Let $\mathbf{A} \in \mathcal{M}_{mn}$. Prove that $\mathbf{A}^T\mathbf{A}$ is orthogonally diagonalizable.

**16.** Let $\mathbf{A}$ be an orthogonally diagonalizable $n \times n$ matrix, and let $L$ be the linear operator $L(\mathbf{v}) = \mathbf{A}\mathbf{v}$ on $\mathbb{R}^n$. Prove that $\text{range}(L) = (\ker(L))^\perp$. (Hint: Use Exercise 24 in Section 6.2.)

★ **17.** True or False:

**(a)** A set of nonzero mutually orthogonal vectors in $\mathbb{R}^n$ is linearly independent.

**(b)** When applying the Gram-Schmidt Process to a set of nonzero vectors, the first vector produced for the orthogonal set is a scalar multiple of the first vector in the original set of vectors.

**(c)** If $S = \{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ is a subset of $\mathbb{R}^n$ such that $\mathbf{w}_k = \mathbf{w}_1 + \cdots + \mathbf{w}_{k-1}$, then attempting to apply the Gram-Schmidt Process to $S$ will result in the zero vector for $\mathbf{v}_k$, the $k$th vector obtained by the process.

**(d)** $\begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$ is an orthogonal matrix.

**(e)** If $\mathbf{A}$ is a matrix such that $\mathbf{A}\mathbf{A}^T = \mathbf{I}$, then $\mathbf{A}$ is an orthogonal matrix.

**(f)** All diagonal matrices are orthogonal matrices since the rows of diagonal matrices clearly form a mutually orthogonal set of vectors.

**(g)** Every orthogonal matrix is nonsingular.

**(h)** If $\mathbf{A}$ is an orthogonal $n \times n$ matrix, and $L: \mathbb{R}^n \to \mathbb{R}^n$ is the linear transformation $L(\mathbf{X}) = \mathbf{A}\mathbf{X}$, then for all nonzero $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, $\mathbf{v} \perp \mathbf{w}$ implies that $L(\mathbf{v}) \perp L(\mathbf{w})$. (Notice that $L(\mathbf{v})$ and $L(\mathbf{w})$ are also nonzero.)

**(i)** Every subspace of $\mathbb{R}^n$ has an orthogonal complement.

**(j)** If $\mathcal{W}$ is a subspace of $\mathbb{R}^n$ and $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^n$, then $\mathbf{proj}_{\mathcal{W}}\mathbf{v}_1 + \mathbf{proj}_{\mathcal{W}}\mathbf{v}_2 = \mathbf{proj}_{\mathcal{W}}(\mathbf{v}_1 + \mathbf{v}_2)$.

**(k)** If $\mathcal{W}$ is a subspace of $\mathbb{R}^n$, and $L: \mathbb{R}^n \to \mathbb{R}^n$ is given by $L(\mathbf{v}) = \mathbf{proj}_{\mathcal{W}}\mathbf{v}$, then $\text{range}(L) = \mathcal{W}$.

**(l)** If $\mathcal{W}$ is a subspace of $\mathbb{R}^n$, and $L: \mathbb{R}^n \to \mathbb{R}^n$ is given by $L(\mathbf{v}) = \mathbf{proj}_{\mathcal{W}^\perp}\mathbf{v}$, then $\ker(L) = \mathcal{W}$.

**(m)** If $\mathcal{W}$ is a nontrivial subspace of $\mathbb{R}^n$, and $L: \mathbb{R}^n \to \mathbb{R}^n$ is given by $L(\mathbf{v}) = \mathbf{proj}_{\mathcal{W}}\mathbf{v}$, then the matrix for $L$ with respect to the standard basis is an orthogonal matrix.

**(n)** If $\mathcal{W}$ is a subspace of $\mathbb{R}^n$, and $L: \mathbb{R}^n \to \mathbb{R}^n$ is given by $L(\mathbf{v}) = \mathbf{proj}_{\mathcal{W}}\mathbf{v}$, then $L \circ L = L$.

**(o)** If $\mathcal{W}$ is a plane through the origin in $\mathbb{R}^3$, then the linear operator $L$ on $\mathbb{R}^3$ representing an orthogonal reflection through $\mathcal{W}$ has exactly two distinct eigenvalues.

**(p)** If $\mathcal{W}$ is a nontrivial subspace of $\mathbb{R}^n$, and $L: \mathbb{R}^n \to \mathbb{R}^n$ is given by $L(\mathbf{v}) = \mathbf{proj}_{\mathcal{W}}\mathbf{v}$, then $L$ is a symmetric operator on $\mathbb{R}^n$.

**(q)** The composition of two symmetric linear operators on $\mathbb{R}^n$ is a symmetric linear operator.

**(r)** If $L$ is a symmetric linear operator on $\mathbb{R}^n$ and $B$ is an ordered orthonormal basis for $\mathbb{R}^n$, then the matrix for $L$ with respect to $B$ is diagonal.

**(s)** If $\mathbf{A}$ is a symmetric matrix, then $\mathbf{A}$ has at least one eigenvalue $\lambda$, and the algebraic multiplicity of $\lambda$ equals its geometric multiplicity.

**(t)** Every orthogonally diagonalizable matrix is symmetric.

**(u)** Every $n \times n$ orthogonal matrix is the matrix for some symmetric linear operator on $\mathbb{R}^n$.

**(v)** If $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors corresponding to two distinct eigenvalues of a symmetric matrix $\mathbf{A}$, then $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$.

**(w)** If $\mathbf{A}$ is an $n \times n$ orthogonally diagonalizable matrix, then $\mathbf{A}$ can be expressed as any linear combination of the form $\lambda_1\mathbf{u}_1\mathbf{u}_1^T + \lambda_2\mathbf{u}_2\mathbf{u}_2^T + \cdots + \lambda_n\mathbf{u}_n\mathbf{u}_n^T$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ represent the (not necessarily distinct) eigenvalues of $\mathbf{A}$, and each $\mathbf{u}_i$ represents a column eigenvector for $\mathbf{A}$ corresponding to $\lambda_i$ such that $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is a basis for $\mathbb{R}^n$.

Chapter 7

# Complex Vector Spaces and General Inner Products

**A Complex Situation**

Until now, we have kept our theory of linear algebra within the real number system. But many practical mathematical problems, especially in physics and electronics, involve square roots of negative numbers (that is, complex numbers). For example, modern theories of heat transfer, fluid flow, damped harmonic oscillation, alternating current circuit theory, and quantum mechanics—all beyond the scope of this text—depend on the use of complex quantities. Therefore, our next goal is to extend many of our previous results to the realm of complex numbers.

We begin by examining complex vectors and matrices and their fundamental properties. Using the Fundamental Theorem of Algebra, which states that every $n$th-degree polynomial can be factored completely when complex roots are permitted, we can find additional (nonreal) solutions to eigenvalue problems. We also compare the properties of general complex vector spaces to their real counterparts. Finally we discuss inner product spaces, which are complex vector spaces that possess an additional operation analogous to the dot product.

Section 7.1 can be covered any time after finishing Section 1.5. Each remaining section of Chapter 7 depends on preceding ones, along with indicated prerequisites from earlier chapters. Section 7.5 can be covered without going through Sections 7.1 through 7.4 if attention is paid only to real inner products.

We let $\mathbb{C}$ represent the set of complex numbers throughout this chapter. For quick reference, Appendix C lists the definition of a complex number and the rules for complex addition, multiplication, conjugation, magnitude, and reciprocal.

## 7.1 Complex $n$-Vectors and Matrices

*Prerequisite: Section 1.5, Matrix Multiplication*

Until now, our scalars and entries in vectors and matrices have always been real numbers. In this section, however, we use the complex numbers to define and study complex $n$-vectors and matrices, emphasizing their differences with real vectors and matrices from Chapter 1.

### Complex $n$-Vectors

**Definition** A **complex $n$-vector** is an ordered sequence (or ordered $n$-tuple) of $n$ complex numbers. The set of all complex $n$-vectors is denoted by $\mathbb{C}^n$.

For example, $[3-2i, 4+3i, -i]$ is a vector in $\mathbb{C}^3$. We often write $\mathbf{z} = [z_1, z_2, \ldots, z_n]$ (where $z_1, z_2, \ldots, z_n \in \mathbb{C}$) to represent an arbitrary vector in $\mathbb{C}^n$.

For complex vectors, we usually need to extend our definition of **scalar** to include complex numbers instead of only real numbers. In what follows, it will always be clear from the context whether we are using complex scalars or real scalars.

**Scalar multiplication** and **addition** of complex vectors are defined coordinate-wise, just as for real vectors. For example, $(-2+i)[4+i, -1-2i] + [-3-2i, -2+i] = [-9+2i, 4+3i] + [-3-2i, -2+i] = [-12, 2+4i]$. You can verify that all the properties in Theorem 1.3 carry over to complex vectors (with real or complex scalars).

The **complex conjugate** of a vector $\mathbf{z} = [z_1, z_2, \ldots, z_n] \in \mathbb{C}^n$ is defined, using the complex conjugate operation, to be $\bar{\mathbf{z}} = [\overline{z_1}, \overline{z_2}, \ldots, \overline{z_n}]$. For example, if $\mathbf{z} = [3-2i, -5-4i, -2i]$, then $\bar{\mathbf{z}} = [3+2i, -5+4i, 2i]$.

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We define the complex dot product of two vectors as follows:

**Definition** Let $\mathbf{z} = [z_1, z_2, \ldots, z_n]$ and $\mathbf{w} = [w_1, w_2, \ldots, w_n]$ be vectors in $\mathbb{C}^n$. The **complex dot (inner) product** of $\mathbf{z}$ and $\mathbf{w}$ is given by $\mathbf{z} \cdot \mathbf{w} = z_1\overline{w_1} + z_2\overline{w_2} + \cdots + z_n\overline{w_n}$.

Notice that if $\mathbf{z}$ and $\mathbf{w}$ are both real vectors, then $\mathbf{z} \cdot \mathbf{w}$ is the familiar dot product in $\mathbb{R}^n$. The next example illustrates the complex dot product.

**Example 1**

Let $\mathbf{z} = [3-2i, -2+i, -4-3i]$ and $\mathbf{w} = [-2+4i, 5-i, -2i]$. Then

$$\begin{aligned}\mathbf{z} \cdot \mathbf{w} &= (3-2i)\overline{(-2+4i)} + (-2+i)\overline{(5-i)} + (-4-3i)\overline{(-2i)} \\ &= (3-2i)(-2-4i) + (-2+i)(5+i) + (-4-3i)(+2i) = -19-13i.\end{aligned}$$

However,

$$\mathbf{w} \cdot \mathbf{z} = (-2+4i)\,\overline{(3-2i)} + (5-i)\,\overline{(-2+i)} + (-2i)\overline{(-4-3i)} = -19+13i.$$

Notice that $\mathbf{z} \cdot \mathbf{w} = \overline{\mathbf{w} \cdot \mathbf{z}}$. (This is true in general, as we will see shortly.)

Now, if $\mathbf{z} = [z_1, \ldots, z_n]$, then $\mathbf{z} \cdot \mathbf{z} = z_1\overline{z_1} + \cdots + z_n\overline{z_n} = |z_1|^2 + \cdots + |z_n|^2$, a nonnegative real number. We define the **length** of a complex vector $\mathbf{z} = [z_1, \ldots, z_n]$ as $\|\mathbf{z}\| = \sqrt{\mathbf{z} \cdot \mathbf{z}}$. For example, if $\mathbf{z} = [3-i, -2i, 4+3i]$, then

$$\|\mathbf{z}\| = \sqrt{(3-i)(3+i) + (-2i)(2i) + (4+3i)(4-3i)} = \sqrt{10+4+25} = \sqrt{39}.$$

As with real $n$-vectors, a complex vector having length 1 is called a **unit vector**.

The following theorem lists the most important properties of the complex dot product. You are asked to prove parts of this theorem in Exercise 2. Notice the use of the complex conjugate in parts (1) and (5).

**Theorem 7.1** *Let $\mathbf{z}_1$, $\mathbf{z}_2$, and $\mathbf{z}_3$ be vectors in $\mathbb{C}^n$, and let $k \in \mathbb{C}$ be any scalar. Then*

| | | |
|---|---|---|
| (1) | $\mathbf{z}_1 \cdot \mathbf{z}_2 = \overline{\mathbf{z}_2 \cdot \mathbf{z}_1}$ | *Conjugate-Commutativity of Complex Dot Product* |
| (2) | $\mathbf{z}_1 \cdot \mathbf{z}_1 = \|\mathbf{z}_1\|^2 \geq 0$ | *Relationships Between Complex* |
| (3) | $\mathbf{z}_1 \cdot \mathbf{z}_1 = 0$ *if and only if* $\mathbf{z}_1 = \mathbf{0}$ | *Dot Product and Length* |
| (4) | $k(\mathbf{z}_1 \cdot \mathbf{z}_2) = (k\mathbf{z}_1) \cdot \mathbf{z}_2$ | *Relationships Between Scalar* |
| (5) | $\overline{k}(\mathbf{z}_1 \cdot \mathbf{z}_2) = \mathbf{z}_1 \cdot (k\mathbf{z}_2)$ | *Multiplication and Complex Dot Product* |
| (6) | $\mathbf{z}_1 \cdot (\mathbf{z}_2 + \mathbf{z}_3) = (\mathbf{z}_1 \cdot \mathbf{z}_2) + (\mathbf{z}_1 \cdot \mathbf{z}_3)$ | *Distributive Laws of Complex* |
| (7) | $(\mathbf{z}_1 + \mathbf{z}_2) \cdot \mathbf{z}_3 = (\mathbf{z}_1 \cdot \mathbf{z}_3) + (\mathbf{z}_2 \cdot \mathbf{z}_3)$ | *Dot Product Over Addition* |

Unfortunately, we cannot define the angle between two complex $n$-vectors as we did in Section 1.2 for real vectors, since the complex dot product is not necessarily a real number and hence $\frac{\mathbf{z} \cdot \mathbf{w}}{\|\mathbf{z}\|\,\|\mathbf{w}\|}$ does not always represent the cosine of an angle.

## Complex Matrices

**Definition** An $m \times n$ **complex matrix** is a rectangular array of complex numbers arranged in $m$ rows and $n$ columns. The set of all $m \times n$ complex matrices is denoted as $\mathcal{M}_{mn}^{\mathbb{C}}$, or **complex** $\mathcal{M}_{mn}$.

**Addition** and **scalar multiplication** of complex matrices are defined entrywise in the usual manner, and the properties in Theorem 1.12 also hold for complex matrices. We next define **multiplication** of complex matrices.

**Definition** If $\mathbf{Z}$ is an $m \times n$ matrix and $\mathbf{W}$ is an $n \times r$ matrix, then $\mathbf{ZW}$ is the $m \times r$ matrix whose $(i, j)$ entry equals

$$(\mathbf{ZW})_{ij} = z_{i1}w_{1j} + z_{i2}w_{2j} + \cdots + z_{in}w_{nj}.$$

Beware! The $(i, j)$ entry of $\mathbf{ZW}$ is *not* the complex dot product of the $i$th row of $\mathbf{Z}$ with the $j$th column of $\mathbf{W}$, because the complex conjugates of $w_{1j}, w_{2j}, \ldots, w_{nj}$ are not involved. Instead, complex matrices are multiplied in the same manner as real matrices.

**Example 2**

Let $\mathbf{Z} = \begin{bmatrix} 1-i & 2i & -2+i \\ -3i & 3-2i & -1-i \end{bmatrix}$ and $\mathbf{W} = \begin{bmatrix} -2i & 1-4i \\ -1+3i & 2-3i \\ -2+i & -4+i \end{bmatrix}$. Then, the

$$\begin{aligned}(1, 1) \text{ entry of } \mathbf{ZW} &= (1-i)(-2i) + (2i)(-1+3i) + (-2+i)(-2+i) \\ &= -2i - 2 - 2i - 6 + 3 - 4i \\ &= -5 - 8i.\end{aligned}$$

You can verify that the entire product is $\mathbf{ZW} = \begin{bmatrix} -5-8i & 10-7i \\ 12i & -7-13i \end{bmatrix}$.

The familiar properties of matrix multiplication carry over to the complex case.

The **complex conjugate** $\overline{\mathbf{Z}}$ of a complex matrix $\mathbf{Z} = [z_{ij}]$ is the matrix whose $(i, j)$ entry is $\overline{z_{ij}}$. The **transpose** $\mathbf{Z}^T$ of an $m \times n$ complex matrix $\mathbf{Z} = [z_{ij}]$ is the $n \times m$ matrix whose $(j, i)$ entry is $z_{ij}$. The familiar rule that $(\mathbf{ZW})^T = \mathbf{W}^T\mathbf{Z}^T$ holds for complex matrices, and can be shown in a manner similar to the proof of Theorem 1.18. You can also verify that $(\overline{\mathbf{Z}})^T = \overline{(\mathbf{Z}^T)}$ for any complex matrix $\mathbf{Z}$, and so we can define the **conjugate transpose** $\mathbf{Z}^*$ of a complex matrix to be

$$\mathbf{Z}^* = \left(\overline{\mathbf{Z}}\right)^T = \overline{(\mathbf{Z}^T)}.$$

**Example 3**

If $\mathbf{Z} = \begin{bmatrix} 2-3i & -i & 5 \\ 4i & 1+2i & -2-4i \end{bmatrix}$, then $\overline{\mathbf{Z}} = \begin{bmatrix} 2+3i & i & 5 \\ -4i & 1-2i & -2+4i \end{bmatrix}$, and

$$\mathbf{Z}^* = \left(\overline{\mathbf{Z}}\right)^T = \begin{bmatrix} 2+3i & -4i \\ i & 1-2i \\ 5 & -2+4i \end{bmatrix}.$$

The following theorem lists the most important properties of the complex conjugate and conjugate transpose operations:

**Theorem 7.2** *Let* $\mathbf{Z}$ *and* $\mathbf{Y}$ *be* $m \times n$ *complex matrices, let* $\mathbf{W}$ *be an* $n \times p$ *complex matrix, and let* $k \in \mathbb{C}$. *Then*

(1) $\overline{\left(\overline{\mathbf{Z}}\right)} = \mathbf{Z}$ *and* $(\mathbf{Z}^*)^* = \mathbf{Z}$

(2) $(\mathbf{Z} + \mathbf{Y})^* = \mathbf{Z}^* + \mathbf{Y}^*$

(3) $(k\mathbf{Z})^* = \overline{k}(\mathbf{Z}^*)$

(4) $\overline{\mathbf{ZW}} = \overline{\mathbf{Z}}\,\overline{\mathbf{W}}$

(5) $(\mathbf{ZW})^* = \mathbf{W}^*\mathbf{Z}^*$

Note the use of $\overline{k}$ in part (3). The proof of this theorem is straightforward, and parts of it are left as Exercise 4. The proof of the next result uses the fact that if $\mathbf{x}$ and $\mathbf{y}$ are complex column vectors, then $\mathbf{x} \cdot \mathbf{y}$ equals the single entry in the matrix $\mathbf{x}^T\overline{\mathbf{y}}$.

**Theorem 7.3** *If* $\mathbf{A}$ *is any* $n \times n$ *complex matrix and* $\mathbf{z}$ *and* $\mathbf{w}$ *are complex* $n$*-vectors, then* $(\mathbf{Az}) \cdot \mathbf{w} = \mathbf{z} \cdot (\mathbf{A}^*\mathbf{w})$.

Compare the following proof of Theorem 7.3 with that of Theorem 6.9.

*Proof.* $(\mathbf{Az}) \cdot \mathbf{w} = (\mathbf{Az})^T \overline{\mathbf{w}} = \mathbf{z}^T \mathbf{A}^T \overline{\mathbf{w}} = \mathbf{z}^T \left(\overline{\mathbf{A}^*\mathbf{w}}\right) = \mathbf{z} \cdot (\mathbf{A}^*\mathbf{w})$. □

## Hermitian, Skew-Hermitian, and Normal Matrices

Real symmetric and skew-symmetric matrices have complex analogs.

**Definition** Let $\mathbf{Z}$ be a square complex matrix. Then $\mathbf{Z}$ is **Hermitian** if and only if $\mathbf{Z}^* = \mathbf{Z}$, and $\mathbf{Z}$ is **skew-Hermitian** if and only if $\mathbf{Z}^* = -\mathbf{Z}$.

Notice that an $n \times n$ complex matrix $\mathbf{Z}$ is Hermitian if and only if $z_{ij} = \overline{z_{ji}}$, for $1 \le i, j \le n$. When $i = j$, we have $z_{ii} = \overline{z_{ii}}$ for all $i$, and so *all main diagonal entries of a Hermitian matrix are real*. Similarly, $\mathbf{Z}$ is skew-Hermitian if and only if $z_{ij} = -\overline{z_{ji}}$ for $1 \le i, j \le n$. When $i = j$, we have $z_{ii} = -\overline{z_{ii}}$ for all $i$, and so *all main diagonal entries of a skew-Hermitian matrix are pure imaginary*.

### Example 4

Consider the matrix

$$\mathbf{H} = \begin{bmatrix} 3 & 2-i & 1-2i \\ 2+i & -1 & -3i \\ 1+2i & 3i & 4 \end{bmatrix}.$$

Notice that

$$\overline{\mathbf{H}} = \begin{bmatrix} 3 & 2+i & 1+2i \\ 2-i & -1 & 3i \\ 1-2i & -3i & 4 \end{bmatrix}, \text{ and so } \mathbf{H}^* = \left(\overline{\mathbf{H}}\right)^T = \begin{bmatrix} 3 & 2-i & 1-2i \\ 2+i & -1 & -3i \\ 1+2i & 3i & 4 \end{bmatrix}.$$

Since $\mathbf{H}^* = \mathbf{H}$, $\mathbf{H}$ is Hermitian. Similarly, you can verify that the matrix

$$\mathbf{K} = \begin{bmatrix} -2i & 5+i & -1-3i \\ -5+i & i & 6 \\ 1-3i & -6 & 3i \end{bmatrix}$$

is skew-Hermitian.

Some other useful results concerning Hermitian and skew-Hermitian matrices are left for you to prove in Exercises 6, 7, and 8.

Another very important type of complex matrix is the following:

**Definition** Let $\mathbf{Z}$ be a square complex matrix. Then $\mathbf{Z}$ is **normal** if and only if $\mathbf{ZZ}^* = \mathbf{Z}^*\mathbf{Z}$.

The next theorem gives two important classes of normal matrices.

**Theorem 7.4** *If* $\mathbf{Z}$ *is a Hermitian or skew-Hermitian matrix, then* $\mathbf{Z}$ *is normal.*

The proof is left as Exercise 9. The next example gives a normal matrix that is neither Hermitian nor skew-Hermitian, thus illustrating that the converse to Theorem 7.4 is false.

**Example 5**

Consider $\mathbf{Z} = \begin{bmatrix} 1-2i & -i \\ 1 & 2-3i \end{bmatrix}$. Now, $\mathbf{Z}^* = \begin{bmatrix} 1+2i & 1 \\ i & 2+3i \end{bmatrix}$, and so

$$\mathbf{Z}\mathbf{Z}^* = \begin{bmatrix} 1-2i & -i \\ 1 & 2-3i \end{bmatrix}\begin{bmatrix} 1+2i & 1 \\ i & 2+3i \end{bmatrix} = \begin{bmatrix} 6 & 4-4i \\ 4+4i & 14 \end{bmatrix}.$$

$$\text{Also,} \quad \mathbf{Z}^*\mathbf{Z} = \begin{bmatrix} 1+2i & 1 \\ i & 2+3i \end{bmatrix}\begin{bmatrix} 1-2i & -i \\ 1 & 2-3i \end{bmatrix} = \begin{bmatrix} 6 & 4-4i \\ 4+4i & 14 \end{bmatrix}.$$

Since $\mathbf{Z}\mathbf{Z}^* = \mathbf{Z}^*\mathbf{Z}$, $\mathbf{Z}$ is normal.

In Exercise 10 you are asked to prove that a matrix $\mathbf{Z}$ is normal if and only if $\mathbf{Z} = \mathbf{H}_1 + \mathbf{H}_2$, where $\mathbf{H}_1$ is Hermitian, $\mathbf{H}_2$ is skew-Hermitian, and $\mathbf{H}_1\mathbf{H}_2 = \mathbf{H}_2\mathbf{H}_1$. For example, the normal matrix $\mathbf{Z}$ from Example 5 equals

$$\begin{bmatrix} 1 & \frac{1}{2} - \frac{1}{2}i \\ \frac{1}{2} + \frac{1}{2}i & 2 \end{bmatrix} + \begin{bmatrix} -2i & -\frac{1}{2} - \frac{1}{2}i \\ \frac{1}{2} - \frac{1}{2}i & -3i \end{bmatrix}.$$

## New Vocabulary

addition (of complex vectors or matrices)
complex conjugate (of a vector or matrix)
complex matrix
complex scalar
complex vector
conjugate transpose (of a complex matrix)
dot product (of complex vectors)
Hermitian matrix
length (of a complex vector)
multiplication (of complex matrices)
normal matrix
scalar multiplication (of complex vectors or matrices)
skew-Hermitian matrix
transpose (of a complex matrix)
unit complex vector

## Highlights

- Scalar multiplication and addition are defined coordinate-wise for complex vectors and matrices (as with real vectors and matrices).
- The dot product of complex vectors $\mathbf{z} = [z_1, z_2, \ldots, z_n]$ and $\mathbf{w} = [w_1, w_2, \ldots, w_n]$ is given by $\mathbf{z} \cdot \mathbf{w} = z_1\overline{w_1} + z_2\overline{w_2} + \cdots + z_n\overline{w_n}$.
- The length of a complex vector $\mathbf{z}$ is defined as $\|\mathbf{z}\| = \sqrt{\mathbf{z} \cdot \mathbf{z}}$.
- If $\mathbf{Z}$ is an $m \times n$ complex matrix and $\mathbf{W}$ is an $n \times r$ complex matrix, then $\mathbf{ZW}$ is the $m \times r$ matrix whose $(i, j)$ entry equals $z_{i1}w_{1j} + z_{i2}w_{2j} + \cdots + z_{in}w_{nj}$. (Note that this formula is *not* a complex dot product since the complex conjugates of $w_{1j}, w_{2j}, \ldots, w_{nj}$ are not involved.)
- The complex conjugate of a complex matrix $\mathbf{Z}$ is the matrix $\overline{\mathbf{Z}}$ whose $(i, j)$ entry equals $\overline{z_{ij}}$.
- If $\mathbf{Z}$ is an $m \times n$ complex matrix, then its transpose $\mathbf{Z}^T$ is the $n \times m$ matrix whose $(j, i)$ entry is $z_{ij}$. The conjugate transpose $\mathbf{Z}^*$ of $\mathbf{Z}$ is the complex matrix $\left(\overline{\mathbf{Z}}\right)^T = \overline{\left(\mathbf{Z}^T\right)}$.
- For an $m \times n$ complex matrix $\mathbf{Z}$, an $n \times p$ complex matrix $\mathbf{W}$, and $k \in \mathbb{C}$, we have $(k\mathbf{Z})^* = \overline{k}(\mathbf{Z}^*)$, $\overline{\mathbf{ZW}} = \overline{\mathbf{Z}}\,\overline{\mathbf{W}}$, and $(\mathbf{ZW})^* = \mathbf{W}^*\mathbf{Z}^*$.
- If $\mathbf{x}$ and $\mathbf{y}$ are complex column vectors, then $\mathbf{x} \cdot \mathbf{y}$ equals the single entry in the matrix $\mathbf{x}^T\overline{\mathbf{y}}$.
- If $\mathbf{A}$ is any $n \times n$ complex matrix and $\mathbf{z}$ and $\mathbf{w}$ are complex $n$-vectors, then $(\mathbf{Az}) \cdot \mathbf{w} = \mathbf{z} \cdot (\mathbf{A}^*\mathbf{w})$.
- A complex matrix $\mathbf{Z}$ is Hermitian iff $\mathbf{Z}^* = \mathbf{Z}$, skew-Hermitian iff $\mathbf{Z}^* = -\mathbf{Z}$, and normal iff $\mathbf{Z}\mathbf{Z}^* = \mathbf{Z}^*\mathbf{Z}$.
- If $\mathbf{Z}$ is Hermitian, then all main diagonal entries of $\mathbf{Z}$ are real. If $\mathbf{Z}$ is skew-Hermitian, then all main diagonal entries of $\mathbf{Z}$ are pure imaginary.
- If $\mathbf{Z}$ is Hermitian or skew-Hermitian, then $\mathbf{Z}$ is normal.

## Exercises for Section 7.1

1. Perform the following computations involving complex vectors.
   - ★ (a) $[2+i, 3, -i] + [-1+3i, -2+i, 6]$
   - ★ (b) $(-8+3i)\,[4i, 2-3i, -7+i]$
   - (c) $\overline{[5-i, 2+i, -3i]}$
   - ★ (d) $\overline{(-4)[6-3i, 7-2i, -8i]}$
   - ★ (e) $[-2+i, 5-2i, 3+4i] \cdot [1+i, 4-3i, -6i]$
   - (f) $[5+2i, 6i, -2+i] \cdot [3-6i, 8+i, 1-4i]$
2. This exercise asks for proofs of various parts of Theorem 7.1.
   - (a) Prove parts (1) and (2) of Theorem 7.1.
   - (b) Prove part (5) of Theorem 7.1.
3. Perform the indicated operations with the following matrices:

$$\mathbf{A} = \begin{bmatrix} 2+5i & -4+i \\ -3-6i & 8-3i \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 9-i & -3i \\ 5+2i & 4+3i \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1+i & -2i & 6+4i \\ 0 & 3+i & 5 \\ -10i & 0 & 7-3i \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 5-i & -i & -3 \\ 2+3i & 0 & -4+i \end{bmatrix}$$

   - ★ (a) $\mathbf{A}+\mathbf{B}$
   - (b) $\overline{\mathbf{C}}$
   - ★ (c) $\mathbf{C}^*$
   - ★ (d) $(-3i)\mathbf{D}$
   - (e) $\mathbf{A}-\mathbf{B}^T$
   - ★ (f) $\mathbf{AB}$
   - (g) $\mathbf{D}\,(\mathbf{C}^*)$
   - (h) $\mathbf{B}^2$
   - ★ (i) $\mathbf{C}^T\mathbf{D}^*$
   - (j) $(\mathbf{C}^*)^2$
4. This exercise asks for proofs of various parts of Theorem 7.2.
   - (a) Prove part (3) of Theorem 7.2.
   - ▶ (b) Prove part (5) of Theorem 7.2.

★ 5. Determine which of the following matrices are Hermitian or skew-Hermitian.

   (a) $\begin{bmatrix} -4i & 6-2i & 8 \\ -6-2i & 0 & -2-i \\ -8 & 2-i & 5i \end{bmatrix}$

   (b) $\begin{bmatrix} 2+3i & 6i & 1+i \\ -6i & 4 & 8-3i \\ 1-i & -8+3i & 5i \end{bmatrix}$

   (c) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

   (d) $\begin{bmatrix} 5i & 0 & 0 \\ 0 & -2i & 0 \\ 0 & 0 & 6i \end{bmatrix}$

   (e) $\begin{bmatrix} 2 & -2i & 2 \\ 2i & -2 & -2i \\ 2 & 2i & -2 \end{bmatrix}$

6. Let $\mathbf{Z}$ be any square complex matrix.
   - (a) Prove that $\mathbf{H} = \frac{1}{2}(\mathbf{Z}+\mathbf{Z}^*)$ is a Hermitian matrix and $\mathbf{K} = \frac{1}{2}(\mathbf{Z}-\mathbf{Z}^*)$ is skew-Hermitian.
   - (b) Prove that $\mathbf{Z}$ can be expressed uniquely as the sum of a Hermitian matrix $\mathbf{H}$ and a skew-Hermitian matrix $\mathbf{K}$. (Hint: Use part (a).)
7. Let $\mathbf{H}$ be an $n \times n$ Hermitian matrix.
   - (a) Suppose $\mathbf{J}$ is an $n \times n$ Hermitian matrix. Prove that $\mathbf{HJ}$ is Hermitian if and only if $\mathbf{HJ} = \mathbf{JH}$.
   - (b) Prove that $\mathbf{H}^k$ is Hermitian for all integers $k \geq 1$. (Hint: Use part (a) and a proof by induction.)
   - (c) Prove that $\mathbf{P}^*\mathbf{HP}$ is Hermitian for any $n \times n$ complex matrix $\mathbf{P}$.
8. Prove that for any complex matrix $\mathbf{A}$, both $\mathbf{AA}^*$ and $\mathbf{A}^*\mathbf{A}$ are Hermitian.

▶ 9. Prove Theorem 7.4.

10. Let $\mathbf{Z}$ be a square complex matrix. Prove that $\mathbf{Z}$ is normal if and only if there exists a Hermitian matrix $\mathbf{H}_1$ and a skew-Hermitian matrix $\mathbf{H}_2$ such that $\mathbf{Z} = \mathbf{H}_1 + \mathbf{H}_2$ and $\mathbf{H}_1\mathbf{H}_2 = \mathbf{H}_2\mathbf{H}_1$. (Hint: If $\mathbf{Z}$ is normal, let $\mathbf{H}_1 = (\mathbf{Z}+\mathbf{Z}^*)/2$.)

★ 11. True or False:
   - (a) The dot product of two complex $n$-vectors is always a real number.
   - (b) The $(i, j)$ entry of the product $\mathbf{ZW}$ is the complex dot product of the $i$th row of $\mathbf{Z}$ with the $j$th column of $\mathbf{W}$.

**(c)** The complex conjugate of the transpose of $\mathbf{Z}$ is equal to the transpose of the complex conjugate of $\mathbf{Z}$.
**(d)** If $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{C}^n$ and $k \in \mathbb{C}$, then $k(\mathbf{v}_1 \cdot \mathbf{v}_2) = (k\mathbf{v}_1) \cdot \mathbf{v}_2 = \mathbf{v}_1 \cdot (k\mathbf{v}_2)$.
**(e)** Every Hermitian matrix is symmetric.
**(f)** The transpose of a skew-Hermitian matrix is normal.

# 7.2 Complex Eigenvalues and Complex Eigenvectors

*Prerequisite: Section 3.4, Eigenvalues and Diagonalization*

In this section, we consider row reduction and determinants using complex numbers and matrices and then extend the concept of eigenvalues and eigenvectors to complex $n \times n$ matrices.

## Complex Linear Systems and Determinants

The **Gaussian Elimination** and **Gauss-Jordan row reduction** methods can both be used to solve **systems of complex linear equations** just as described in Sections 2.1 and 2.2 for real linear systems. However, the arithmetic involved is typically more tedious.

### Example 1

Let us solve the system

$$\begin{cases} (2-3i)w + (19+4i)z = -35+59i \\ (2+i)w + (-4+13i)z = -40-30i \\ (1-i)w + (9+6i)z = -32+25i \end{cases}$$

using Gaussian Elimination. We begin with the augmented matrix

$$\left[\begin{array}{cc|c} 2-3i & 19+4i & -35+59i \\ 2+i & -4+13i & -40-30i \\ 1-i & 9+6i & -32+25i \end{array}\right].$$

Working on the first column, we have

| Row Operations | Resulting Matrix |
|---|---|
| (I): $\langle 1\rangle \leftarrow \frac{1}{2-3i}\langle 1\rangle$, or, $\langle 1\rangle \leftarrow \left(\frac{2}{13}+\frac{3}{13}i\right)\langle 1\rangle$ <br> (II): $\langle 2\rangle \leftarrow -(2+i)\langle 1\rangle + \langle 2\rangle$ <br> (II): $\langle 3\rangle \leftarrow -(1-i)\langle 1\rangle + \langle 3\rangle$ | $\left[\begin{array}{cc\|c} 1 & 2+5i & -19+i \\ 0 & -3+i & -1-13i \\ 0 & 2+3i & -14+5i \end{array}\right].$ |

Continuing to the second column, we obtain

| Row Operations | Resulting Matrix |
|---|---|
| (I): $\langle 2\rangle \leftarrow \frac{1}{-3+i}\langle 2\rangle$, or, $\langle 2\rangle \leftarrow \left(-\frac{3}{10}-\frac{1}{10}i\right)\langle 2\rangle$ <br> (II): $\langle 3\rangle \leftarrow -(2+3i)\langle 2\rangle + \langle 3\rangle$ | $\left[\begin{array}{cc\|c} 1 & 2+5i & -19+i \\ 0 & 1 & -1+4i \\ 0 & 0 & 0 \end{array}\right].$ |

Hence,

$$w + (2+5i)z = -19+i, \quad \text{and}$$
$$z = -1+4i.$$

Thus, $w = -19+i-(2+5i)(-1+4i) = 3-2i$. Therefore, the unique solution to the system is $(w,z) = (3-2i, -1+4i)$.

All of our results for real matrices involving **reduced row echelon form**, **rank**, **row space**, **homogeneous systems**, and **inverse matrices** carry over to complex matrices. Similarly, **determinants** of complex matrices are computed in the same manner as for real matrices, and the following results, which we state without proof, are true:

**Theorem 7.5** *Let* $\mathbf{W}$ *and* $\mathbf{Z}$ *be complex* $n \times n$ *matrices. Then*

(1) $|\mathbf{WZ}| = |\mathbf{W}||\mathbf{Z}|$

(2) $|\mathbf{W}| = |\mathbf{W}^T|$

(3) $|\overline{\mathbf{W}}| = |\mathbf{W}^*| = \overline{|\mathbf{W}|}$

(4) $|\mathbf{W}| \neq 0$ *iff* $\mathbf{W}$ *is nonsingular iff* $\text{rank}(\mathbf{W}) = n$

In addition, all the equivalences in Table 3.1 also hold for complex $n \times n$ matrices.

## Complex Eigenvalues and Complex Eigenvectors

If $\mathbf{A}$ is an $n \times n$ complex matrix, then $\lambda \in \mathbb{C}$ is an **eigenvalue** for $\mathbf{A}$ if and only if there is a nonzero vector $\mathbf{v} \in \mathbb{C}^n$ such that $\mathbf{Av} = \lambda\mathbf{v}$. As before, the nonzero vector $\mathbf{v}$ is called an **eigenvector** for $\mathbf{A}$ associated with $\lambda$. The **characteristic polynomial** of $\mathbf{A}$, defined as $p_{\mathbf{A}}(x) = |x\mathbf{I}_n - \mathbf{A}|$, is used to find the eigenvalues of $\mathbf{A}$, just as in Section 3.4.

### Example 2

For the matrix

$$\mathbf{A} = \begin{bmatrix} -4+7i & 2+i & 7+7i \\ 1-3i & 1-i & -3-i \\ 5+4i & 1-2i & 7-5i \end{bmatrix},$$

we have

$$x\mathbf{I}_3 - \mathbf{A} = \begin{bmatrix} x+4-7i & -2-i & -7-7i \\ -1+3i & x-1+i & 3+i \\ -5-4i & -1+2i & x-7+5i \end{bmatrix}.$$

After some calculation, you can verify that $p_{\mathbf{A}}(x) = |x\mathbf{I}_3 - \mathbf{A}| = x^3 - (4+i)x^2 + (5+5i)x - (6+6i)$. You can also check that $p_{\mathbf{A}}(x)$ factors as $(x-(1-i))(x-2i)(x-3)$. Hence, the eigenvalues of $\mathbf{A}$ are $\lambda_1 = 1-i$, $\lambda_2 = 2i$, and $\lambda_3 = 3$. To find an eigenvector for $\lambda_1$, we look for a nontrivial solution $\mathbf{v}$ of the system $((1-i)\mathbf{I}_3 - \mathbf{A})\mathbf{v} = \mathbf{0}$. Hence, we row reduce

$$\left[\begin{array}{ccc|c} 5-8i & -2-i & -7-7i & 0 \\ -1+3i & 0 & 3+i & 0 \\ -5-4i & -1+2i & -6+4i & 0 \end{array}\right] \quad \text{to obtain} \quad \left[\begin{array}{ccc|c} 1 & 0 & -i & 0 \\ 0 & 1 & i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus, we get the fundamental eigenvector $[i, -i, 1]$ corresponding to $\lambda_1$. A similar analysis shows that $[3i, -i, 2]$ is a fundamental eigenvector corresponding to $\lambda_2$, and $[i, 0, 1]$ is a fundamental eigenvector corresponding to $\lambda_3$.

## Diagonalizable Complex Matrices and Algebraic Multiplicity

We say a complex matrix $\mathbf{A}$ is **diagonalizable** if and only if there is a nonsingular complex matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$ is a diagonal matrix. Just as with real matrices, the matrix $\mathbf{P}$ has fundamental eigenvectors for $\mathbf{A}$ as its columns, and the diagonal matrix $\mathbf{D}$ has the eigenvalues for $\mathbf{A}$ on its main diagonal, with $d_{ii}$ being an eigenvalue corresponding to the fundamental eigenvector that is the $i$th column of $\mathbf{P}$. The six-step method for diagonalizing a matrix given in Section 3.4 works just as well for complex matrices.

The **algebraic multiplicity** of an eigenvalue of a complex matrix is defined just as for real matrices—that is, $k$ is the algebraic multiplicity of an eigenvalue $\lambda$ for a matrix $\mathbf{A}$ if and only if $(x-\lambda)^k$ is the highest power of $(x-\lambda)$ that divides $p_{\mathbf{A}}(x)$. However, an important property of complex polynomials makes the situation for complex matrices a bit different than for real matrices. In particular, the **Fundamental Theorem of Algebra** states that any complex polynomial of degree $n$ factors into a product of $n$ *linear* factors. Thus, for every $n \times n$ matrix A, $p_{\mathbf{A}}(x)$ can be expressed as a product of $n$ *linear* factors. Therefore, the algebraic multiplicities of the eigenvalues of $\mathbf{A}$ must add up to $n$. This eliminates one of the two reasons that some real matrices are not diagonalizable. However, there are still some complex matrices that are not diagonalizable, as we will see later in Example 4.

**Example 3**

Consider the matrix

$$\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

from Example 8 in Section 3.4 for a fixed value of $\theta$ such that $\sin\theta \neq 0$. In that example, we computed $p_{\mathbf{A}}(x) = x^2 - 2(\cos\theta)x + 1$, which factors into complex linear factors as $p_{\mathbf{A}}(x) = (x - (\cos\theta + i\sin\theta))(x - (\cos\theta - i\sin\theta))$. Thus, the two complex eigenvalues[1] for $\mathbf{A}$ are $\lambda_1 = \cos\theta + i\sin\theta$ and $\lambda_2 = \cos\theta - i\sin\theta$.

Row reducing $\lambda_1\mathbf{I}_2 - \mathbf{A}$ yields $\begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$, thus giving the fundamental eigenvector $[i, 1]$. Similarly, row reducing $\lambda_2\mathbf{I}_2 - \mathbf{A}$ produces the fundamental eigenvector $[-i, 1]$. Hence, $\mathbf{P} = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$. You can verify that

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \left(\frac{1}{2}\begin{bmatrix} -i & 1 \\ i & 1 \end{bmatrix}\right)\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \cos\theta + i\sin\theta & 0 \\ 0 & \cos\theta - i\sin\theta \end{bmatrix} = \mathbf{D}.$$

For example, if $\theta = \frac{\pi}{6}$, then $\mathbf{A} = \frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$ and $\mathbf{D} = \begin{bmatrix} \frac{\sqrt{3}+i}{2} & 0 \\ 0 & \frac{\sqrt{3}-i}{2} \end{bmatrix}$. Note that the fundamental eigenvectors for $\mathbf{A}$ are independent of $\theta$, and hence so is the matrix $\mathbf{P}$. However, $\mathbf{D}$ and the eigenvalues of $\mathbf{A}$ change as $\theta$ changes.

This example illustrates how a real matrix could be diagonalizable when thought of as a complex matrix, even though it is not diagonalizable when considered as a real matrix.

## Nondiagonalizable Complex Matrices

It is still possible for a complex matrix to be nondiagonalizable. This occurs whenever the number of fundamental eigenvectors for a given eigenvalue produced in Step 3 of the diagonalization process is less than the algebraic multiplicity of that eigenvalue.

**Example 4**

Consider the matrix

$$\mathbf{A} = \begin{bmatrix} -3-15i & -6+25i & 43+18i \\ 2-2i & -4+i & 1+8i \\ 2-5i & -7+6i & 9+14i \end{bmatrix},$$

whose characteristic polynomial is $p_{\mathbf{A}}(x) = x^3 - 2x^2 + x = x(x-1)^2$. The eigenvalue $\lambda_1 = 1$ has algebraic multiplicity 2. However, $\lambda_1\mathbf{I}_2 - \mathbf{A} = \mathbf{I}_2 - \mathbf{A}$ row reduces to

$$\begin{bmatrix} 1 & 0 & -\frac{3}{2} - \frac{7}{2}i \\ 0 & 1 & \frac{1}{2} + \frac{1}{2}i \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence, Step 3 produces only one fundamental eigenvector: $\left[\frac{3}{2} - \frac{7}{2}i, -\frac{1}{2} - \frac{1}{2}i, 1\right]$. Since the number of fundamental eigenvectors produced for $\lambda_1$ is less than the algebraic multiplicity of $\lambda_1$, $\mathbf{A}$ cannot be diagonalized.

[1] We could have solved for $\lambda_1$ and $\lambda_2$ by using the quadratic formula instead of factoring.

## New Vocabulary

algebraic multiplicity of a complex eigenvalue
characteristic polynomial (of a complex matrix)
determinant (of a complex matrix)
diagonalizable (complex) matrix
eigenvalue (of a complex matrix)
eigenvector (of a complex matrix)
homogeneous system (of complex linear equations)
inverse (of a complex matrix)
rank (of a complex matrix)
row space (of a complex matrix)
system of complex linear equations (= complex linear system)

## Highlights

- Gaussian Elimination and the Gauss-Jordan Method can be used to solve systems of complex linear equations.
- For a complex matrix, its rank, row space, inverse (if it exists) and determinant can all be computed using the same techniques valid for real matrices.
- An $n \times n$ complex matrix $\mathbf{W}$ is nonsingular iff $|\mathbf{W}| \neq 0$ iff rank($\mathbf{W}$) $= n$.
- If $\mathbf{W}$, $\mathbf{Z}$ are $n \times n$ complex matrices, then $|\mathbf{WZ}| = |\mathbf{W}||\mathbf{Z}|$, $|\mathbf{W}^T| = |\mathbf{W}|$, and $|\mathbf{W}^*| = |\overline{\mathbf{W}}| = \overline{|\mathbf{W}|}$.
- If $\mathbf{A}$ is an $n \times n$ complex matrix, then $\lambda \in \mathbb{C}$ is an eigenvalue for $\mathbf{A}$ if and only if there is a nonzero vector $\mathbf{v} \in \mathbb{C}^n$ such that $\mathbf{Av} = \lambda\mathbf{v}$. Such a nonzero complex vector $\mathbf{v}$ is an eigenvector for $\mathbf{A}$ associated with $\lambda$.
- If $\mathbf{A}$ is an $n \times n$ complex matrix, the eigenvalues of $\mathbf{A}$ are the complex roots of the characteristic polynomial $p_{\mathbf{A}}(x) = |x\mathbf{I}_n - \mathbf{A}|$ of $\mathbf{A}$, which factors over the complex numbers into $n$ linear factors. That is, the sum of the algebraic multiplicities of the eigenvalues of $\mathbf{A}$ equals $n$.
- A complex matrix $\mathbf{A}$ is diagonalizable if and only if there is a nonsingular complex matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$ is a diagonal matrix. The Diagonalization Method applies to complex matrices.
- A complex $n \times n$ matrix $\mathbf{A}$ is not diagonalizable if the number of fundamental eigenvectors obtained in the Diagonalization Method for $\mathbf{A}$ does not equal $n$.
- Let $\mathbf{A}$ be a complex matrix with all entries real. If $\mathbf{A}$ has nonreal eigenvalues, then $\mathbf{A}$ is not diagonalizable when considered as a real matrix, but $\mathbf{A}$ may be diagonalizable when considered as a complex matrix.

## Exercises for Section 7.2

1. Give the complete solution set for each of the following complex linear systems:

★ (a) $\begin{cases} (3+i)w + (5+5i)z = 29+33i \\ (1+i)w + (6-2i)z = 30-12i \end{cases}$

(b) $\begin{cases} (1+2i)x + (-1+3i)y + (9+3i)z = 18+46i \\ (2+3i)x + (-1+5i)y + (15+5i)z = 30+76i \\ (5-2i)x + (7+3i)y + (11-20i)z = 120-25i \end{cases}$

★ (c) $\begin{cases} 3ix + (-6+3i)y + (12+18i)z = -51+9i \\ (3+2i)x + (1+7i)y + (25-2i)z = -13+56i \\ (1+i)x + 2iy + (9+i)z = -7+17i \end{cases}$

(d) $\begin{cases} (1+3i)w + 10iz = -46-38i \\ (4+2i)w + (12+13i)z = -111 \end{cases}$

★ (e) $\begin{cases} (3-2i)w + (12+5i)z = 3+11i \\ (5+4i)w + (-2+23i)z = -14+15i \end{cases}$

(f) $\begin{cases} (2-i)x + (1-3i)y + (21+2i)z = -14-13i \\ (1+2i)x + (6+2i)y + (3+46i)z = 24-27i \end{cases}$

2. In each part, compute the determinant of the given matrix $\mathbf{A}$, determine whether $\mathbf{A}$ is nonsingular, and then calculate $|\mathbf{A}^*|$ to verify that $|\mathbf{A}^*| = \overline{|\mathbf{A}|}$.

**(a)** $\mathbf{A} = \begin{bmatrix} 2+i & -3+2i \\ 4-3i & 1+8i \end{bmatrix}$

★ **(b)** $\mathbf{A} = \begin{bmatrix} i & 2 & 5i \\ 1+i & 1-i & i \\ 4 & -2 & 2-i \end{bmatrix}$

**(c)** $\mathbf{A} = \begin{bmatrix} 0 & i & 0 & 1 \\ -i & 0 & 0 & 0 \\ 0 & -1 & 2 & 1 \\ 1 & 0 & 3i & 4i \end{bmatrix}$

**3.** For each of the following matrices, find all eigenvalues and express each eigenspace as a set of linear combinations of fundamental eigenvectors:

★ **(a)** $\begin{bmatrix} 4+3i & -1-3i \\ 8-2i & -5-2i \end{bmatrix}$

**(b)** $\begin{bmatrix} 11 & 2 & -7 \\ 0 & 6 & -5 \\ 10 & 2 & -6 \end{bmatrix}$

★ **(c)** $\begin{bmatrix} 4+3i & -4-2i & 4+7i \\ 2-4i & -2+5i & 7-4i \\ -4-2i & 4+2i & -4-6i \end{bmatrix}$

**(d)** $\begin{bmatrix} -i & 2i & -1+2i \\ 1 & -1+i & -i \\ -2+i & 2-i & 3+2i \end{bmatrix}$

**4.** This exercise explores whether some matrices in Exercise 3 are diagonalizable.

★ **(a)** Explain why the matrix $\mathbf{A}$ in part (a) of Exercise 3 is diagonalizable. Find a nonsingular $\mathbf{P}$ and diagonal $\mathbf{D}$ such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$.

**(b)** Show that the matrix in part (d) of Exercise 3 is not diagonalizable.

**(c)** Show that the matrix from part (b) of Exercise 3 is diagonalizable as a complex matrix, but not as a real matrix.

**5.** Give a convincing argument that if the algebraic multiplicity of every eigenvalue of a complex $n \times n$ matrix is 1, then the matrix is diagonalizable.

★ **6.** True or False:

**(a)** If $\mathbf{A}$ is a $4 \times 4$ complex matrix whose second row is $i$ times its first row, then $|\mathbf{A}| = 0$.

**(b)** The algebraic multiplicity of any eigenvalue of an $n \times n$ complex matrix must equal $n$.

**(c)** Every real $n \times n$ matrix is diagonalizable when thought of as a complex matrix.

**(d)** The Fundamental Theorem of Algebra guarantees that every $n$th degree complex polynomial has $n$ distinct roots.

## 7.3 Complex Vector Spaces

*Prerequisite: Section 5.2, the Matrix of a Linear Transformation*

In this section, we examine complex vector spaces and their similarities and differences with real vector spaces. We also discuss linear transformations from one complex vector space to another.

### Complex Vector Spaces

We define **complex vector spaces** exactly the same way that we defined real vector spaces in Section 4.1, except that the set of scalars is enlarged to allow the use of complex numbers rather than just real numbers. Naturally, $\mathbb{C}^n$ is an example (in fact, the most important one) of a complex vector space. Also, under regular addition and complex scalar multiplication, both $\mathcal{M}_{mn}^{\mathbb{C}}$ and $\mathcal{P}_n^{\mathbb{C}}$ (polynomials of degree $\leq n$ with complex coefficients) are complex vector spaces (see Exercise 1).

The concepts of **subspace**, **span**, **linear independence**, **basis**, and **dimension** for real vector spaces carry over to complex vector spaces in an analogous way. All of the results in Chapter 4 have complex counterparts. In particular, if $\mathcal{W}$ is any subspace of a finite $n$-dimensional complex vector space (for example, $\mathbb{C}^n$), then $\mathcal{W}$ has a finite basis, and $\dim(\mathcal{W}) \leq n$.

Because every real scalar is also a complex number, every complex vector space is also a real vector space. Therefore, we must be careful about whether a vector space is being considered as a *real* or a *complex* vector space, that is, whether complex scalars are to be used or just real scalars. For example, $\mathbb{C}^3$ is both a real vector space and a complex vector space. As a *real* vector space, $\mathbb{C}^3$ has $\{[1,0,0], [i,0,0], [0,1,0], [0,i,0], [0,0,1], [0,0,i]\}$ as a basis and $\dim(\mathbb{C}^3) = 6$. But as a *complex* vector space, $\mathbb{C}^3$ has $\{[1,0,0], [0,1,0], [0,0,1]\}$ as a basis (since $i$ can now be used as a scalar) and so $\dim(\mathbb{C}^3) = 3$. In general, $\dim(\mathbb{C}^n) = 2n$ as a *real* vector space, but $\dim(\mathbb{C}^n) = n$ as a *complex* vector space. In Exercise 6, you are asked to prove that if $\mathcal{V}$ is an $n$-dimensional complex vector space, then $\mathcal{V}$ is a $2n$-dimensional real vector space.[2]

[2] The two different dimensions are sometimes distinguished by calling them the **real dimension** and the **complex dimension**.

As usual, we let $\mathbf{e}_i = [1, 0, 0, \ldots, 0]$, $\mathbf{e}_2 = [0, 1, 0, \ldots, 0]$, $\ldots$, $\mathbf{e}_n = [0, 0, 0, \ldots, 1]$ represent the **standard basis vectors** for the complex vector space $\mathbb{C}^n$.

**Coordinatization** in a complex vector space is done in the usual manner, as the following example indicates:

### Example 1

Consider the subspace $\mathcal{W}$ of the complex vector space $\mathbb{C}^4$ spanned by the vectors $\mathbf{x}_1 = [1+i, 3, 0, -2i]$ and $\mathbf{x}_2 = [-i, 1-i, 3i, 1+2i]$. Since these vectors are linearly independent (why?), the set $B = (\mathbf{x}_1, \mathbf{x}_2)$ is an ordered basis for $\mathcal{W}$ and $\dim(\mathcal{W}) = 2$. The linear combination $\mathbf{z} = (1-i)\mathbf{x}_1 + 3\mathbf{x}_2$ of these basis vectors is equal to

$$\begin{aligned}\mathbf{z} &= (1-i)\mathbf{x}_1 + 3\mathbf{x}_2 = [2, 3-3i, 0, -2-2i] + [-3i, 3-3i, 9i, 3+6i] \\ &= [2-3i, 6-6i, 9i, 1+4i].\end{aligned}$$

Of course, the coordinatization of $\mathbf{z}$ with respect to $B$ is $[\mathbf{z}]_B = [1-i, 3]$.

## Linear Transformations

Linear transformations from one complex vector space to another are defined just as for real vector spaces, except that complex scalars are used in the rule $L(k\mathbf{v}) = kL(\mathbf{v})$. The properties of complex linear transformations are completely analogous to those for linear transformations between real vector spaces.

Now every complex vector space is also a real vector space. Therefore, if $\mathcal{V}$ and $\mathcal{W}$ are complex vector spaces, and $L: \mathcal{V} \to \mathcal{W}$ is a complex linear transformation, then $L$ is also a real linear transformation when we consider $\mathcal{V}$ and $\mathcal{W}$ to be real vector spaces. Beware! The converse is not true. It is possible to have a real linear transformation $T: \mathcal{V} \to \mathcal{W}$ that is not a complex linear transformation, as in the next example.

### Example 2

Let $T: \mathbb{C}^2 \to \mathbb{C}^2$ be given by $T([z_1, z_2]) = [\overline{z_2}, \overline{z_1}]$. Then $T$ is a real linear transformation because it satisfies the two properties, as follows:

**(1)** $T([z_1, z_2] + [z_3, z_4]) = T([z_1 + z_3, z_2 + z_4]) = [\overline{z_2 + z_4}, \overline{z_1 + z_3}] = [\overline{z_2} + \overline{z_4}, \overline{z_1} + \overline{z_3}] = [\overline{z_2}, \overline{z_1}] + [\overline{z_4}, \overline{z_3}] = T([z_1, z_2]) + T([z_3, z_4])$.

**(2)** If $k \in \mathbb{R}$, then $T(k[z_1, z_2]) = T([kz_1, kz_2]) = [\overline{kz_2}, \overline{kz_1}] = [\overline{k}\overline{z_2}, \overline{k}\overline{z_1}] = [k\overline{z_2}, k\overline{z_1}] = k[\overline{z_2}, \overline{z_1}] = kT([z_1, z_2])$.

However, $T$ is not a complex linear transformation. Consider $T(i[1, i]) = T([i, -1]) = [-1, -i]$, while $iT([1, i]) = i[-i, 1] = [1, i]$ instead. Hence, $T$ is not a complex linear transformation.

## New Vocabulary

basis (for a complex vector space)
complex dimension (of a complex vector space)
complex vector spaces
coordinatization of a vector with respect to a basis (in a complex vector space)
linear independence (of a set of complex vectors)
linear transformation (from one complex vector space to another)
matrix for a linear transformation (from one complex vector space to another)
real dimension (of a complex vector space)
span (in a complex vector space)
standard basis vectors in $\mathbb{C}^n$
subspace (of a complex vector space)

## Highlights

- Complex vector spaces and subspaces are defined in a manner analogous to real vector spaces using the operations of complex vector addition and complex scalar multiplication.
- Span, linear independence, basis, dimension, and coordinatization are defined for complex vector spaces in the same manner as for real vector spaces.
- The standard basis vectors for $\mathbb{C}^n$ are the same as those for $\mathbb{R}^n$.
- When $\mathbb{C}^n$ is considered as a real vector space, $\dim(\mathbb{C}^n) = 2n$, but when $\mathbb{C}^n$ is considered as a complex vector space, $\dim(\mathbb{C}^n) = n$.
- Linear transformations between complex vector spaces are defined just as between real vector spaces, except that complex scalars may be used.

- Every complex linear transformation from a complex vector space $\mathcal{V}$ to a complex vector space $\mathcal{W}$ can be considered a real linear transformation when $\mathcal{V}$ and $\mathcal{W}$ are considered as real vector spaces. However, not every real linear transformation is a complex linear transformation.

## Exercises for Section 7.3

1. This exercise concerns $\mathcal{P}_n^{\mathbb{C}}$ and $\mathcal{M}_{mn}^{\mathbb{C}}$.
   (a) Show that the set $\mathcal{P}_n^{\mathbb{C}}$ of all polynomials of degree $\leq n$ under addition and complex scalar multiplication is a complex vector space.
   (b) Show that the set $\mathcal{M}_{mn}^{\mathbb{C}}$ of all $m \times n$ complex matrices under addition and complex scalar multiplication is a complex vector space.
2. Determine which of the following subsets of the complex vector space $\mathbb{C}^3$ are linearly independent. Also, in each case find the dimension of the span of the subset.
   (a) $\{[2+i, -i, 3], [-i, 3+i, -1]\}$
   ★ (b) $\{[2+i, -i, 3], [-3+6i, 3, 9i]\}$
   (c) $\{[3-i, 1+2i, -i], [1+i, -2, 4+i], [1-3i, 5+2i, -8-3i]\}$
   ★ (d) $\{[3-i, 1+2i, -i], [1+i, -2, 4+i], [3+i, -2+5i, 3-8i]\}$
3. Repeat Exercise 2 considering $\mathbb{C}^3$ as a *real* vector space. (Hint: First coordinatize the given vectors with respect to the basis $\{[1,0,0], [i,0,0], [0,1,0], [0,i,0], [0,0,1], [0,0,i]\}$ for $\mathbb{C}^3$. This essentially replaces the original vectors with vectors in $\mathbb{R}^6$, a more intuitive setting.)
4. This exercise explores a particular ordered basis for $\mathbb{C}^3$.
   (a) Show that $B = ([2i, -1+3i, 4], [3+i, -2, 1-i], [-3+5i, 2i, -5+3i])$ is an ordered basis for the complex vector space $\mathbb{C}^3$.
   ★ (b) Let $\mathbf{z} = [3-i, -5-5i, 7+i]$. For the ordered basis $B$ in part (a), find $[\mathbf{z}]_B$.
5. ★ With $\mathbb{C}^2$ as a real vector space, give an ordered basis for $\mathbb{C}^2$ and a matrix with respect to this basis for the linear transformation $L: \mathbb{C}^2 \to \mathbb{C}^2$ given by $L([z_1, z_2]) = [\overline{z_2}, \overline{z_1}]$. (Hint: What is the dimension of $\mathbb{C}^2$ as a real vector space?)
6. Let $\mathcal{V}$ be an $n$-dimensional complex vector space with basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. Prove that $\{\mathbf{v}_1, i\mathbf{v}_1, \mathbf{v}_2, i\mathbf{v}_2, \ldots, \mathbf{v}_n, i\mathbf{v}_n\}$ is a basis for $\mathcal{V}$ when considered as a real vector space.
7. Prove that not every real vector space can be considered to be a complex vector space. (Hint: Consider $\mathbb{R}^3$ and Exercise 6.)
8. ★ Give the matrix with respect to the standard bases for the linear transformation $L: \mathbb{C}^2 \to \mathbb{C}^3$ (considered as complex vector spaces) such that $L([1+i, -1+3i]) = [3-i, 5, -i]$ and $L([1-i, 1+2i]) = [2+i, 1-3i, 3]$.
9. ★ True or False:
   (a) Every linearly independent subset of a complex vector space $\mathcal{V}$ is contained in a basis for $\mathcal{V}$.
   (b) The function $L: \mathbb{C} \to \mathbb{C}$ given by $L(z) = \overline{z}$ is a complex linear transformation.
   (c) If $\mathcal{V}$ is an $n$-dimensional complex vector space with ordered basis $B$, then $L: \mathcal{V} \to \mathbb{C}^n$ given by $L(\mathbf{v}) = [\mathbf{v}]_B$ is a complex linear transformation.
   (d) Every complex subspace of a finite dimensional complex vector space has even (complex) dimension.

# 7.4 Orthogonality in $\mathbb{C}^n$

*Prerequisite: Section 6.3, Orthogonal Diagonalization*

In this section, we study orthogonality and the Gram-Schmidt Process in $\mathbb{C}^n$, and the complex analog of orthogonal diagonalization.

## Orthogonal Bases and the Gram-Schmidt Process

**Definition** A subset $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of vectors of $\mathbb{C}^n$ is **orthogonal** if and only if the *complex* dot product of any two distinct vectors in the set is zero. An orthogonal set of vectors in $\mathbb{C}^n$ is **orthonormal** if and only if each vector in the set is a unit vector.

As with real vector spaces, any set of orthogonal nonzero vectors in a complex vector space is linearly independent. The **Gram-Schmidt Process** for finding an orthogonal basis extends to the complex case, as in the next example.

**Example 1**

We find an orthogonal basis for the complex vector space $\mathbb{C}^3$ containing $\mathbf{w}_1 = [i, 1+i, 1]$. First, we use the Enlarging Method of Section 4.6 to find a basis for $\mathbb{C}^3$ containing $\mathbf{w}_1$. Row reducing

$$\begin{bmatrix} i & 1 & 0 & 0 \\ 1+i & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \quad \text{to obtain} \quad \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -i \\ 0 & 0 & 1 & -1-i \end{bmatrix}$$

shows that if $\mathbf{w}_2 = [1, 0, 0]$ and $\mathbf{w}_3 = [0, 1, 0]$, then $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is a basis for $\mathbb{C}^3$.

Let $\mathbf{v}_1 = \mathbf{w}_1$. Following the steps of the Gram-Schmidt Process, we obtain

$$\mathbf{v}_2 = \mathbf{w}_2 - \left(\frac{\mathbf{w}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 = [1, 0, 0] - \left(\frac{-i}{4}\right)[i, 1+i, 1].$$

Multiplying by 4 to avoid fractions, we get

$$\mathbf{v}_2 = [4, 0, 0] + i[i, 1+i, 1] = [3, -1+i, i].$$

Continuing, we get

$$\mathbf{v}_3 = \mathbf{w}_3 - \left(\frac{\mathbf{w}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{w}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 = [0, 1, 0] - \left(\frac{1-i}{4}\right)[i, 1+i, 1] - \left(\frac{-1-i}{12}\right)[3, -1+i, 1].$$

Multiplying by 12 to avoid fractions, we get

$$\mathbf{v}_3 = [0, 12, 0] + 3(-1+i)[i, 1+i, 1] + (1+i)[3, -1+i, i] = [0, 4, -4+4i].$$

We can divide by 4 to avoid multiples, and so finally get $\mathbf{v}_3 = [0, 1, -1+i]$. Hence, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{[i, 1+i, 1], [3, -1+i, i], [0, 1, -1+i]\}$ is an orthogonal basis for $\mathbb{C}^3$ containing $\mathbf{w}_1$. (You should verify that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are mutually orthogonal.)

We can normalize $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ to obtain the following orthonormal basis for $\mathbb{C}^3$:

$$\left\{\left[\frac{i}{2}, \frac{1+i}{2}, \frac{1}{2}\right], \left[\frac{3}{2\sqrt{3}}, \frac{-1+i}{2\sqrt{3}}, \frac{i}{2\sqrt{3}}\right], \left[0, \frac{1}{\sqrt{3}}, \frac{-1+i}{\sqrt{3}}\right]\right\}.$$

Recall that the complex dot product is not symmetric. Hence, in Example 1 we were careful in the Gram-Schmidt Process to compute the dot products $\mathbf{w}_2 \cdot \mathbf{v}_1$, $\mathbf{w}_3 \cdot \mathbf{v}_1$, and $\mathbf{w}_3 \cdot \mathbf{v}_2$ in the correct order. If we had computed $\mathbf{v}_1 \cdot \mathbf{w}_2$, $\mathbf{v}_1 \cdot \mathbf{w}_3$, and $\mathbf{v}_2 \cdot \mathbf{w}_3$ instead, the vectors obtained would not be orthogonal.

## Unitary Matrices

We now examine the complex analog of orthogonal matrices.

**Definition** A nonsingular (square) complex matrix $\mathbf{A}$ is **unitary** if and only if $\mathbf{A}^* = \mathbf{A}^{-1}$ (that is, if and only if $(\overline{\mathbf{A}})^T = \mathbf{A}^{-1}$).

It follows immediately that every unitary matrix is a normal matrix (why?).

**Example 2**

For

$$\mathbf{A} = \begin{bmatrix} \frac{1-i}{\sqrt{3}} & 0 & \frac{i}{\sqrt{3}} \\ \frac{-1+i}{\sqrt{15}} & \frac{3}{\sqrt{15}} & \frac{2i}{\sqrt{15}} \\ \frac{1-i}{\sqrt{10}} & \frac{2}{\sqrt{10}} & \frac{-2i}{\sqrt{10}} \end{bmatrix}, \quad \text{we have} \quad \mathbf{A}^* = \left(\overline{\mathbf{A}}\right)^T = \begin{bmatrix} \frac{1+i}{\sqrt{3}} & \frac{-1-i}{\sqrt{15}} & \frac{1+i}{\sqrt{10}} \\ 0 & \frac{3}{\sqrt{15}} & \frac{2}{\sqrt{10}} \\ -\frac{i}{\sqrt{3}} & -\frac{2i}{\sqrt{15}} & \frac{2i}{\sqrt{10}} \end{bmatrix}.$$

A quick calculation shows that $\mathbf{A}\mathbf{A}^* = \mathbf{I}_3$ (verify!), so $\mathbf{A}$ is unitary.

The following theorem gives some basic properties of unitary matrices, and is analogous to Theorem 6.6.

**Theorem 7.6** *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are unitary matrices of the same size, then*

(1) *The absolute value of* $|\mathbf{A}|$ *equals* 1 *(that is,* $\big||\mathbf{A}|\big| = 1$*),*

(2) $\mathbf{A}^* = \mathbf{A}^{-1} = \left(\overline{\mathbf{A}}\right)^T$ *is unitary, and*

(3) $\mathbf{AB}$ *is unitary.*

The proof of part (1) is left as Exercise 4, while the proof of parts (2) and (3) are left as Exercise 5. The next two theorems are the analogs of Theorems 6.7 and 6.8. They are left for you to prove in Exercises 7 and 8. You should verify that the unitary matrix of Example 2 satisfies Theorem 7.7.

**Theorem 7.7** *Let* $\mathbf{A}$ *be an* $n \times n$ *complex matrix. Then* $\mathbf{A}$ *is unitary*

(1) *if and only if the rows of* $\mathbf{A}$ *form an orthonormal basis for* $\mathbb{C}^n$

(2) *if and only if the columns of* $\mathbf{A}$ *form an orthonormal basis for* $\mathbb{C}^n$.

**Theorem 7.8** *Let B and C be ordered orthonormal bases for* $\mathbb{C}^n$. *Then the transition matrix from B to C is a unitary matrix.*

## Unitarily Diagonalizable Matrices

We now consider the complex analog of orthogonal diagonalization.

**Definition** A (square) complex matrix $\mathbf{A}$ is **unitarily diagonalizable** if and only if there is a unitary matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{AP}$ is diagonal.

### Example 3

Consider the matrix

$$\mathbf{P} = \frac{1}{3}\begin{bmatrix} -2i & 2 & 1 \\ 2i & 1 & 2 \\ 1 & -2i & 2i \end{bmatrix}.$$

Notice that $\mathbf{P}$ is a unitary matrix, since the columns of $\mathbf{P}$ form an orthonormal basis for $\mathbb{C}^3$.

Next, consider the matrix

$$\mathbf{A} = \frac{1}{3}\begin{bmatrix} -1+3i & 2+2i & -2 \\ 2+2i & 2i & -2i \\ 2 & 2i & 1+4i \end{bmatrix}.$$

Now, $\mathbf{A}$ is unitarily diagonalizable because

$$\mathbf{P}^{-1}\mathbf{AP} = \mathbf{P}^*\mathbf{AP} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2i & 0 \\ 0 & 0 & 1+i \end{bmatrix},$$

a diagonal matrix.

We saw in Section 6.3 that a matrix is orthogonally diagonalizable if and only if it is symmetric. The following theorem, stated without proof, characterizes unitarily diagonalizable matrices:

**Theorem 7.9** *A complex matrix* $\mathbf{A}$ *is unitarily diagonalizable if and only if* $\mathbf{A}$ *is normal.*

A quick calculation shows that the matrix $\mathbf{A}$ in Example 3 is normal (see Exercise 9).

**Example 4**

Let $\mathbf{A} = \begin{bmatrix} -48+18i & -24+36i \\ 24-36i & -27+32i \end{bmatrix}$. A direct computation of $\mathbf{AA}^*$ and $\mathbf{A}^*\mathbf{A}$ shows that $\mathbf{A}$ is normal (verify!). Therefore, $\mathbf{A}$ is unitarily diagonalizable by Theorem 7.9. After some calculation, you can verify that the eigenvalues of $\mathbf{A}$ are $\lambda_1 = 50i$ and $\lambda_2 = -75$. Hence, $\mathbf{A}$ is unitarily diagonalizable to $\mathbf{D} = \begin{bmatrix} 50i & 0 \\ 0 & -75 \end{bmatrix}$.

In fact, $\lambda_1$ and $\lambda_2$ have associated eigenvectors $\mathbf{v}_1 = \left[\frac{3}{5}, -\frac{4}{5}i\right]$ and $\mathbf{v}_2 = \left[-\frac{4}{5}i, \frac{3}{5}\right]$. Since $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthonormal set, the matrix $\mathbf{P} = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5}i \\ -\frac{4}{5}i & \frac{3}{5} \end{bmatrix}$, whose columns are $\mathbf{v}_1$ and $\mathbf{v}_2$, is a unitary matrix, and $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{P}^*\mathbf{AP} = \mathbf{D}$.

## Self-Adjoint Operators and Hermitian Matrices

An immediate corollary of Theorems 7.4 and 7.9 is

**Corollary 7.10** *If* $\mathbf{A}$ *is a Hermitian or skew-Hermitian matrix, then* $\mathbf{A}$ *is unitarily diagonalizable.*

We can prove even more about Hermitian matrices. First, we introduce some new terminology. If linear operators $L$ and $M$ on $\mathbb{C}^n$ have the property $L(\mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot M(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$, then $M$ is called an **adjoint** of $L$. Now, suppose that $L$: $\mathbb{C}^n \to \mathbb{C}^n$ is the linear operator $L(\mathbf{x}) = \mathbf{Ax}$, where $\mathbf{A}$ is an $n \times n$ matrix, and let $L^*$: $\mathbb{C}^n \to \mathbb{C}^n$ be given by $L^*(\mathbf{x}) = \mathbf{A}^*\mathbf{x}$. By Theorem 7.3, $(L(\mathbf{x})) \cdot \mathbf{y} = \mathbf{x} \cdot (L^*(\mathbf{y}))$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$, and so $L^*$ is an adjoint of $L$.

Now, if $\mathbf{A}$ is a Hermitian matrix, then $\mathbf{A} = \mathbf{A}^*$, and so $L = L^*$. Thus, $(L(\mathbf{x})) \cdot \mathbf{y} = \mathbf{x} \cdot (L(\mathbf{y}))$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$. Such an operator is called **self-adjoint**, since it is its own adjoint. It can be shown that every self-adjoint operator on $\mathbb{C}^n$ has a Hermitian matrix representation with respect to any orthonormal basis. Self-adjoint operators are the complex analogs of the symmetric operators in Section 6.3. Corollary 7.10 asserts that all self-adjoint operators are unitarily diagonalizable. The converse to Corollary 7.10 is *not* true because there are unitarily diagonalizable (= normal) matrices that are not Hermitian. This differs from the situation with linear operators on real vector spaces where the analog of the converse of Corollary 7.10 *is* true; that is, every orthogonally diagonalizable linear operator is symmetric.

The final theorem of this section shows that any diagonal matrix representation for a self-adjoint operator has all real entries.

**Theorem 7.11** *All eigenvalues of a Hermitian matrix are real.*

*Proof.* Let $\lambda$ be an eigenvalue for a Hermitian matrix $\mathbf{A}$, and let $\mathbf{u}$ be a unit eigenvector for $\lambda$. Then $\lambda = \lambda\,\|\mathbf{u}\|^2 = \lambda(\mathbf{u} \cdot \mathbf{u}) = (\lambda\mathbf{u}) \cdot \mathbf{u} = (\mathbf{Au}) \cdot \mathbf{u} = \mathbf{u} \cdot (\mathbf{Au})$ (by Theorem 7.3) $= \mathbf{u} \cdot \lambda\mathbf{u} = \overline{\lambda}(\mathbf{u} \cdot \mathbf{u})$ (by part (5) of Theorem 7.1) $= \overline{\lambda}$. Hence, $\lambda$ is real. □

**Example 5**

Consider the Hermitian matrix

$$\mathbf{A} = \begin{bmatrix} 17 & -24+8i & -24-32i \\ -24-8i & 53 & 4+12i \\ -24+32i & 4-12i & 11 \end{bmatrix}.$$

By Theorem 7.11, all eigenvalues of $\mathbf{A}$ are real. It can be shown that these eigenvalues are $\lambda_1 = 27$, $\lambda_2 = -27$, and $\lambda_3 = 81$. By Corollary 7.10, $\mathbf{A}$ is unitarily diagonalizable. In fact, the unitary matrix

$$\mathbf{P} = \frac{1}{9}\begin{bmatrix} 4 & 6-2i & -3-4i \\ 6+2i & 1 & 2+6i \\ -3+4i & 2-6i & 4 \end{bmatrix}$$

has the property that $\mathbf{P}^{-1}\mathbf{AP}$ is the diagonal matrix with eigenvalues $\lambda_1$, $\lambda_2$, and $\lambda_3$ on the main diagonal (verify!).

Every real symmetric matrix $\mathbf{A}$ can be thought of as a complex Hermitian matrix. Now $p_{\mathbf{A}}(x)$ must have at least one complex root. But by Theorem 7.11, this eigenvalue for $\mathbf{A}$ must be real. This gives us a shorter proof of Lemma 6.21 in Section 6.3. (We did not use this method of proof in Section 6.3 since it entails complex numbers.)

## New Vocabulary

adjoint linear operator
Gram-Schmidt Process (for finding an orthogonal basis for a subspace of $\mathbb{C}^n$)
orthogonal set (of complex vectors)
orthonormal set (of complex vectors)
self-adjoint linear operator
unitarily diagonalizable matrix
unitary matrix

## Highlights

- A subset $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of vectors of $\mathbb{C}^n$ is orthogonal if and only if the *complex* dot product of any two distinct vectors in the set is zero. An orthogonal set of vectors in $\mathbb{C}^n$ is orthonormal if and only if each vector in the set is a unit vector.
- The Gram-Schmidt Process can be used to convert a linearly independent set $T$ of vectors in $\mathbb{C}^n$ to an orthogonal basis for span($T$), but must be applied with care since the complex dot product is not commutative.
- A (square) complex matrix $\mathbf{A}$ is unitary iff $\mathbf{A}^* = \mathbf{A}^{-1}$.
- If $\mathbf{A}$ and $\mathbf{B}$ are unitary, then $\left| |\mathbf{A}| \right| = 1$, $\mathbf{A}^*$ is unitary, and $\mathbf{AB}$ is unitary.
- An $n \times n$ complex matrix $\mathbf{A}$ is unitary iff the rows of $\mathbf{A}$ form an orthonormal basis for $\mathbb{C}^n$ iff the columns of $\mathbf{A}$ form an orthonormal basis for $\mathbb{C}^n$.
- If $B$ and $C$ are any ordered orthonormal bases for $\mathbb{C}^n$, then the transition matrix from $B$ to $C$ is a unitary matrix.
- A complex matrix $\mathbf{A}$ is unitarily diagonalizable iff there is a unitary matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{AP}$ is diagonal.
- If $\mathbf{A}$ is Hermitian or skew-Hermitian, then $\mathbf{A}$ is unitarily diagonalizable.
- If $\mathbf{A}$ is Hermitian, all eigenvalues of $\mathbf{A}$ are real.
- A complex matrix $\mathbf{A}$ is unitarily diagonalizable iff $\mathbf{A}$ is normal.
- If $L\colon \mathbb{C}^n \to \mathbb{C}^n$ is a self-adjoint operator, then $L$ is unitarily diagonalizable. However, not every unitarily diagonalizable operator is self-adjoint.

# Exercises for Section 7.4

1. Determine whether the following sets of vectors are orthogonal.
   ★ (a) In $\mathbb{C}^2$: $\{[1+2i, -3-i], [4-2i, 3+i]\}$
   (b) In $\mathbb{C}^3$: $\{[1-i, -1+i, 1-i], [i, -2i, 2i]\}$
   ★ (c) In $\mathbb{C}^3$: $\{[2i, -1, i], [1, -i, -1], [0, 1, i]\}$
   (d) In $\mathbb{C}^4$: $\{[1, i, -1, 1+i], [4, -i, 1, -1-i], [0, 3, -i, -1+i]\}$
2. Suppose $\{\mathbf{z}_1, \ldots, \mathbf{z}_k\}$ is an orthonormal subset of $\mathbb{C}^n$, and $c_1, \ldots, c_k \in \mathbb{C}$ with $|c_i| = 1$ for $1 \le i \le k$. Prove that $\{c_1\mathbf{z}_1, \ldots, c_k\mathbf{z}_k\}$ is an orthonormal subset of $\mathbb{C}^n$.

★ 3. This exercise concerns a particular orthogonal basis for $\mathbb{C}^3$.
   (a) Use the Gram-Schmidt Process to find an orthogonal basis for $\mathbb{C}^3$ containing $[1+i, i, 1]$.
   (b) Find a $3 \times 3$ unitary matrix having a multiple of $[1+i, i, 1]$ as its first row.

► 4. Prove part (1) of Theorem 7.6.

5. This exercise asks for proofs for parts of Theorem 7.6.
   (a) Prove part (2) of Theorem 7.6.
   (b) Prove part (3) of Theorem 7.6.
6. This exercise establishes certain properties of unitary matrices.
   (a) Prove that a complex matrix $\mathbf{A}$ is unitary if and only if $\overline{\mathbf{A}}$ is unitary.
   (b) Let $\mathbf{A}$ be a unitary matrix. Prove that $\mathbf{A}^k$ is unitary for all integers $k \ge 1$.
   (c) Let $\mathbf{A}$ be a unitary matrix. Prove that $\mathbf{A}^2 = \mathbf{I}_n$ if and only if $\mathbf{A}$ is Hermitian.
7. This exercise is related to Theorem 7.7.
   ► (a) Without using Theorem 7.7, prove that $\mathbf{A}$ is a unitary matrix if and only if $\mathbf{A}^T$ is unitary.
   (b) Prove Theorem 7.7. (Hint: First prove part (1) of Theorem 7.7, and then use part (a) of this exercise to prove part (2). Modify the proof of Theorem 6.7. For instance, when $i \ne j$, to show that the $i$th row of $\mathbf{A}$ is orthogonal to the $j$th column of $\mathbf{A}$, we must show that the *complex* dot product of the $i$th row of $\mathbf{A}$ with the $j$th column of $\mathbf{A}$ equals zero.)

8. Prove Theorem 7.8. (Hint: Modify the proof of Theorem 6.8.)
9. Show that the matrix **A** in Example 3 is normal.
10. This exercise investigates a particular linear operator on $\mathbb{C}^2$.
    (a) Show that the linear operator $L: \mathbb{C}^2 \to \mathbb{C}^2$ given by $L\left(\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}\right) = \begin{bmatrix} 1-6i & -10-2i \\ 2-10i & 5 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ is unitarily diagonalizable.
    ★ (b) If **A** is the matrix for $L$ (with respect to the standard basis for $\mathbb{C}^2$), find a unitary matrix **P** such that $\mathbf{P}^{-1}\mathbf{AP}$ is diagonal.
11. This exercise investigates a particular $3 \times 3$ complex matrix.
    (a) Show that the following matrix is unitarily diagonalizable:

$$\mathbf{A} = \begin{bmatrix} -4+5i & 2+2i & 4+4i \\ 2+2i & -1+8i & -2-2i \\ 4+4i & -2-2i & -4+5i \end{bmatrix}.$$

    (b) Find a unitary matrix **P** such that $\mathbf{P}^{-1}\mathbf{AP}$ is diagonal.
12. This exercise establishes properties of certain unitary matrices.
    (a) Let **A** be a unitary matrix. Show that $|\lambda| = 1$ for every eigenvalue $\lambda$ of **A**. (Hint: Suppose $\mathbf{Az} = \lambda\mathbf{z}$, for some $\mathbf{z} \neq \mathbf{0}$. Use Theorem 7.3 to calculate $\mathbf{Az} \cdot \mathbf{Az}$ two different ways to show that $\lambda\overline{\lambda} = 1$.)
    (b) Prove that a unitary matrix **A** is Hermitian if and only if all the eigenvalues of **A** are 1 and/or $-1$.

★ 13. Verify directly that all of the eigenvalues of the following Hermitian matrix are real:

$$\begin{bmatrix} 1 & 2+i & 1-2i \\ 2-i & -3 & -i \\ 1+2i & i & 2 \end{bmatrix}.$$

14. This exercise establishes certain results concerning normal matrices.
    (a) Prove that if **A** is normal and all eigenvalues of **A** are real, then **A** is Hermitian. (Hint: Use Theorem 7.9 to express **A** as $\mathbf{PDP}^*$ for some unitary **P** and diagonal **D**. Calculate $\mathbf{A}^*$.)
    (b) Prove that if **A** is normal and all eigenvalues have absolute value equal to 1, then **A** is unitary. (Hint: With $\mathbf{A} = \mathbf{PDP}^*$ as in part (a), show $\mathbf{DD}^* = \mathbf{I}$ and use this to calculate $\mathbf{AA}^*$.)
    (c) Prove that if **A** is unitary, then **A** is normal.

★ 15. True or False:
    (a) Every Hermitian matrix is unitary.
    (b) Every orthonormal basis for $\mathbb{R}^n$ is also an orthonormal basis for $\mathbb{C}^n$.
    (c) An $n \times n$ complex matrix **A** is unitarily diagonalizable if and only if there is a unitary matrix **P** such that $\mathbf{PAP}^*$ is diagonal.
    (d) If the columns of an $n \times n$ matrix **A** form an orthonormal basis for $\mathbb{C}^n$, then the rows of **A** also form an orthonormal basis for $\mathbb{C}^n$.
    (e) If **A** is the matrix with respect to the standard basis for a linear operator $L$ on $\mathbb{C}^n$, then $\mathbf{A}^T$ is the matrix for the adjoint of $L$ with respect to the standard basis.

## 7.5 Inner Product Spaces

*Prerequisite: Section 6.3, Orthogonal Diagonalization*

In $\mathbb{R}^n$ and $\mathbb{C}^n$, we have the dot product along with the operations of vector addition and scalar multiplication. In other vector spaces, we can often create a similar type of product, known as an inner product.

## Inner Products

**Definition** Let $\mathcal{V}$ be a real [complex] vector space with operations $+$ and $\cdot$, and let $\langle\, , \,\rangle$ be an operation that assigns to each pair of vectors $\mathbf{x}, \mathbf{y} \in \mathcal{V}$ a real [complex] number, denoted $\langle \mathbf{x}, \mathbf{y} \rangle$. Then $\langle\, , \,\rangle$ is a **real [complex] inner product** for $\mathcal{V}$ if and only if the following properties hold for all $\mathbf{x}, \mathbf{y} \in \mathcal{V}$ and all $k \in \mathbb{R}\,[k \in \mathbb{C}]$:

**(1)** $\langle \mathbf{x}, \mathbf{x} \rangle$ is always real, and $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$

**(2)** $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ if and only if $\mathbf{x} = \mathbf{0}$

**(3)** $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle \ \left[\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}\,\right]$

**(4)** $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$

**(5)** $\langle k\mathbf{x}, \mathbf{y} \rangle = k\,\langle \mathbf{x}, \mathbf{y} \rangle$

A vector space together with a real [complex] inner product operation is known as a **real [complex] inner product space**.

### Example 1

Consider the real vector space $\mathbb{R}^n$. Let $\mathbf{x} = [x_1, \ldots, x_n]$ and $\mathbf{y} = [y_1, \ldots, y_n]$ be vectors in $\mathbb{R}^n$. By Theorem 1.5, the operation $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \cdots + x_n y_n$ (usual real dot product) is a real inner product (verify!). Hence, $\mathbb{R}^n$ together with the dot product is a real inner product space.

Similarly, let $\mathbf{x} = [x_1, \ldots, x_n]$ and $\mathbf{y} = [y_1, \ldots, y_n]$ be vectors in the complex vector space $\mathbb{C}^n$. By Theorem 7.1, the operation $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y} = x_1 \overline{y_1} + \cdots + x_n \overline{y_n}$ (usual complex dot product) is an inner product on $\mathbb{C}^n$. Thus, $\mathbb{C}^n$ together with the complex dot product is a complex inner product space.

### Example 2

Consider the real vector space $\mathbb{R}^2$. For $\mathbf{x} = [x_1, x_2]$ and $\mathbf{y} = [y_1, y_2]$ in $\mathbb{R}^2$, define $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 - x_1 y_2 - x_2 y_1 + 2x_2 y_2$. We verify the five properties in the definition of an inner product space.

**Property (1)**: $\langle \mathbf{x}, \mathbf{x} \rangle = x_1 x_1 - x_1 x_2 - x_2 x_1 + 2x_2 x_2 = x_1^2 - 2x_1 x_2 + x_2^2 + x_2^2 = (x_1 - x_2)^2 + x_2^2 \geq 0$.

**Property (2):** $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ exactly when $x_1 = x_2 = 0$ (that is, when $\mathbf{x} = \mathbf{0}$).

**Property (3):** $\langle \mathbf{y}, \mathbf{x} \rangle = y_1 x_1 - y_1 x_2 - y_2 x_1 + 2y_2 x_2 = x_1 y_1 - x_1 y_2 - x_2 y_1 + 2x_2 y_2 = \langle \mathbf{x}, \mathbf{y} \rangle$.

**Property (4):** Let $\mathbf{z} = [z_1, z_2]$. Then

$$\begin{aligned}
\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle &= (x_1 + y_1)z_1 - (x_1 + y_1)z_2 - (x_2 + y_2)z_1 + 2(x_2 + y_2)z_2 \\
&= x_1 z_1 + y_1 z_1 - x_1 z_2 - y_1 z_2 - x_2 z_1 - y_2 z_1 + 2x_2 z_2 + 2y_2 z_2 \\
&= (x_1 z_1 - x_1 z_2 - x_2 z_1 + 2x_2 z_2) + (y_1 z_1 - y_1 z_2 - y_2 z_1 + 2y_2 z_2) \\
&= \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle\,.
\end{aligned}$$

**Property (5):** $\langle k\mathbf{x}, \mathbf{y} \rangle = (kx_1)y_1 - (kx_1)y_2 - (kx_2)y_1 + 2(kx_2)y_2 = k(x_1 y_1 - x_1 y_2 - x_2 y_1 + 2x_2 y_2) = k\,\langle \mathbf{x}, \mathbf{y} \rangle$.

Hence, $\langle\, , \,\rangle$ is a real inner product on $\mathbb{R}^2$, and $\mathbb{R}^2$ together with this operation $\langle\, , \,\rangle$ is a real inner product space.

### Example 3

Consider the real vector space $\mathbb{R}^n$. Let $\mathbf{A}$ be a nonsingular $n \times n$ real matrix. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and define $\langle \mathbf{x}, \mathbf{y} \rangle = (\mathbf{A}\mathbf{x}) \cdot (\mathbf{A}\mathbf{y})$ (the usual dot product of $\mathbf{A}\mathbf{x}$ and $\mathbf{A}\mathbf{y}$). It can be shown (see Exercise 1) that $\langle\, , \,\rangle$ is a real inner product on $\mathbb{R}^n$, and so $\mathbb{R}^n$ together with this operation $\langle\, , \,\rangle$ is a real inner product space.

### Example 4

Consider the real vector space $\mathcal{P}_n$. Let $\mathbf{p}_1 = a_n x^n + \cdots + a_1 x + a_0$ and $\mathbf{p}_2 = b_n x^n + \cdots + b_1 x + b_0$ be in $\mathcal{P}_n$. Define $\langle \mathbf{p}_1, \mathbf{p}_2 \rangle = a_n b_n + \cdots + a_1 b_1 + a_0 b_0$. It can be shown (see Exercise 2) that $\langle\, , \,\rangle$ is a real inner product on $\mathcal{P}_n$, and so $\mathcal{P}_n$ together with this operation $\langle\, , \,\rangle$ is a real inner product space.

---

**Example 5**

Let $a, b \in \mathbb{R}$, with $a < b$, and consider the real vector space $\mathcal{V}$ of all real-valued continuous functions defined on the interval $[a, b]$ (for example, polynomials, $\sin x$, $e^x$). Let $\mathbf{f}, \mathbf{g} \in \mathcal{V}$. Define $\langle \mathbf{f}, \mathbf{g} \rangle = \int_a^b \mathbf{f}(t)\mathbf{g}(t)\,dt$. It can be shown (see Exercise 3) that $\langle\, , \,\rangle$ is a real inner product on $\mathcal{V}$, and so $\mathcal{V}$ together with this operation $\langle\, , \,\rangle$ is a real inner product space.

Analogously, the operation $\langle \mathbf{f}, \mathbf{g} \rangle = \int_a^b \mathbf{f}(t)\overline{\mathbf{g}(t)}\,dt$ makes the complex vector space of all complex-valued continuous functions on $[a, b]$ into a complex inner product space.

---

Of course, not every operation is an inner product. For example, for the vectors $\mathbf{x} = [x_1, x_2]$ and $\mathbf{y} = [y_1, y_2]$ in $\mathbb{R}^2$, consider the operation $\langle \mathbf{x}, \mathbf{y} \rangle = x_1^2 + y_1^2$. Now, with $\mathbf{x} = \mathbf{y} = [1, 0]$, we have $\langle 2\mathbf{x}, \mathbf{y} \rangle = 2^2 + 1^2 = 5$, but $2\,\langle \mathbf{x}, \mathbf{y} \rangle = 2(1^2 + 1^2) = 4$. Thus, property (5) fails to hold.

The next theorem lists some useful results for inner product spaces.

**Theorem 7.12** *Let $\mathcal{V}$ be a real [complex] inner product space with inner product $\langle\, , \,\rangle$. Then for all $\mathbf{x}, \mathbf{y} \in \mathcal{V}$ and all $k \in \mathbb{R}$ [$k \in \mathbb{C}$], we have*

**(1)** $\langle \mathbf{0}, \mathbf{x} \rangle = \langle \mathbf{x}, \mathbf{0} \rangle = 0$.

**(2)** $\langle \mathbf{x}, \mathbf{y} + \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle$.

**(3)** $\langle \mathbf{x}, k\mathbf{y} \rangle = k\,\langle \mathbf{x}, \mathbf{y} \rangle \quad [\langle \mathbf{x}, k\mathbf{y} \rangle = \overline{k}\,\langle \mathbf{x}, \mathbf{y} \rangle]$.

Note the use of $\overline{k}$ in part (3) for complex vector spaces. The proof of this theorem is straightforward, and parts are left for you to do in Exercise 5.

## Length, Distance, and Angles in Inner Product Spaces

The next definition extends the concept of the length of a vector to any inner product space.

**Definition** If $\mathbf{x}$ is a vector in an inner product space, then the **norm (length)** of $\mathbf{x}$ is $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$.

This definition yields a nonnegative real number for $\|\mathbf{x}\|$, since by definition, $\langle \mathbf{x}, \mathbf{x} \rangle$ is always real and nonnegative for any vector $\mathbf{x}$. Also note that this definition agrees with the earlier definition of length in $\mathbb{R}^n$ based on the usual dot product in $\mathbb{R}^n$. We also have the following result:

**Theorem 7.13** *Let $\mathcal{V}$ be a real [complex] inner product space, with $\mathbf{x} \in \mathcal{V}$. Let $k \in \mathbb{R}$ [$k \in \mathbb{C}$]. Then, $\|k\mathbf{x}\| = |k|\,\|\mathbf{x}\|$.*

The proof of this theorem is left for you to do in Exercise 6.

As before, we say that a vector of length 1 in an inner product space is a **unit vector**. For instance, in the inner product space of Example 4, the polynomial $\mathbf{p} = \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}$ is a unit vector since $\|\mathbf{p}\| = \sqrt{\langle \mathbf{p}, \mathbf{p} \rangle} = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2} = 1$.

We define the distance between two vectors in the general inner product space setting as we did for $\mathbb{R}^n$:

**Definition** Let $\mathbf{x}, \mathbf{y} \in \mathcal{V}$, an inner product space. Then the **distance between x and y** is $\|\mathbf{x} - \mathbf{y}\|$.

---

**Example 6**

Consider the real vector space $\mathcal{V}$ of real continuous functions from Example 5, with $a = 0$ and $b = \pi$. That is, $\langle \mathbf{f}, \mathbf{g} \rangle = \int_0^\pi \mathbf{f}(t)\mathbf{g}(t)\,dt$ for all $\mathbf{f}, \mathbf{g} \in \mathcal{V}$. Let $\mathbf{f} = \cos t$ and $\mathbf{g} = \sin t$. Then the distance between $\mathbf{f}$ and $\mathbf{g}$ is

$$\|\mathbf{f} - \mathbf{g}\| = \sqrt{\langle \cos t - \sin t,\ \cos t - \sin t \rangle} = \sqrt{\int_0^\pi (\cos t - \sin t)^2\,dt}$$

$$= \sqrt{\int_0^\pi \left(\cos^2 t - 2\cos t \sin t + \sin^2 t\right)\,dt} = \sqrt{\int_0^\pi (1 - \sin 2t)\,dt} = \sqrt{\left(t + \frac{1}{2}\cos 2t\right)\Big|_0^\pi} = \sqrt{\pi}.$$

Hence, the distance between $\cos t$ and $\sin t$ is $\sqrt{\pi}$ under this inner product.

---

The next theorem shows that some other familiar results from the ordinary dot product carry over to the general inner product.

**Theorem 7.14** *Let* $\mathbf{x}, \mathbf{y} \in \mathcal{V}$*, an inner product space, with inner product* $\langle\, , \,\rangle$*. Then*
(1) $|\langle \mathbf{x}, \mathbf{y}\rangle| \leq \|\mathbf{x}\|\,\|\mathbf{y}\|$ *Cauchy-Schwarz Inequality*
(2) $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ *Triangle Inequality.*

The proofs of these statements are analogous to the proofs for the ordinary dot product and are left for you to do in Exercise 11.

From the Cauchy-Schwarz Inequality, we have $-1 \leq \langle \mathbf{x}, \mathbf{y}\rangle / (\|\mathbf{x}\|\,\|\mathbf{y}\|) \leq 1$, for any nonzero vectors $\mathbf{x}$ and $\mathbf{y}$ in a *real* inner product space. Hence, we can make the following definition:

**Definition** Let $\mathbf{x}, \mathbf{y} \in \mathcal{V}$, a *real* inner product space. Then the **angle between x and y** is the angle $\theta$ from 0 to $\pi$ such that $\cos\theta = \langle \mathbf{x}, \mathbf{y}\rangle / (\|\mathbf{x}\|\,\|\mathbf{y}\|)$.

**Example 7**

Consider again the inner product space of Example 6, where $\langle \mathbf{f}, \mathbf{g}\rangle = \int_0^\pi \mathbf{f}(t)\mathbf{g}(t)\,dt$. Let $\mathbf{f} = t$ and $\mathbf{g} = \sin t$. Then $\langle \mathbf{f}, \mathbf{g}\rangle = \int_0^\pi t \sin t\,dt$. Using integration by parts, we get

$$\langle \mathbf{f}, \mathbf{g}\rangle = (-t\cos t)\big|_0^\pi + \int_0^\pi \cos t\,dt = \pi + (\sin t)\big|_0^\pi = \pi.$$

Also,

$$\|\mathbf{f}\|^2 = \langle \mathbf{f}, \mathbf{f}\rangle = \int_0^\pi (\mathbf{f}(t))^2\,dt = \int_0^\pi t^2\,dt = \left(t^3/3\right)\Big|_0^\pi = \pi^3/3,$$

and so $\|\mathbf{f}\| = \sqrt{\pi^3/3}$. Similarly,

$$\begin{aligned}\|\mathbf{g}\|^2 = \langle \mathbf{g}, \mathbf{g}\rangle &= \int_0^\pi (\mathbf{g}(t))^2\,dt = \int_0^\pi \sin^2 t\,dt \\ &= \int_0^\pi \frac{1}{2}(1-\cos 2t)\,dt = \left(\frac{1}{2}t - \frac{1}{4}\sin 2t\right)\Big|_0^\pi = \pi/2,\end{aligned}$$

and so $\|\mathbf{g}\| = \sqrt{\pi/2}$. Hence, the cosine of the angle $\theta$ between $t$ and $\sin t$ equals

$$\langle \mathbf{f}, \mathbf{g}\rangle / (\|\mathbf{f}\|\,\|\mathbf{g}\|) = \pi/\left(\left(\sqrt{\pi^3/3}\right)\left(\sqrt{\pi/2}\right)\right) = \sqrt{6}/\pi \approx 0.78.$$

Hence, $\theta \approx 0.68$ radians (38.8°).

## Orthogonality in Inner Product Spaces

We next define orthogonal vectors in a general inner product space setting and show that nonzero orthogonal vectors are linearly independent.

**Definition** A subset $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ of vectors in an inner product space $\mathcal{V}$ with inner product $\langle\, , \,\rangle$ is **orthogonal** if and only if $\langle \mathbf{x}_i, \mathbf{x}_j\rangle = 0$ for $1 \leq i, j \leq n$, with $i \neq j$. Also, an orthogonal set of vectors in $\mathcal{V}$ is **orthonormal** if and only if each vector in the set is a unit vector.

The next theorem is the analog of Theorem 6.1, and its proof is left for you to do in Exercise 15.

**Theorem 7.15** *If* $\mathcal{V}$ *is an inner product space and* $T$ *is an orthogonal set of nonzero vectors in* $\mathcal{V}$*, then* $T$ *is a linearly independent set.*

**Example 8**

Consider again the inner product space $\mathcal{V}$ of Example 5 of real continuous functions with inner product $\langle \mathbf{f}, \mathbf{g} \rangle = \int_a^b \mathbf{f}(t)\mathbf{g}(t)\, dt$, with $a = -\pi$ and $b = \pi$. The set $\{1, \cos t, \sin t\}$ is an orthogonal set in $\mathcal{V}$, since each of the following definite integrals equals zero (verify!):

$$\int_{-\pi}^{\pi} (1)\cos t\, dt, \quad \int_{-\pi}^{\pi} (1)\sin t\, dt, \quad \int_{-\pi}^{\pi} (\cos t)(\sin t)\, dt.$$

Also, note that $\|1\|^2 = \langle 1, 1 \rangle = \int_{-\pi}^{\pi} (1)(1)dt = 2\pi$, $\|\cos t\|^2 = \langle \cos t, \cos t \rangle = \int_{-\pi}^{\pi} \cos^2 t\, dt = \pi$ (why?), and $\|\sin t\|^2 = \langle \sin t, \sin t \rangle = \int_{-\pi}^{\pi} \sin^2 t\, dt = \pi$ (why?). Therefore, the set

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos t}{\sqrt{\pi}}, \frac{\sin t}{\sqrt{\pi}} \right\}$$

is an orthonormal set in $\mathcal{V}$.

Example 8 can be generalized. The set $\{1, \cos t, \sin t, \cos 2t, \sin 2t, \cos 3t, \sin 3t, \ldots\}$ is an orthogonal set (see Exercise 16) and therefore linearly independent by Theorem 7.15. The functions in this set are important in the theory of partial differential equations. It can be shown that every continuously differentiable function on the interval $[-\pi, \pi]$ can be represented as the (infinite) sum of constant multiples of these functions. Such a sum is known as the **Fourier series** of the function.

A basis for an inner product space $\mathcal{V}$ is an **orthogonal [orthonormal] basis** if the vectors in the basis form an orthogonal [orthonormal] set.

**Example 9**

Consider again the inner product space $\mathcal{P}_n$ with the inner product of Example 4; that is, if $\mathbf{p}_1 = a_n x^n + \cdots + a_1 x + a_0$ and $\mathbf{p}_2 = b_n x^n + \cdots + b_1 x + b_0$ are in $\mathcal{P}_n$, then $\langle \mathbf{p}_1, \mathbf{p}_2 \rangle = a_n b_n + \cdots + a_1 b_1 + a_0 b_0$. Now, $\left\{ x^n, x^{n-1}, \ldots, x, 1 \right\}$ is an orthogonal basis for $\mathcal{P}_n$ with this inner product, since $\left\langle x^k, x^l \right\rangle = 0$, for $0 \le k, l \le n$, with $k \ne l$ (why?). Since $\left\| x^k \right\| = \sqrt{\left\langle x^k, x^k \right\rangle} = 1$, for all $k$, $0 \le k \le n$ (why?), the set $\left\{ x^n, x^{n-1}, \ldots, x, 1 \right\}$ is also an orthonormal basis for this inner product space.

A proof analogous to that of Theorem 6.3 gives us the next theorem (see Exercise 17).

**Theorem 7.16** *If $B = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$ is an orthogonal ordered basis for a subspace $\mathcal{W}$ of an inner product space $\mathcal{V}$, and if $\mathbf{v}$ is any vector in $\mathcal{W}$, then*

$$[\mathbf{v}]_B = \left[ \frac{\langle \mathbf{v}, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}, \frac{\langle \mathbf{v}, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}, \ldots, \frac{\langle \mathbf{v}, \mathbf{v}_k \rangle}{\langle \mathbf{v}_k, \mathbf{v}_k \rangle} \right].$$

*In particular, if $B$ is an orthonormal ordered basis for $\mathcal{W}$, then $[\mathbf{v}]_B = [\langle \mathbf{v}, \mathbf{v}_1 \rangle, \langle \mathbf{v}, \mathbf{v}_2 \rangle, \ldots, \langle \mathbf{v}, \mathbf{v}_k \rangle]$.*

**Example 10**

Recall the inner product space $\mathbb{R}^2$ in Example 2, with inner product given as follows: if $\mathbf{x} = [x_1, x_2]$ and $\mathbf{y} = [y_1, y_2]$, then $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 - x_1 y_2 - x_2 y_1 + 2x_2 y_2$. An ordered orthogonal basis for this space is $B = (\mathbf{v}_1, \mathbf{v}_2) = ([2, 1], [0, 1])$ (verify!). Recall from Example 2 that $\langle \mathbf{x}, \mathbf{x} \rangle = (x_1 - x_2)^2 + x_2^2$. Thus, $\langle \mathbf{v}_1, \mathbf{v}_1 \rangle = (2-1)^2 + 1^2 = 2$, and $\langle \mathbf{v}_2, \mathbf{v}_2 \rangle = (0-1)^2 + 1^2 = 2$.

Next, suppose that $\mathbf{v} = [a, b]$ is any vector in $\mathbb{R}^2$. Now, $\langle \mathbf{v}, \mathbf{v}_1 \rangle = \langle [a, b], [2, 1] \rangle = (a)(2) - (a)(1) - (b)(2) + 2(b)(1) = a$. Also, $\langle \mathbf{v}, \mathbf{v}_2 \rangle = \langle [a, b], [0, 1] \rangle = (a)(0) - (a)(1) - (b)(0) + 2(b)(1) = -a + 2b$. Then,

$$[\mathbf{v}]_B = \left[ \frac{\langle \mathbf{v}, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}, \frac{\langle \mathbf{v}, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle} \right] = \left[ \frac{a}{2}, \frac{-a + 2b}{2} \right].$$

Notice that $\frac{a}{2}[2, 1] + \left( \frac{-a+2b}{2} \right)[0, 1]$ does equal $[a, b] = \mathbf{v}$.

## The Generalized Gram-Schmidt Process

We can generalize the Gram-Schmidt Process of Section 6.1 to any inner product space. That is, we can replace any linearly independent set of $k$ vectors with an orthogonal set of $k$ vectors that spans the same subspace.

**Method for Finding an Orthogonal Basis for the Span of a Linearly Independent Subset of an Inner Product Space (Generalized Gram-Schmidt Process)**
Let $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ be a linearly independent subset of an inner product space $\mathcal{V}$, with inner product $\langle\, , \rangle$. We create a new set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ of vectors as follows:

Let $\mathbf{v}_1 = \mathbf{w}_1$.

Let $\mathbf{v}_2 = \mathbf{w}_2 - \left(\dfrac{\langle \mathbf{w}_2, \mathbf{v}_1\rangle}{\langle \mathbf{v}_1, \mathbf{v}_1\rangle}\right)\mathbf{v}_1$.

Let $\mathbf{v}_3 = \mathbf{w}_3 - \left(\dfrac{\langle \mathbf{w}_3, \mathbf{v}_1\rangle}{\langle \mathbf{v}_1, \mathbf{v}_1\rangle}\right)\mathbf{v}_1 - \left(\dfrac{\langle \mathbf{w}_3, \mathbf{v}_2\rangle}{\langle \mathbf{v}_2, \mathbf{v}_2\rangle}\right)\mathbf{v}_2$.

$\vdots$

Let $\mathbf{v}_k = \mathbf{w}_k - \left(\dfrac{\langle \mathbf{w}_k, \mathbf{v}_1\rangle}{\langle \mathbf{v}_1, \mathbf{v}_1\rangle}\right)\mathbf{v}_1 - \left(\dfrac{\langle \mathbf{w}_k, \mathbf{v}_2\rangle}{\langle \mathbf{v}_2, \mathbf{v}_2\rangle}\right)\mathbf{v}_2 - \cdots - \left(\dfrac{\langle \mathbf{w}_k, \mathbf{v}_{k-1}\rangle}{\langle \mathbf{v}_{k-1}, \mathbf{v}_{k-1}\rangle}\right)\mathbf{v}_{k-1}$.

A proof similar to that of Theorem 6.4 (see Exercise 21) gives

**Theorem 7.17** *Let $B = \{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ be a basis for a finite dimensional inner product space $\mathcal{V}$. Then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ obtained by applying the Generalized Gram-Schmidt Process to $B$ is an orthogonal basis for $\mathcal{V}$.*

*Hence, every nontrivial finite dimensional inner product space has an orthogonal basis.*

**Example 11**

Recall the inner product space $\mathcal{V}$ from Example 5 of real continuous functions using $a = -1$ and $b = 1$; that is, with inner product $\langle \mathbf{f}, \mathbf{g}\rangle = \int_{-1}^{1} \mathbf{f}(t)\mathbf{g}(t)\, dt$. Now, $\left\{1, t, t^2, t^3\right\}$ is a linearly independent set in $\mathcal{V}$. We use this set to find four orthogonal vectors in $\mathcal{V}$.

Let $\mathbf{w}_1 = 1$, $\mathbf{w}_2 = t$, $\mathbf{w}_3 = t^2$, and $\mathbf{w}_4 = t^3$. Using the Generalized Gram-Schmidt Process, we start with $\mathbf{v}_1 = \mathbf{w}_1 = 1$ and obtain

$$\mathbf{v}_2 = \mathbf{w}_2 - \left(\frac{\langle \mathbf{w}_2, \mathbf{v}_1\rangle}{\langle \mathbf{v}_1, \mathbf{v}_1\rangle}\right)\mathbf{v}_1 = t - \left(\frac{\langle t, 1\rangle}{\langle 1, 1\rangle}\right)1.$$

Now, $\langle t, 1\rangle = \int_{-1}^{1} (t)(1)\, dt = \left(t^2/2\right)\Big|_{-1}^{1} = 0$. Hence, $\mathbf{v}_2 = t$. Next,

$$\mathbf{v}_3 = \mathbf{w}_3 - \left(\frac{\langle \mathbf{w}_3, \mathbf{v}_1\rangle}{\langle \mathbf{v}_1, \mathbf{v}_1\rangle}\right)\mathbf{v}_1 - \left(\frac{\langle \mathbf{w}_3, \mathbf{v}_2\rangle}{\langle \mathbf{v}_2, \mathbf{v}_2\rangle}\right)\mathbf{v}_2 = t^2 - \left(\frac{\left\langle t^2, 1\right\rangle}{\langle 1, 1\rangle}\right)1 - \left(\frac{\left\langle t^2, t\right\rangle}{\langle t, t\rangle}\right)t.$$

After a little calculation, we obtain $\left\langle t^2, 1\right\rangle = \frac{2}{3}$, $\langle 1, 1\rangle = 2$, and $\left\langle t^2, t\right\rangle = 0$. Hence, $\mathbf{v}_3 = t^2 - \left(\left(\frac{2}{3}\right)/2\right)1 = t^2 - \frac{1}{3}$. Finally,

$$\begin{aligned}\mathbf{v}_4 &= \mathbf{w}_4 - \left(\frac{\langle \mathbf{w}_4, \mathbf{v}_1\rangle}{\langle \mathbf{v}_1, \mathbf{v}_1\rangle}\right)\mathbf{v}_1 - \left(\frac{\langle \mathbf{w}_4, \mathbf{v}_2\rangle}{\langle \mathbf{v}_2, \mathbf{v}_2\rangle}\right)\mathbf{v}_2 - \left(\frac{\langle \mathbf{w}_4, \mathbf{v}_3\rangle}{\langle \mathbf{v}_3, \mathbf{v}_3\rangle}\right)\mathbf{v}_3 \\ &= t^3 - \left(\frac{\left\langle t^3, 1\right\rangle}{\langle 1, 1\rangle}\right)1 - \left(\frac{\left\langle t^3, t\right\rangle}{\langle t, t\rangle}\right)t - \left(\frac{\left\langle t^3, t^2\right\rangle}{\left\langle t^2, t^2\right\rangle}\right)t^2.\end{aligned}$$

Now, $\left\langle t^3, 1\right\rangle = 0$, $\left\langle t^3, t\right\rangle = \frac{2}{5}$, $\langle t, t\rangle = \frac{2}{3}$, and $\left\langle t^3, t^2\right\rangle = 0$. Hence, $\mathbf{v}_4 = t^3 - \left(\left(\frac{2}{5}\right)/\left(\frac{2}{3}\right)\right)t = t^3 - \frac{3}{5}t$.

Thus, the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} = \left\{1,\ t,\ t^2 - \frac{1}{3},\ t^3 - \frac{3}{5}t\right\}$ is an orthogonal set of vectors in this inner product space.[3]

We saw in Theorem 6.8 that the transition matrix between orthonormal bases of $\mathbb{R}^n$ is an orthogonal matrix. This result generalizes to inner product spaces as follows:

[3] The polynomials $1$, $t$, $t^2 - \frac{1}{3}$, and $t^3 - \frac{3}{5}t$ from Example 11 are multiples of the first four **Legendre polynomials**: $1$, $t$, $\frac{3}{2}t^2 - \frac{1}{2}$, $\frac{5}{2}t^3 - \frac{3}{2}t$. All Legendre polynomials equal 1 when $t = 1$. To find the complete set of Legendre polynomials, we can continue the Generalized Gram-Schmidt Process with $t^4$, $t^5$, $t^6$, and so on, and take appropriate multiples so that the resulting polynomials equal 1 when $t = 1$. These polynomials form an (infinite) orthogonal set for the inner product space of Example 11.

**Theorem 7.18** *Let $\mathcal{V}$ be a nontrivial finite dimensional real [complex] inner product space, and let B and C be ordered orthonormal bases for $\mathcal{V}$. Then the transition matrix from B to C is an orthogonal [unitary] matrix.*

## Orthogonal Complements and Orthogonal Projections in Inner Product Spaces

We can generalize the notion of an orthogonal complement of a subspace to inner product spaces as follows:

**Definition** Let $\mathcal{W}$ be a subspace of a real (or complex) inner product space $\mathcal{V}$. Then the **orthogonal complement** $\mathcal{W}^{\perp}$ of $\mathcal{W}$ in $\mathcal{V}$ is the set of all vectors $\mathbf{x} \in \mathcal{V}$ with the property that $\langle \mathbf{x}, \mathbf{w} \rangle = 0$, for all $\mathbf{w} \in \mathcal{W}$.

### Example 12

Consider again the real vector space $\mathcal{P}_n$, with the inner product of Example 4—for $\mathbf{p}_1 = a_n x^n + \cdots + a_1 x + a_0$ and $\mathbf{p}_2 = b_n x^n + \cdots + b_1 x + b_0$, $\langle \mathbf{p}_1, \mathbf{p}_2 \rangle = a_n b_n + \cdots + a_1 b_1 + a_0 b_0$. Example 9 showed that $\left\{x^n, x^{n-1}, \ldots, x, 1\right\}$ is an orthogonal basis for $\mathcal{P}_n$ under this inner product. Now, consider the subspace $\mathcal{W}$ spanned by $\{x, 1\}$. A little thought will convince you that $\mathcal{W}^{\perp} = \text{span}\left\{x^n, x^{n-1}, \ldots, x^2\right\}$ and so, $\dim(\mathcal{W}) + \dim\left(\mathcal{W}^{\perp}\right) = 2 + (n-1) = n + 1 = \dim(\mathcal{P}_n)$.

The following properties of orthogonal complements are the analogs to Theorems 6.11 and 6.12 and Corollaries 6.13 and 6.14 and are proved in a similar manner (see Exercise 22):

**Theorem 7.19** *Let $\mathcal{W}$ be a subspace of a real (or complex) inner product space $\mathcal{V}$. Then*

**(1)** *$\mathcal{W}^{\perp}$ is a subspace of $\mathcal{V}$.*
**(2)** $\mathcal{W} \cap \mathcal{W}^{\perp} = \{\mathbf{0}\}$.
**(3)** $\mathcal{W} \subseteq \left(\mathcal{W}^{\perp}\right)^{\perp}$.

*Furthermore, if $\mathcal{V}$ is finite dimensional, then*

**(4)** *If $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an orthogonal basis for $\mathcal{W}$ contained in an orthogonal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$ for $\mathcal{V}$, then $\{\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$ is an orthogonal basis for $\mathcal{W}^{\perp}$.*
**(5)** $\dim(\mathcal{W}) + \dim\left(\mathcal{W}^{\perp}\right) = \dim(\mathcal{V})$.
**(6)** $\left(\mathcal{W}^{\perp}\right)^{\perp} = \mathcal{W}$.

Note that if $\mathcal{V}$ is not finite dimensional, $\left(\mathcal{W}^{\perp}\right)^{\perp}$ is not necessarily[4] equal to $\mathcal{W}$, although it is always true that $\mathcal{W} \subseteq \left(\mathcal{W}^{\perp}\right)^{\perp}$.

The next theorem is the analog of Theorem 6.15. It holds for any inner product space $\mathcal{V}$ where the subspace $\mathcal{W}$ is finite dimensional. The proof is left for you to do in Exercise 25.

**Theorem 7.20** **(Projection Theorem)** *Let $\mathcal{W}$ be a finite dimensional subspace of an inner product space $\mathcal{V}$. Then every vector $\mathbf{v} \in \mathcal{V}$ can be expressed in a unique way as $\mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1 \in \mathcal{W}$ and $\mathbf{w}_2 \in \mathcal{W}^{\perp}$.*

As before, we define the **orthogonal projection** of a vector $\mathbf{v}$ onto a subspace $\mathcal{W}$ as follows:

**Definition** If $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an orthonormal basis for $\mathcal{W}$, a subspace of an inner product space $\mathcal{V}$, then the vector $\mathbf{proj}_{\mathcal{W}}\mathbf{v} = \langle \mathbf{v}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \cdots + \langle \mathbf{v}, \mathbf{v}_k \rangle \mathbf{v}_k$ is called the **orthogonal projection of v onto** $\mathcal{W}$. If $\mathcal{W}$ is the trivial subspace of $\mathcal{V}$, then $\mathbf{proj}_{\mathcal{W}}\mathbf{v} = \mathbf{0}$.

[4] The following is an example of a subspace $\mathcal{W}$ of an infinite dimensional inner product space such that $\mathcal{W} \neq (\mathcal{W}^{\perp})^{\perp}$. Let $\mathcal{V}$ be the inner product space of Example 5 with $a = 0$, $b = 1$, and let $\mathbf{f}_n(x) = \begin{cases} 1, & \text{if } x > \frac{1}{n} \\ nx, & \text{if } 0 \leq x \leq \frac{1}{n} \end{cases}$. Let $\mathcal{W}$ be the subspace of $\mathcal{V}$ spanned by $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \ldots\}$. It can be shown that $\mathbf{f}(x) = 1$ is not in $\mathcal{W}$, but $\mathbf{f}(x) \in (\mathcal{W}^{\perp})^{\perp}$. Hence, $\mathcal{W} \neq (\mathcal{W}^{\perp})^{\perp}$.

From the proof of the Projection Theorem (the solution to Exercise 25), we find that this formula for $\mathbf{proj}_{\mathcal{W}}\mathbf{v}$ yields the unique vector $\mathbf{w}_1$ in that theorem. Therefore, the choice of orthonormal basis in the definition does not matter because any choice leads to the same vector for $\mathbf{proj}_{\mathcal{W}}\mathbf{v}$. Hence, the Projection Theorem can be restated as follows:

**Corollary 7.21** *If $\mathcal{W}$ is a finite dimensional subspace of an inner product space $\mathcal{V}$, and if $\mathbf{v} \in \mathcal{V}$, then there are unique vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ with $\mathbf{w}_1 \in \mathcal{W}$ and $\mathbf{w}_2 \in \mathcal{W}^{\perp}$ such that $\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2$. Moreover, $\mathbf{w}_1 = \mathbf{proj}_{\mathcal{W}}\mathbf{v}$ and $\mathbf{w}_2 = \mathbf{v} - \mathbf{proj}_{\mathcal{W}}\mathbf{v}$ ($= \mathbf{proj}_{\mathcal{W}^{\perp}}\mathbf{v}$ when $\mathcal{W}^{\perp}$ is finite dimensional).*

### Example 13

Consider again the real vector space $\mathcal{V}$ of real continuous functions in Example 8, where $\langle \mathbf{f}, \mathbf{g} \rangle = \int_{-\pi}^{\pi} \mathbf{f}(t)\mathbf{g}(t)\,dt$. Notice from that example that the set $\left\{1/\sqrt{2\pi},\ (\sin t)/\sqrt{\pi}\right\}$ is an orthonormal (and hence, linearly independent) set of vectors in $\mathcal{V}$. Let $\mathcal{W} = \text{span}\left(\left\{1/\sqrt{2\pi},\ (\sin t)/\sqrt{\pi}\right\}\right)$ in $\mathcal{V}$. Then any continuous function $\mathbf{f}$ in $\mathcal{V}$ can be expressed uniquely as $\mathbf{f}_1 + \mathbf{f}_2$, where $\mathbf{f}_1 \in \mathcal{W}$ and $\mathbf{f}_2 \in \mathcal{W}^{\perp}$.

We illustrate this decomposition for the function $\mathbf{f} = t + 1$. Now,

$$\mathbf{f}_1 = \mathbf{proj}_{\mathcal{W}}\mathbf{f} = c_1\left(\frac{1}{\sqrt{2\pi}}\right) + c_2\left(\frac{\sin t}{\sqrt{\pi}}\right),$$

where $c_1 = \left\langle (t+1),\ 1/\sqrt{2\pi} \right\rangle$ and $c_2 = \left\langle (t+1),\ (\sin t)/\sqrt{\pi} \right\rangle$. Then

$$c_1 = \int_{-\pi}^{\pi} (t+1)\left(\frac{1}{\sqrt{2\pi}}\right) dt = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} (t+1)\,dt = \frac{1}{\sqrt{2\pi}}\left(\frac{t^2}{2} + t\right)\Bigg|_{-\pi}^{\pi} = \frac{2\pi}{\sqrt{2\pi}} = \sqrt{2\pi}.$$

Also,

$$c_2 = \int_{-\pi}^{\pi} (t+1)\left(\frac{\sin t}{\sqrt{\pi}}\right) dt = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} (t+1)\ \sin t\ dt = \frac{1}{\sqrt{\pi}}\left(\int_{-\pi}^{\pi} t\sin t\,dt + \int_{-\pi}^{\pi} \sin t\,dt\right).$$

The very last integral equals zero. Using integration by parts on the other integral, we obtain

$$c_2 = \frac{1}{\sqrt{\pi}}\left((-t\cos t)|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \cos t\,dt\right) = \left(\frac{1}{\sqrt{\pi}}\right)2\pi = 2\sqrt{\pi}.$$

Hence,

$$\mathbf{f}_1 = c_1\left(\frac{1}{\sqrt{2\pi}}\right) + c_2\left(\frac{\sin t}{\sqrt{\pi}}\right) = \sqrt{2\pi}\left(\frac{1}{\sqrt{2\pi}}\right) + 2\sqrt{\pi}\left(\frac{\sin t}{\sqrt{\pi}}\right) = 1 + 2\sin t.$$

Then by the Projection Theorem, $\mathbf{f}_2 = \mathbf{f} - \mathbf{f}_1 = (t+1) - (1 + 2\sin t) = t - 2\sin t$ is orthogonal to $\mathcal{W}$. We check that $\mathbf{f}_2 \in \mathcal{W}^{\perp}$ by showing that $\mathbf{f}_2$ is orthogonal to both $1/\sqrt{2\pi}$ and $(\sin t)/\sqrt{\pi}$.

$$\left\langle \mathbf{f}_2, \frac{1}{\sqrt{2\pi}} \right\rangle = \int_{-\pi}^{\pi} (t - 2\sin t)\left(\frac{1}{\sqrt{2\pi}}\right) dt = \left(\frac{1}{\sqrt{2\pi}}\right)\left(\frac{t^2}{2} + 2\cos t\right)\Bigg|_{-\pi}^{\pi} = 0.$$

Also,

$$\left\langle \mathbf{f}_2, \frac{\sin t}{\sqrt{\pi}} \right\rangle = \int_{-\pi}^{\pi} (t - 2\sin t)\left(\frac{\sin t}{\sqrt{\pi}}\right) dt = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} t\sin t\,dt - \frac{2}{\sqrt{\pi}} \int_{-\pi}^{\pi} \sin^2 t\,dt,$$

which equals $2\sqrt{\pi} - 2\sqrt{\pi} = 0$.

## New Vocabulary

angle between vectors (in a real inner product space)
Cauchy-Schwarz Inequality (in an inner product space)
complex inner product (on a complex vector space)
complex inner product space
distance between vectors (in an inner product space)
Fourier series
Generalized Gram-Schmidt Process (in an inner product space)
Legendre polynomials
norm (length) of a vector (in an inner product space)
orthogonal basis (in an inner product space)
orthogonal complement (of a subspace in an inner product space)

orthogonal projection (of a vector onto a subspace of an inner product space)
orthogonal set of vectors (in an inner product space)
orthonormal basis (in an inner product space)
orthonormal set of vectors (in an inner product space)
real inner product (on a real vector space)
real inner product space
Triangle Inequality (in an inner product space)
unit vector (in an inner product space)

## Highlights

- If $\mathcal{V}$ is a real [complex] vector space, then an inner product for $\mathcal{V}$ is an operation that assigns a real [complex] number $\langle \mathbf{x}, \mathbf{y} \rangle$ to each pair of vectors $\mathbf{x}, \mathbf{y} \in \mathcal{V}$ such that: $\langle \mathbf{x}, \mathbf{x} \rangle$ is real, $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$, $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ iff $\mathbf{x} = \mathbf{0}$, $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$ $\left[\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}\right]$, $\langle \mathbf{x}+\mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$, and $\langle k\mathbf{x}, \mathbf{y} \rangle = k \langle \mathbf{x}, \mathbf{y} \rangle$, for all $k \in \mathbb{R}$ [$k \in \mathbb{C}$].
- A vector space $\mathcal{V}$ is an inner product space if $\mathcal{V}$ has an inner product operation (along with addition and scalar multiplication).
- For vectors $\mathbf{x}$, $\mathbf{y}$ and scalar $k$ in a real [complex] inner product space, $\langle \mathbf{x}, \mathbf{y}+\mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle$, and $\langle \mathbf{x}, k\mathbf{y} \rangle = k \langle \mathbf{x}, \mathbf{y} \rangle$ [$\langle \mathbf{x}, k\mathbf{y} \rangle = \overline{k} \langle \mathbf{x}, \mathbf{y} \rangle$].
- The length of a vector $\mathbf{x}$ in an inner product space is $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$, and the distance between vectors $\mathbf{x}$ and $\mathbf{y}$ in an inner product space is $||\mathbf{x}-\mathbf{y}||$.
- For vectors $\mathbf{x}$, $\mathbf{y}$ and scalar $k$ in a real [complex] inner product space, $||k\mathbf{x}|| = |k| \, \|\mathbf{x}\|$, $|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\| \, \|\mathbf{y}\|$, and $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.
- The angle $\theta$ between two vectors in a real inner product space is defined as the angle between 0 and $\pi$ such that $\cos\theta = \langle \mathbf{x}, \mathbf{y} \rangle \, / \, (\|\mathbf{x}\| \, \|\mathbf{y}\|)$.
- A subset $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ of vectors in an inner product space $\mathcal{V}$ with inner product $\langle \, , \, \rangle$ is orthogonal iff $\left\langle \mathbf{x}_i, \mathbf{x}_j \right\rangle = 0$ for $1 \leq i, j \leq n$, with $i \neq j$. Also, an orthogonal set of vectors in $\mathcal{V}$ is orthonormal iff each vector is a unit vector.
- If $\mathcal{V}$ is an inner product space and $T$ is an orthogonal set of nonzero vectors in $\mathcal{V}$, then $T$ is a linearly independent set.
- If $B = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$ is an orthogonal ordered basis for a subspace $\mathcal{W}$ of an inner product space $\mathcal{V}$, and if $\mathbf{v}$ is any vector in $\mathcal{W}$, then $[\mathbf{v}]_B = \left[\frac{\langle \mathbf{v}, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}, \frac{\langle \mathbf{v}, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}, \ldots, \frac{\langle \mathbf{v}, \mathbf{v}_k \rangle}{\langle \mathbf{v}_k, \mathbf{v}_k \rangle}\right]$.
- The Generalized Gram-Schmidt Process can be used to convert a linearly independent set $T$ of vectors in an inner product space to an orthogonal basis for span($T$).
- If $B$ and $C$ are any ordered orthonormal bases for a nontrivial finite dimensional real [complex] inner product space, then the transition matrix from $B$ to $C$ is an orthogonal [unitary] matrix.
- If $\mathcal{W}$ is a subspace of an inner product space $\mathcal{V}$, then the orthogonal complement $\mathcal{W}^\perp$ of $\mathcal{W}$ in $\mathcal{V}$ is the subspace of $\mathcal{V}$ consisting of all vectors $\mathbf{x} \in \mathcal{V}$ with the property that $\langle \mathbf{x}, \mathbf{w} \rangle = 0$, for all $\mathbf{w} \in \mathcal{W}$.
- If $\mathcal{W}$ is a subspace of an inner product space $\mathcal{V}$, then $\mathcal{W} \cap \mathcal{W}^\perp = \{\mathbf{0}\}$, and $\mathcal{W} \subseteq \left(\mathcal{W}^\perp\right)^\perp$.
- Let $\mathcal{V}$ be a finite dimensional inner product space. If $\mathcal{W}$ is a subspace with orthogonal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$, and $\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$ is an orthogonal basis for $\mathcal{V}$, then $\{\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$ is an orthogonal basis for $\mathcal{W}^\perp$. Furthermore, $\dim(\mathcal{W}) + \dim\left(\mathcal{W}^\perp\right) = \dim(\mathcal{V})$, and $\left(\mathcal{W}^\perp\right)^\perp = \mathcal{W}$.
- If $\mathcal{W}$ is a subspace of an inner product space $\mathcal{V}$, $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an orthonormal basis for $\mathcal{W}$, and $\mathbf{v} \in \mathcal{V}$, then the orthogonal projection of $\mathbf{v}$ onto $\mathcal{W}$ is $\mathbf{proj}_{\mathcal{W}}\mathbf{v} = \langle \mathbf{v}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \cdots + \langle \mathbf{v}, \mathbf{v}_k \rangle \mathbf{v}_k$.
- If $\mathcal{W}$ is a finite dimensional subspace of an inner product space $\mathcal{V}$, and if $\mathbf{v} \in \mathcal{V}$, then there are unique vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ with $\mathbf{w}_1 \in \mathcal{W}$ and $\mathbf{w}_2 \in \mathcal{W}^\perp$ such that $\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2$. Also, $\mathbf{w}_1 = \mathbf{proj}_{\mathcal{W}}\mathbf{v}$ and $\mathbf{w}_2 = \mathbf{v} - \mathbf{proj}_{\mathcal{W}}\mathbf{v}$ ($= \mathbf{proj}_{\mathcal{W}^\perp}\mathbf{v}$ when $\mathcal{W}^\perp$ is finite dimensional).

## Exercises for Section 7.5

1. This exercise introduces a particular inner product on $\mathbb{R}^n$.
   (a) Let $\mathbf{A}$ be a nonsingular $n \times n$ real matrix. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, define an operation $\langle \mathbf{x}, \mathbf{y} \rangle = (\mathbf{A}\mathbf{x}) \cdot (\mathbf{A}\mathbf{y})$ (dot product). Prove that this operation is a real inner product on $\mathbb{R}^n$.
   ★ (b) For the inner product in part (a) with $\mathbf{A} = \begin{bmatrix} 5 & 4 & 2 \\ -2 & 3 & 1 \\ 1 & -1 & 0 \end{bmatrix}$, find $\langle \mathbf{x}, \mathbf{y} \rangle$ and $\|\mathbf{x}\|$, for $\mathbf{x} = [3, -2, 4]$ and $\mathbf{y} = [-2, 1, -1]$.
2. Define an operation $\langle \, , \, \rangle$ on $\mathcal{P}_n$ as follows: if $\mathbf{p}_1 = a_n x^n + \cdots + a_1 x + a_0$ and $\mathbf{p}_2 = b_n x^n + \cdots + b_1 x + b_0$, let $\langle \mathbf{p}_1, \mathbf{p}_2 \rangle = a_n b_n + \cdots + a_1 b_1 + a_0 b_0$. Prove that this operation is a real inner product on $\mathcal{P}_n$.

**3.** This exercise concerns the set of real continuous functions on $[a, b]$.

**(a)** Let $a$ and $b$ be fixed real numbers with $a < b$, and let $\mathcal{V}$ be the set of all real continuous functions on $[a, b]$. Define $\langle\, ,\, \rangle$ on $\mathcal{V}$ by $\langle \mathbf{f}, \mathbf{g} \rangle = \int_a^b \mathbf{f}(t)\mathbf{g}(t)\,dt$. Prove that this operation is a real inner product on $\mathcal{V}$.

★ **(b)** For the inner product of part (a) with $a = 0$ and $b = \pi$, find $\langle \mathbf{f}, \mathbf{g} \rangle$ and $\|\mathbf{f}\|$, for $\mathbf{f} = e^t$ and $\mathbf{g} = \sin t$.

**4.** Define $\langle\, ,\, \rangle$ on the real vector space $\mathcal{M}_{mn}$ by $\langle \mathbf{A}, \mathbf{B} \rangle = \text{trace}(\mathbf{A}^T\mathbf{B})$. Prove that this operation is a real inner product on $\mathcal{M}_{mn}$. (Hint: Refer to Exercise 13 in Section 1.4 and Exercise 28 in Section 1.5.)

**5.** This exercise asks for proofs for parts of Theorem 7.12.

**(a)** Prove part (1) of Theorem 7.12. (Hint: $\mathbf{0} = \mathbf{0} + \mathbf{0}$. Use property (4) in the definition of an inner product space.)

**(b)** Prove part (3) of Theorem 7.12. (Be sure to give a proof for both real and complex inner product spaces.)

▶ **6.** Prove Theorem 7.13.

**7.** Let $\mathbf{x}, \mathbf{y} \in \mathcal{V}$, a real inner product space.

**(a)** Prove that $\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + 2\langle \mathbf{x}, \mathbf{y} \rangle + \|\mathbf{y}\|^2$.

**(b)** Show that $\mathbf{x}$ and $\mathbf{y}$ are orthogonal in $\mathcal{V}$ if and only if $\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$.

**(c)** Show that $\frac{1}{2}\left(\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2\right) = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$.

**8.** The following formulas show how the value of the inner product can be derived from the norm (length):

**(a)** Let $\mathbf{x}, \mathbf{y} \in \mathcal{V}$, a real inner product space. Prove the following (real) **Polarization Identity:**

$$\langle \mathbf{x}, \mathbf{y} \rangle = \frac{1}{4}\left(\|\mathbf{x} + \mathbf{y}\|^2 - \|\mathbf{x} - \mathbf{y}\|^2\right).$$

**(b)** Let $\mathbf{x}, \mathbf{y} \in \mathcal{V}$, a complex inner product space. Prove the following **Complex Polarization Identity:**

$$\langle \mathbf{x}, \mathbf{y} \rangle = \frac{1}{4}\left(\left(\|\mathbf{x} + \mathbf{y}\|^2 - \|\mathbf{x} - \mathbf{y}\|^2\right) + i\left(\|\mathbf{x} + i\mathbf{y}\|^2 - \|\mathbf{x} - i\mathbf{y}\|^2\right)\right).$$

**9.** Consider the inner product space $\mathcal{V}$ of Example 5, with $a = 0$ and $b = \pi$.

★ **(a)** Find the distance between $\mathbf{f} = t$ and $\mathbf{g} = \sin t$ in $\mathcal{V}$.

**(b)** Find the angle between $\mathbf{f} = e^t$ and $\mathbf{g} = \sin t$ in $\mathcal{V}$.

**10.** Consider the inner product space $\mathcal{V}$ of Example 3, using

$$\mathbf{A} = \begin{bmatrix} -2 & 0 & 1 \\ 1 & -1 & 2 \\ 3 & -1 & -1 \end{bmatrix}.$$

**(a)** Find the distance between $\mathbf{x} = [2, -1, 3]$ and $\mathbf{y} = [5, -2, 2]$ in $\mathcal{V}$.

★ **(b)** Find the angle between $\mathbf{x} = [2, -1, 3]$ and $\mathbf{y} = [5, -2, 2]$ in $\mathcal{V}$.

**11.** Let $\mathcal{V}$ be an inner product space.

**(a)** Prove part (1) of Theorem 7.14. (Hint: Modify the proof of Theorem 1.7.)

**(b)** Prove part (2) of Theorem 7.14. (Hint: Modify the proof of Theorem 1.8.)

**12.** Let $f$ and $g$ be continuous real-valued functions defined on a closed interval $[a, b]$. Show that

$$\left(\int_a^b f(t)g(t)\,dt\right)^2 \le \int_a^b (f(t))^2\,dt \int_a^b (g(t))^2\,dt.$$

(Hint: Use the Cauchy-Schwarz Inequality in an appropriate inner product space.)

**13.** A **metric space** is a set in which every pair of elements $x$, $y$ has been assigned a real number distance $d$ with the following properties:

**(i)** $d(x, y) = d(y, x)$.

**(ii)** $d(x, y) \ge 0$, with $d(x, y) = 0$ if and only if $x = y$.

**(iii)** $d(x, y) \le d(x, z) + d(z, y)$, for all $z$ in the set.

Prove that every inner product space is a metric space with $d(\mathbf{x}, \mathbf{y})$ taken to be $\|\mathbf{x} - \mathbf{y}\|$ for all vectors $\mathbf{x}$ and $\mathbf{y}$ in the space.

**14.** Determine whether the following sets of vectors are orthogonal:

★ **(a)** $\left\{t^2,\ t+1,\ t-1\right\}$ in $\mathcal{P}_3$, under the inner product of Example 4

**(b)** $\{[15, 9, 19], [-2, -1, -2], [-12, -9, -14]\}$ in $\mathbb{R}^3$, under the inner product of Example 3, with

$$\mathbf{A} = \begin{bmatrix} -3 & 1 & 2 \\ 0 & -2 & 1 \\ 2 & -1 & -1 \end{bmatrix}$$

★ **(c)** $\{[5,-2],[3,4]\}$ in $\mathbb{R}^2$, under the inner product of Example 2
**(d)** $\left\{3t^2-1,\ 4t,\ 5t^3-3t\right\}$ in $\mathcal{P}_3$, under the inner product of Example 11

**15.** Prove Theorem 7.15. (Hint: Modify the proof of Result 7 in Section 1.3.)

**16.** This exercise establishes the claim made after Example 8.
**(a)** Show that $\int_{-\pi}^{\pi}\cos mt\,dt=0$ and $\int_{-\pi}^{\pi}\sin nt\ dt=0$, for all integers $m,n\geq 1$.
**(b)** Show that $\int_{-\pi}^{\pi}\cos mt\cos nt\,dt=0$ and $\int_{-\pi}^{\pi}\sin mt\sin nt\,dt=0$, for any *distinct* integers $m,n\geq 1$. (Hint: Use trigonometric identities.)
**(c)** Show that $\int_{-\pi}^{\pi}\cos mt\sin nt\,dt=0$, for any integers $m,n\geq 1$.
**(d)** Conclude from parts (a), (b), and (c) that $\{1,\cos t,\sin t,\cos 2t,\sin 2t,\cos 3t,\sin 3t,\ldots\}$ is an orthogonal set of real continuous functions on $[-\pi,\pi]$, as claimed after Example 8.

**17.** Prove Theorem 7.16. (Hint: Modify the proof of Theorem 6.3.)

**18.** Let $\{\mathbf{v}_1,\ldots,\mathbf{v}_k\}$ be an orthonormal basis for a complex inner product space $\mathcal{V}$. Prove that for all $\mathbf{v},\mathbf{w}\in\mathcal{V}$,

$$\langle\mathbf{v},\mathbf{w}\rangle=\langle\mathbf{v},\mathbf{v}_1\rangle\,\overline{\langle\mathbf{w},\mathbf{v}_1\rangle}+\langle\mathbf{v},\mathbf{v}_2\rangle\,\overline{\langle\mathbf{w},\mathbf{v}_2\rangle}+\cdots+\langle\mathbf{v},\mathbf{v}_k\rangle\,\overline{\langle\mathbf{w},\mathbf{v}_k\rangle}.$$

(Compare this with Exercise 9(a) in Section 6.1.)

★ **19.** Use the Generalized Gram-Schmidt Process to find an orthogonal basis for $\mathcal{P}_2$ containing $t^2-t+1$ under the inner product of Example 11.

**20.** Use the Generalized Gram-Schmidt Process to find an orthogonal basis for $\mathbb{R}^3$ containing $[-9,-4,8]$ under the inner product of Example 3 with the matrix

$$\mathbf{A}=\begin{bmatrix}2&1&3\\3&-1&3\\2&-1&2\end{bmatrix}.$$

**21.** Prove Theorem 7.17. (Hint: Modify the proof of Theorem 6.4.)

**22.** This exercise asks for proofs for parts of Theorem 7.19.
**(a)** Prove parts (1) and (2) of Theorem 7.19. (Hint: Modify the proof of Theorem 6.11.)
▶ **(b)** Prove parts (4) and (5) of Theorem 7.19. (Hint: Modify the proofs of Theorem 6.12 and Corollary 6.13.)
**(c)** Prove part (3) of Theorem 7.19.
▶ **(d)** Prove part (6) of Theorem 7.19. (Hint: Use part (5) of Theorem 7.19 to show that $\dim(\mathcal{W})=\dim\left(\left(\mathcal{W}^{\perp}\right)^{\perp}\right)$. Then use part (c) and apply Theorem 4.13, or its complex analog.)

★ **23.** Find $\mathcal{W}^{\perp}$ if $\mathcal{W}=\operatorname{span}\left(\left\{t^3+t^2,\ t-1\right\}\right)$ in $\mathcal{P}_3$ with the inner product of Example 4.

**24.** Find an orthogonal basis for $\mathcal{W}^{\perp}$ if $\mathcal{W}=\operatorname{span}(\{(t-1)^2\})$ in $\mathcal{P}_2$, with the inner product $\langle\mathbf{f},\mathbf{g}\rangle=\int_0^1\mathbf{f}(t)\mathbf{g}(t)\,dt$, for all $\mathbf{f},\mathbf{g}\in\mathcal{P}_2$.

▶ **25.** Prove Theorem 7.20. (Hint: Choose an orthonormal basis $\{\mathbf{v}_1,\ldots,\mathbf{v}_k\}$ for $\mathcal{W}$. Then define $\mathbf{w}_1=\mathbf{proj}_{\mathcal{W}}\mathbf{v}=\langle\mathbf{v},\mathbf{v}_1\rangle\,\mathbf{v}_1+\cdots+\langle\mathbf{v},\mathbf{v}_k\rangle\,\mathbf{v}_k$. Let $\mathbf{w}_2=\mathbf{v}-\mathbf{w}_1$, and prove $\mathbf{w}_2\in\mathcal{W}^{\perp}$. Finally, see the proof of Theorem 6.15 for uniqueness.)

★ **26.** In the inner product space of Example 8, decompose $\mathbf{f}=\frac{1}{k}e^t$, where $k=e^{\pi}-e^{-\pi}$, as $\mathbf{w}_1+\mathbf{w}_2$, where $\mathbf{w}_1\in\mathcal{W}=\operatorname{span}(\{\cos t,\sin t\})$ and $\mathbf{w}_2\in\mathcal{W}^{\perp}$. Check that $\langle\mathbf{w}_1,\mathbf{w}_2\rangle=0$. (Hint: First find an orthonormal basis for $\mathcal{W}$.)

**27.** Decompose $\mathbf{v}=4t^2-t+3$ in $\mathcal{P}_2$ as $\mathbf{w}_1+\mathbf{w}_2$, where $\mathbf{w}_1\in\mathcal{W}=\operatorname{span}(\{2t^2-1,\ t+1\})$ and $\mathbf{w}_2\in\mathcal{W}^{\perp}$, under the inner product of Example 11. Check that $\langle\mathbf{w}_1,\mathbf{w}_2\rangle=0$. (Hint: First find an orthonormal basis for $\mathcal{W}$.)

**28.** **Bessel's Inequality:** Let $\mathcal{V}$ be a real inner product space, and let $\{\mathbf{v}_1,\ldots,\mathbf{v}_k\}$ be an orthonormal set in $\mathcal{V}$. Prove that for any vector $\mathbf{v}\in\mathcal{V}$, $\sum_{i=1}^{k}\langle\mathbf{v},\mathbf{v}_i\rangle^2\leq\|\mathbf{v}\|^2$. (Hint: Let $\mathcal{W}=\operatorname{span}(\{\mathbf{v}_1,\ldots,\mathbf{v}_k\})$. Now, $\mathbf{v}=\mathbf{w}_1+\mathbf{w}_2$, where $\mathbf{w}_1=\mathbf{proj}_{\mathcal{W}}\mathbf{v}\in\mathcal{W}$ and $\mathbf{w}_2\in\mathcal{W}^{\perp}$. Expand $\langle\mathbf{v},\mathbf{v}\rangle=\langle\mathbf{w}_1+\mathbf{w}_2,\mathbf{w}_1+\mathbf{w}_2\rangle$. Show that $\|\mathbf{v}\|^2\geq\|\mathbf{w}_1\|^2$, and use the definition of $\mathbf{proj}_{\mathcal{W}}\mathbf{v}$.)

**29.** Let $\mathcal{W}$ be a finite dimensional subspace of an inner product space $\mathcal{V}$. Consider the mapping $L\colon\mathcal{V}\to\mathcal{W}$ given by $L(\mathbf{v})=\mathbf{proj}_{\mathcal{W}}\mathbf{v}$.
**(a)** Prove that $L$ is a linear transformation.
★ **(b)** What are the kernel and range of $L$?
**(c)** Show that $L\circ L=L$.

★ **30.** True or False:

**(a)** If $\mathcal{V}$ is a complex inner product space, then for all $\mathbf{x} \in \mathcal{V}$ and all $k \in \mathbb{C}$, $\|k\mathbf{x}\| = \overline{k}\|\mathbf{x}\|$.

**(b)** In a complex inner product space, the distance between two distinct vectors can be a pure imaginary number.

**(c)** Every linearly independent set of unit vectors in an inner product space is an orthonormal set.

**(d)** It is possible to define more than one inner product on the same vector space.

**(e)** The uniqueness proof of the Projection Theorem shows that if $\mathcal{W}$ is a subspace of $\mathbb{R}^n$, then $\mathbf{proj}_{\mathcal{W}}\mathbf{v}$ is independent of the particular inner product used on $\mathbb{R}^n$.

## Review Exercises for Chapter 7

**1.** Let $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{z} \in \mathbb{C}^3$ be given by $\mathbf{v} = [i,\ 3-i,\ 2+3i]$, $\mathbf{w} = [-4-4i,\ 1+2i,\ 3-i]$, and $\mathbf{z} = [2+5i,\ 2-5i,\ -i]$.

★ **(a)** Compute $\mathbf{v} \cdot \mathbf{w}$.

★ **(b)** Compute $(1+2i)(\mathbf{v} \cdot \mathbf{z})$, $((1+2i)\mathbf{v}) \cdot \mathbf{z}$, and $\mathbf{v} \cdot ((1+2i)\mathbf{z})$.

**(c)** Explain why not all of the answers to part (b) are identical.

**(d)** Compute $\mathbf{w} \cdot \mathbf{z}$ and $\mathbf{w} \cdot (\mathbf{v} + \mathbf{z})$.

**2.** This exercise concerns a particular $2 \times 3$ complex matrix.

**(a)** Compute $\mathbf{H} = \mathbf{A}^*\mathbf{A}$, where $\mathbf{A} = \begin{bmatrix} 1-i & 2+i & 3-4i \\ 0 & 5-2i & -2+i \end{bmatrix}$, and show that $\mathbf{H}$ is Hermitian.

**(b)** Show that $\mathbf{A}\mathbf{A}^*$ is also Hermitian.

**3.** Prove that if $\mathbf{A}$ is a skew-Hermitian $n \times n$ matrix and $\mathbf{w}, \mathbf{z} \in \mathbb{C}^n$, then $(\mathbf{A}\mathbf{z}) \cdot \mathbf{w} = -\mathbf{z} \cdot (\mathbf{A}\mathbf{w})$.

**4.** In each part, solve the given system of linear equations.

★ **(a)** $$\begin{cases} (i)w + (1+i)z = -1+2i \\ (1+i)w + (5+2i)z = 5-3i \\ (2-i)w + (2-5i)z = 1-2i \end{cases}$$

**(b)** $$\begin{cases} (1+i)x + (-1+i)y + (-2+8i)z = 5+37i \\ (4-3i)x + (6+3i)y + (37+i)z = 142-49i \\ (2+i)x + (-1+i)y + (2+13i)z = 29+51i \end{cases}$$

**(c)** $$\begin{cases} x - y - z = 2i \\ (3+i)x - 3y - (3-i)z = -1+7i \\ (2+3i)y + (4+6i)z = 6+i \end{cases}$$

★ **(d)** $$\begin{cases} (1+i)x + (3-i)y + (5+5i)z = 23-i \\ (4+3i)x + (11-4i)y + (19+16i)z = 86-7i \end{cases}$$

**5.** Prove that if $\mathbf{A}$ is a square matrix, then $|\mathbf{A}^*\mathbf{A}|$ is a nonnegative real number which equals zero if and only if $\mathbf{A}$ is singular.

**6.** In each part, if possible, diagonalize the given matrix $\mathbf{A}$. Be sure to compute a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$.

★ **(a)** $\mathbf{A} = \begin{bmatrix} -3 & 5 & -10 \\ 2 & -3 & 8 \\ 2 & -3 & 7 \end{bmatrix}$

**(b)** $\mathbf{A} = \begin{bmatrix} 1-5i & -6-4i & 11+5i \\ -2-i & -2+2i & 3-4i \\ 2-i & -3i & 1+5i \end{bmatrix}$

★ **7.** This example explores the properties of a linear operator.

**(a)** Give an example of a function $L\colon \mathcal{V} \to \mathcal{V}$, where $\mathcal{V}$ is a complex vector space, such that $L(\mathbf{v}+\mathbf{w}) = L(\mathbf{v}) + L(\mathbf{w})$ for all $\mathbf{v}, \mathbf{w} \in \mathcal{V}$, but $L$ is not a linear operator on $\mathcal{V}$.

**(b)** Is your example from part (a) a linear operator on $\mathcal{V}$ if $\mathcal{V}$ is considered to be a real vector space?

**8.** This exercise concerns a particular orthogonal basis for $\mathbb{C}^4$.

★ **(a)** Find an ordered orthogonal basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ for $\mathbb{C}^4$ such that $\{\mathbf{v}_1, \mathbf{v}_2\}$ spans the same subspace as $\{[1, i, 1, -i], [1+i, 2-i, 0, 0]\}$.

**(b)** Normalize the vectors in $B$ to produce an orthonormal basis $C$ for $\mathbb{C}^4$.

**(c)** Find the transition matrix from standard coordinates to $C$-coordinates without using row reduction. (Hint: The transition matrix from $C$-coordinates to standard coordinates is unitary.)

**9.** In each part, if possible, unitarily diagonalize the given matrix $\mathbf{A}$. Be sure to compute the unitary matrix $\mathbf{P}$ and the diagonal matrix $\mathbf{D}$ such that $\mathbf{D} = \mathbf{P}^*\mathbf{AP}$.

★ **(a)** $\mathbf{A} = \begin{bmatrix} 3 & 1+i \\ 1-i & 2 \end{bmatrix}$

**(b)** $\mathbf{A} = \begin{bmatrix} 13-13i & 18+18i & -12+12i \\ -18-18i & 40-40i & -6-6i \\ -12+12i & 6+6i & 45-45i \end{bmatrix}$

(Hint: $p_{\mathbf{A}}(x) = x^3 + (-98+98i)x^2 - 4802ix = x(x-49+49i)^2$)

★ **10.** Prove that $\mathbf{A} = \begin{bmatrix} 1+5i & -1+7i & 2i \\ 3+5i & 2+11i & 5+i \\ 2+4i & -1+3i & -1+8i \end{bmatrix}$ is unitarily diagonalizable.

**11.** Prove that $\mathbf{A} = \begin{bmatrix} -16+i & 2-16i & 16-4i & 4+32i & -1-77i \\ 5i & -5+2i & 2-5i & 10+i & -24+3i \\ -8-3i & 4-8i & 8+2i & -7+16i & 18-39i \\ 2-8i & 8+2i & -2+8i & -16-5i & 39+11i \\ -6i & 6 & 6i & -12 & 29 \end{bmatrix}$ is not unitarily diagonalizable.

**12.** Prove that every unitary matrix is normal.

★ **13.** Find the distance between $\mathbf{f}(x) = x$ and $\mathbf{g}(x) = x^3$ in the real inner product space consisting of the set of all real-valued continuous functions defined on the interval [0, 1] with inner product $\langle \mathbf{f}, \mathbf{g} \rangle = \int_0^1 f(t)g(t)\,dt$.

★ **14.** Use the Generalized Gram-Schmidt Process to find an orthogonal basis for $\mathbb{R}^3$, starting with the standard basis using the real inner product given by $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{Ax} \cdot \mathbf{Ay}$, where $\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 3 \\ 2 & -3 & 1 \end{bmatrix}$.

**15.** Decompose $\mathbf{v} = x$ in the real inner product space consisting of the set of all real-valued continuous functions defined on the interval $[-\pi, \pi]$ as $\mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1 \in \mathcal{W} = \text{span}(\{\sin x,\ x\cos x\})$ and $\mathbf{w}_2 \in \mathcal{W}^{\perp}$, using the real inner product given by $\langle \mathbf{f}, \mathbf{g} \rangle = \int_{-\pi}^{\pi} f(t)g(t)\,dt$. (Note: Although it is not required, you may want to use a computer algebra system to help calculate the integrals involved in this problem.)

★ **16.** True or False:

**(a)** Every real $n$-vector can be thought of as a complex $n$-vector as well.

**(b)** The angle $\theta$ between two complex $n$-vectors $\mathbf{v}$ and $\mathbf{w}$ is the angle such that $\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}$.

**(c)** If $\mathbf{w}, \mathbf{z} \in \mathbb{C}^n$ and $\mathbf{A} \in \mathcal{M}_{nn}^{\mathbb{C}}$, then $\mathbf{Aw} \cdot \mathbf{Az} = (\mathbf{A}^*\mathbf{Aw}) \cdot \mathbf{z}$.

**(d)** Every normal $n \times n$ complex matrix is either Hermitian or skew-Hermitian.

**(e)** Every skew-Hermitian matrix has all zeroes on its main diagonal.

**(f)** The sum of the algebraic multiplicities of all eigenvalues for an $n \times n$ complex matrix equals $n$.

**(g)** If $\mathbf{A} \in \mathcal{M}_{nn}^{\mathbb{C}}$ and $\mathbf{w} \in \mathbb{C}^n$, then the linear system $\mathbf{Az} = \mathbf{w}$ has a solution if and only if $|\mathbf{A}| \neq 0$.

**(h)** If $\mathbf{A} \in \mathcal{M}_{33}^{\mathbb{C}}$ has $[i,\ 1+i,\ 1-i]$ as an eigenvector, then it must also have $[-1,\ -1+i,\ 1+i]$ as an eigenvector.

**(i)** The algebraic multiplicity of every eigenvalue of a square complex matrix equals its geometric multiplicity.

**(j)** If $\mathbf{A} \in \mathcal{M}_{nn}^{\mathbb{C}}$, then $|\mathbf{AA}^*| = \left||\mathbf{A}|\right|^2$.

**(k)** Every complex vector space can be thought of as a real vector space.

**(l)** A set of orthogonal nonzero vectors in $\mathbb{C}^n$ must be linearly independent.

**(m)** If the rows of an $n \times n$ complex matrix $\mathbf{A}$ form an orthonormal basis for $\mathbb{C}^n$, then $\mathbf{A}^T\mathbf{A} = \mathbf{I}_n$.

**(n)** Every orthogonal matrix in $\mathcal{M}_{nn}^{\mathbb{R}}$ can be thought of as a unitary matrix in $\mathcal{M}_{nn}^{\mathbb{C}}$.

**(o)** Every Hermitian matrix is unitarily similar to a matrix with all real entries.

**(p)** The algebraic multiplicity of every eigenvalue of a skew-Hermitian matrix equals its geometric multiplicity.

**(q)** Care must be taken when using the Gram-Schmidt Process in $\mathbb{C}^n$ to perform the dot products in the formulas in the correct order because the dot product in $\mathbb{C}^n$ is not commutative.

**(r)** $\mathbb{C}^n$ with its complex dot product is an example of a complex inner product space.

**(s)** If $\mathcal{W}$ is a nontrivial subspace of a finite dimensional complex inner product space $\mathcal{V}$, then the linear operator $L$ on $\mathcal{V}$ given by $L(\mathbf{v}) = \mathbf{proj}_{\mathcal{W}}\mathbf{v}$ is unitarily diagonalizable.

**(t)** If $\mathcal{W}$ is a subspace of a finite dimensional complex inner product space $\mathcal{V}$, then $\left(\mathcal{W}^{\perp}\right)^{\perp} = \mathcal{W}$.

**(u)** If $\mathcal{V}$ is the inner product space $\mathcal{P}_n^{\mathbb{C}}$ with inner product $\langle \mathbf{p}_1, \mathbf{p}_2 \rangle = \int_{-1}^{1} \mathbf{p}_1(t)\overline{\mathbf{p}_2(t)}\, dt$, then $\mathcal{V}$ has an ordered orthonormal basis $\{\mathbf{q}_1, \ldots, \mathbf{q}_{n+1}\}$ such that the degree of $\mathbf{q}_k$ equals $k-1$.

**(v)** Every complex inner product space has a distance function defined on it that gives a nonnegative real number as the distance between any two vectors.

**(w)** If $\mathcal{V}$ is the inner product space of continuous real-valued functions defined on $[-1, 1]$ with inner product $\langle \mathbf{f}, \mathbf{g} \rangle = \int_{-1}^{1} \mathbf{f}(t)\mathbf{g}(t)\, dt$, then the set $\{1, \cos t, \sin t, \cos 2t, \sin 2t, \cos 3t, \sin 3t, \ldots\}$ is an orthogonal set of vectors in $\mathcal{V}$.

Chapter 8

# Additional Applications

**Mathematicians: Apply Within**

Mathematics is everywhere. It is a tool used to analyze and solve problems related to almost every aspect of our physical world and our society. In particular, linear algebra is one of the most useful devices on the mathematician's tool belt, with important applications in almost every discipline, ranging from electronics to psychology. In this chapter, we show how linear algebra is used in graph theory, circuit theory, least-squares analysis, Markov chains, elementary coding theory, linear recurrence relations, computer graphics, and differential equations. The final section on quadratic forms generalizes the orthogonal diagonalization process to a quadratic setting, illustrating that linear algebra can be useful even in certain non-linear situations. In fact, the applications presented in this chapter constitute only a small sample of the myriad problems in which linear algebra is used on a daily basis.

## 8.1 Graph Theory

*Prerequisite: Section 1.5, Matrix Multiplication*

Multiplication of matrices is widely used in graph theory, a branch of mathematics that has come into prominence for modeling many situations in computer science, business, and the social sciences. We begin by introducing graphs and digraphs and then examine their relationship with matrices. Our main goal is to show how matrices are used to calculate the number of paths of a certain length between vertices of a graph or digraph.

### Graphs and Digraphs

**Definition** A **graph** is a finite collection of **vertices** (points) together with a finite collection of **edges** (curves), each of which has two (not necessarily distinct) vertices as endpoints.

Fig. 8.1 depicts two graphs. Note that a graph may have an edge connecting some vertex to itself. Such edges are called **loops**. For example, there is a loop at vertex $P_2$ in graph $G_2$ in Fig. 8.1. A graph with no loops, such as $G_1$ in Fig. 8.1, is said to be **loop-free**. Also, a graph may have more than one edge connecting the same pair of vertices. In $G_1$, the vertices $P_2$ and $P_3$ have two edges connecting them, and in $G_2$, the vertices $P_3$ and $P_4$ have three edges connecting them.

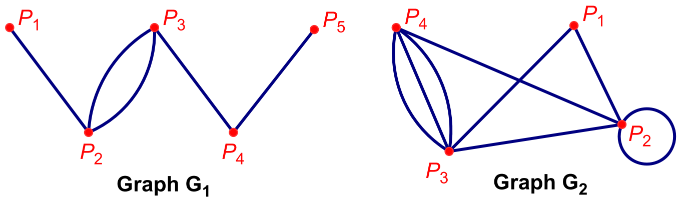


FIGURE 8.1 Two examples of graphs

A **digraph**, or **directed graph**, is a special type of graph in which each edge is assigned a "direction." The "edges" of a digraph are consequently referred to as **directed edges**. Some examples of digraphs appear in Fig. 8.2.

Although the directed edges in a digraph may resemble vectors, they are not necessarily vectors since there is usually no coordinate system present. One interpretation for graphs and digraphs is to consider the vertices as towns and the (possibly directed) edges as roads connecting them. In the case of a digraph, we can think of the roads as one-way streets. Notice

Elementary Linear Algebra. https://doi.org/10.1016/B978-0-12-822978-1.00018-3

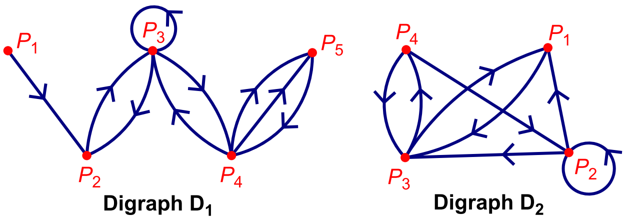


**FIGURE 8.2** Two examples of digraphs

that some pairs of towns may not be connected by roads. Another interpretation for graphs and digraphs is to consider the vertices as relay stations and the (possibly directed) edges as communication channels (for example, phone lines) between the stations. The stations could be individual people, homes, radio/TV installations, or even computer terminals hooked into a network. There are additional interpretations for graphs and digraphs in the exercises.

In Figs. 8.1 and 8.2, some of the edges or directed edges cross each other. Such crossings are analogous to overpasses and underpasses on a system of roads. Edges or directed edges that cross in this manner have no "intersection." There is no way to "get off" one edge or directed edge "onto" another at such a crossing. In fact, the edges for a graph as well as the directed edges for a digraph merely serve to represent which vertices are connected together, and how many such connections exist. The actual shape that an edge or directed edge takes in the figure is irrelevant.

## The Adjacency Matrix

The pattern of [directed] edges between the vertices in a graph [digraph] can be summarized in an algebraic way using matrices.

**Definition** The **adjacency matrix of a graph** having vertices $P_1, P_2, \ldots, P_n$ is the $n \times n$ matrix whose $(i, j)$ entry is the number of edges connecting $P_i$ and $P_j$.

The **adjacency matrix of a digraph** having vertices $P_1, P_2, \ldots, P_n$ is the $n \times n$ matrix whose $(i, j)$ entry is the number of directed edges from $P_i$ to $P_j$.

### Example 1

The adjacency matrices for the two graphs in Fig. 8.1 and the two digraphs in Fig. 8.2 are as follows:

| | $P_1$ | $P_2$ | $P_3$ | $P_4$ | $P_5$ |
|---|---|---|---|---|---|
| $P_1$ | 0 | 1 | 0 | 0 | 0 |
| $P_2$ | 1 | 0 | 2 | 0 | 0 |
| $P_3$ | 0 | 2 | 0 | 1 | 0 |
| $P_4$ | 0 | 0 | 1 | 0 | 1 |
| $P_5$ | 0 | 0 | 0 | 1 | 0 |

Adjacency Matrix for $G_1$

| | $P_1$ | $P_2$ | $P_3$ | $P_4$ |
|---|---|---|---|---|
| $P_1$ | 0 | 1 | 1 | 0 |
| $P_2$ | 1 | 1 | 1 | 1 |
| $P_3$ | 1 | 1 | 0 | 3 |
| $P_4$ | 0 | 1 | 3 | 0 |

Adjacency Matrix for $G_2$

| | $P_1$ | $P_2$ | $P_3$ | $P_4$ | $P_5$ |
|---|---|---|---|---|---|
| $P_1$ | 0 | 1 | 0 | 0 | 0 |
| $P_2$ | 0 | 0 | 1 | 0 | 0 |
| $P_3$ | 0 | 1 | 1 | 1 | 0 |
| $P_4$ | 0 | 0 | 1 | 0 | 2 |
| $P_5$ | 0 | 0 | 0 | 1 | 0 |

Adjacency Matrix for $D_1$

| | $P_1$ | $P_2$ | $P_3$ | $P_4$ |
|---|---|---|---|---|
| $P_1$ | 0 | 0 | 1 | 0 |
| $P_2$ | 1 | 1 | 1 | 0 |
| $P_3$ | 1 | 0 | 0 | 1 |
| $P_4$ | 0 | 1 | 1 | 0 |

Adjacency Matrix for $D_2$

The adjacency matrix of any graph is symmetric, for the obvious reason that there is an edge between $P_i$ and $P_j$ if and only if there is an edge (the same one) between $P_j$ and $P_i$. However, the adjacency matrix for a digraph is usually not symmetric, since the existence of a directed edge from $P_i$ to $P_j$ does not necessarily imply the existence of a directed edge in the reverse direction.

## Paths in a Graph or Digraph

We often want to know how many different routes exist between two given vertices in a graph or digraph.

**Definition** A **path** (or **chain**) between two vertices $P_i$ and $P_j$ in a graph [digraph] is a finite sequence of [directed] edges with the following properties:

**(1)** The first [directed] edge "begins" at $P_i$.
**(2)** The last [directed] edge "ends" at $P_j$.
**(3)** Each [directed] edge after the first one in the sequence "begins" at the vertex where the previous [directed] edge "ended."

The **length** of a path is the number of [directed] edges in the path.

### Example 2

Consider the digraph pictured in Fig. 8.3. There are many different types of paths from $P_1$ to $P_5$. For example,

**(1)** $P_1 \to P_2 \overset{e_1}{\to} P_5$
**(2)** $P_1 \to P_2 \overset{e_2}{\to} P_5$
**(3)** $P_1 \to P_2 \to P_3 \to P_5$
**(4)** $P_1 \to P_4 \to P_3 \to P_5$
**(5)** $P_1 \to P_4 \to P_4 \to P_3 \to P_5$
**(6)** $P_1 \to P_2 \overset{e_1}{\to} P_5 \to P_4 \to P_3 \to P_5$.

(Can you find other paths from $P_1$ to $P_5$?) Path (1) is a path of length 2 (or a 2-chain); path (2) passes through the same vertices as path (1), but uses a different directed edge going from $P_2$ to $P_5$; paths (3), (4), (5), and (6) are paths of lengths 3, 3, 4, and 5, respectively.

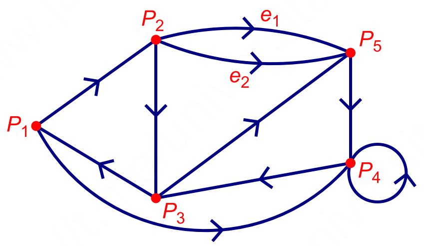


**FIGURE 8.3** Digraph for Examples 2, 3, and 4

## Counting Paths

Our goal is to calculate exactly how many paths of a given length exist between two vertices in a graph or digraph. For example, suppose we want to know precisely how many paths of length 4 from vertex $P_2$ to vertex $P_4$ exist in the digraph of Fig. 8.3. We could attempt to list them, but the chance of making a mistake in counting them all can cast doubt on our final total. However, the next theorem, which you are asked to prove in Exercise 15, gives an algebraic method to get the exact count using the adjacency matrix.

**Theorem 8.1** *Let* $\mathbf{A}$ *be the adjacency matrix for a graph or digraph having vertices* $P_1, P_2, \ldots, P_n$. *Then the total number of paths from* $P_i$ *to* $P_j$ *of length k is given by the* $(i, j)$ *entry in the matrix* $\mathbf{A}^k$.

### Example 3

Consider again the digraph in Fig. 8.3. The adjacency matrix for this digraph is

$$\mathbf{A} = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{array} \begin{array}{c} \begin{array}{ccccc} P_1 & P_2 & P_3 & P_4 & P_5 \end{array} \\ \left[\begin{array}{ccccc} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}.$$

To find the number of paths of length 4 from $P_1$ to $P_4$, we need to calculate the $(1,4)$ entry of $\mathbf{A}^4$. Now,

$$\mathbf{A}^4 = \left(\mathbf{A}^2\right)^2 = \left(\begin{bmatrix} 0 & 0 & 2 & 1 & 2 \\ 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 2 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}\right)^2 = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{array} \begin{array}{c} \begin{array}{ccccc} P_1 & P_2 & P_3 & P_4 & P_5 \end{array} \\ \begin{bmatrix} 1 & 2 & 3 & 7 & 1 \\ 2 & 0 & 5 & 4 & 4 \\ 3 & 0 & 2 & 4 & 3 \\ 1 & 1 & 4 & 5 & 3 \\ 1 & 1 & 1 & 3 & 1 \end{bmatrix} \end{array}.$$

Since the $(1,4)$ entry is 7, there are exactly seven paths of length 4 from $P_1$ to $P_4$. Looking at the digraph, we can see that these paths are

$$\begin{array}{l} P_1 \to P_2 \to P_3 \to P_1 \to P_4 \\ P_1 \to P_2 \to P_3 \to P_5 \to P_4 \\ P_1 \to P_2 \stackrel{e_1}{\to} P_5 \to P_4 \to P_4 \\ P_1 \to P_2 \stackrel{e_2}{\to} P_5 \to P_4 \to P_4 \\ P_1 \to P_4 \to P_3 \to P_1 \to P_4 \\ P_1 \to P_4 \to P_3 \to P_5 \to P_4 \\ P_1 \to P_4 \to P_4 \to P_4 \to P_4. \end{array}$$

We can generalize Theorem 8.1 as follows:

**Corollary 8.2** *The total number of paths of length $\leq k$ from a vertex $P_i$ to a vertex $P_j$ in a graph or digraph is the $(i, j)$ entry of* $\mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 + \cdots + \mathbf{A}^k$.

**Example 4**

For the digraph in Fig. 8.3, we will calculate the total number of paths of length $\leq 4$ from $P_2$ to $P_3$. We listed the adjacency matrix $\mathbf{A}$ for this digraph in Example 3, as well as the products $\mathbf{A}^2$ and $\mathbf{A}^4$. You can verify that $\mathbf{A}^3$ is given by

$$\mathbf{A}^3 = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{array} \begin{array}{c} \begin{array}{ccccc} P_1 & P_2 & P_3 & P_4 & P_5 \end{array} \\ \begin{bmatrix} 2 & 0 & 1 & 3 & 2 \\ 0 & 1 & 2 & 4 & 0 \\ 0 & 0 & 3 & 2 & 2 \\ 1 & 1 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}.$$

Then, a quick calculation gives

$$\mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 + \mathbf{A}^4 = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{array} \begin{array}{c} \begin{array}{ccccc} P_1 & P_2 & P_3 & P_4 & P_5 \end{array} \\ \begin{bmatrix} 3 & 3 & 6 & 12 & 5 \\ 3 & 1 & 8 & 10 & 7 \\ 4 & 1 & 5 & 8 & 6 \\ 3 & 2 & 7 & 10 & 5 \\ 2 & 1 & 3 & 6 & 2 \end{bmatrix} \end{array}.$$

Hence, by Corollary 8.2, the number of paths of length $\leq 4$ from $P_2$ to $P_3$ is the $(2,3)$ entry of this matrix, which is 8. A list of these paths is as follows:

$$\begin{array}{l} P_2 \to P_3 \\ P_2 \stackrel{e_1}{\to} P_5 \to P_4 \to P_3 \\ P_2 \stackrel{e_2}{\to} P_5 \to P_4 \to P_3 \\ P_2 \to P_3 \to P_1 \to P_2 \to P_3 \\ P_2 \to P_3 \to P_1 \to P_4 \to P_3 \\ P_2 \to P_3 \to P_5 \to P_4 \to P_3 \\ P_2 \stackrel{e_1}{\to} P_5 \to P_4 \to P_4 \to P_3 \\ P_2 \stackrel{e_2}{\to} P_5 \to P_4 \to P_4 \to P_3 \end{array}$$

In fact, since we calculated all of the entries of the matrix $\mathbf{A}+\mathbf{A}^2+\mathbf{A}^3+\mathbf{A}^4$, we can now find the total number of paths of length $\leq 4$ between any pair of given vertices. For example, the total number of paths of length $\leq 4$ between $P_3$ and $P_5$ is 6 because that is the $(3,5)$ entry of the sum. Of course, if we only want to know the number of paths of length $\leq 4$ from a particular vertex to another vertex, we would only need a single entry of $\mathbf{A}+\mathbf{A}^2+\mathbf{A}^3+\mathbf{A}^4$ and it would not be necessary to compute all the entries of the sum.

## Connected Graphs

In some graphs, given any pair of distinct vertices, there is a path of some length that connects the two of them. Such graphs are said to be **connected**.[1] For example, both graphs in Fig. 8.1 are connected. However, other graphs may be **disconnected** because some pair of distinct vertices has no path of any length between them. For example, the graph in Fig. 8.4 is disconnected because there are no paths of any length from $P_1$ to $P_4$.

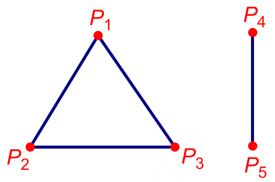


**FIGURE 8.4** A disconnected graph

Now, suppose a graph is connected. Then, for any pair of distinct vertices, there is a path of some length connecting them. How long could the *shortest* path connecting the vertices be? Well, the shortest path connecting a pair of vertices would not pass through the same vertex twice. Otherwise, we could remove the redundant part of the path that circles around from the repeated vertex to itself, and thereby create a shorter overall path. Hence, the shortest path will, at worst, pass through every vertex once. If the graph has $n$ vertices, then such a path would have length $n-1$. Therefore, if there is some path connecting a particular pair of vertices, then there must be a path between those vertices having length $\leq (n-1)$. This means that if $\mathbf{A}$ is the adjacency matrix for a connected graph, then $\mathbf{A}+\mathbf{A}^2+\cdots+\mathbf{A}^{n-1}$ will have no zero entries above and below the main diagonal, since there will be at least one path of length $\leq (n-1)$ connecting each distinct pair of vertices. On the other hand, a disconnected graph would have some zero entries off the main diagonal in the sum $\mathbf{A}+\mathbf{A}^2+\cdots+\mathbf{A}^{n-1}$. Hence,

**Theorem 8.3** *A graph having $n$ vertices (with $n \geq 2$) associated with an adjacency matrix $\mathbf{A}$ is connected if and only if $\mathbf{A}+\mathbf{A}^2+\cdots+\mathbf{A}^{n-1}$ has no zero entries off the main diagonal.*

### Example 5

Consider the graphs associated with the following adjacency matrices $\mathbf{A}$ and $\mathbf{B}$:

$$\mathbf{A} = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{array} \begin{array}{c} \begin{array}{cccc} P_1 & P_2 & P_3 & P_4 \end{array} \\ \left[\begin{array}{cccc} 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right] \end{array} \quad \text{and} \quad \mathbf{B} = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{array} \begin{array}{c} \begin{array}{ccccc} P_1 & P_2 & P_3 & P_4 & P_5 \end{array} \\ \left[\begin{array}{ccccc} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 \end{array}\right] \end{array}.$$

The graph associated with $\mathbf{A}$ is connected because

$$\mathbf{A}+\mathbf{A}^2+\mathbf{A}^3 = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 5 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 5 & 2 \\ 0 & 0 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 12 & 5 \\ 0 & 0 & 5 & 2 \\ 12 & 5 & 0 & 0 \\ 5 & 2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 14 & 6 \\ 2 & 1 & 6 & 2 \\ 14 & 6 & 5 & 2 \\ 6 & 2 & 2 & 1 \end{bmatrix}$$

has no zero entries off the main diagonal.

[1] For simplicity, we will only discuss connectivity for graphs, not for digraphs. See Exercise 13 for a discussion of connectivity with digraphs.

However, the graph associated with the matrix **B** is disconnected because

$$\mathbf{B}+\mathbf{B}^2+\mathbf{B}^3+\mathbf{B}^4=\begin{bmatrix}40&40&0&40&0\\40&45&0&35&0\\0&0&60&0&60\\40&35&0&45&0\\0&0&60&0&60\end{bmatrix}$$

has entries off the main diagonal equal to zero. It is clear from this sum that vertices $P_1$, $P_2$, and $P_4$ all have paths of some length connecting them to each other, but none of these vertices are connected to $P_3$ or $P_5$.

Notice in Example 5 that each of the individual matrices $\mathbf{A}$, $\mathbf{A}^2$, and $\mathbf{A}^3$ do have zero entries. (In fact, every positive integer power of $\mathbf{A}$ has zero entries!) In general, when trying to prove a graph is not connected, it is not enough to show that some particular power of the adjacency matrix $\mathbf{A}$ has a zero entry off the main diagonal. We must, instead, consider the entries off the main diagonal for the entire sum $\mathbf{A}+\mathbf{A}^2+\cdots+\mathbf{A}^{n-1}$. On the other hand, if some particular power of $\mathbf{A}$ has no zero entries off the main diagonal, then neither will the sum $\mathbf{A}+\mathbf{A}^2+\cdots+\mathbf{A}^{n-1}$, and so the associated graph is connected.

In Exercise 12, you are asked to prove that if $\mathbf{A}$ is the adjacency matrix for a connected graph having $n$ vertices, with $n \geq 3$, then $\mathbf{A}^2$ has all nonzero entries on the main diagonal. Therefore, Theorem 8.3 tells us that a graph with 3 or more vertices is connected if and only if *all* entries of $\mathbf{A}+\mathbf{A}^2+\cdots+\mathbf{A}^{n-1}$ are nonzero (*on* as well as *off* the main diagonal).

## New Vocabulary

adjacency matrix (for a graph or digraph)
connected graph
digraph
directed edge
disconnected graph
edge
graph
length of a path
loop
loop-free
path
vertex

## Highlights

- Digraphs differ from graphs in that every edge in a digraph is assigned a direction.
- If $\mathbf{A}$ is an adjacency matrix for a graph [digraph], the $(i, j)$ entry of $\mathbf{A}$ equals the number of [directed] edges from vertex $P_i$ to vertex $P_j$.
- If $\mathbf{A}$ is an adjacency matrix for a graph or digraph, the total number of paths from vertex $P_i$ to vertex $P_j$ of length $k$ is the $(i, j)$ entry of $\mathbf{A}^k$, and the total number of paths of length $\leq k$ is the $(i, j)$ entry of $\mathbf{A}+\mathbf{A}^2+\mathbf{A}^3+\cdots+\mathbf{A}^k$.
- A graph having $n$ vertices (with $n \geq 2$) with adjacency matrix $\mathbf{A}$ is connected if and only if $\mathbf{A}+\mathbf{A}^2+\cdots+\mathbf{A}^{n-1}$ has no zero entries off the main diagonal.

# Exercises for Section 8.1

**Note**: You may want to use a computer or calculator to perform the matrix computations in these exercises.

★ **1.** For each of the graphs and digraphs in Fig. 8.5, give the corresponding adjacency matrix. Which of these matrices are symmetric?

★ **2.** Which of the given matrices could be the adjacency matrix for a graph or digraph? Draw the corresponding graph and/or digraph when appropriate. (If both are appropriate, simply draw the graph.)

$$\mathbf{A}=\begin{bmatrix}1&4\\0&1\\3&0\end{bmatrix}\quad \mathbf{B}=\begin{bmatrix}2&0\\0&-1\end{bmatrix}\quad \mathbf{C}=\begin{bmatrix}2&0&0&1\\0&2&0&0\\0&0&2&0\\0&0&0&2\end{bmatrix}\quad \mathbf{D}=\begin{bmatrix}1&\frac{3}{2}&3\\0&2&1\\0&0&\frac{2}{3}\end{bmatrix}$$

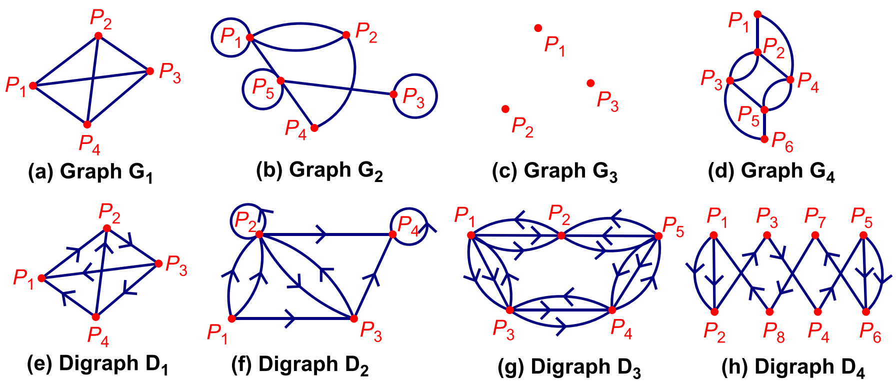


**FIGURE 8.5** Graphs and digraphs for Exercise 1

$$\mathbf{E} = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & -2 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{bmatrix} \quad \mathbf{F} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix} \quad \mathbf{G} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{H} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \quad \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{J} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} \quad \mathbf{K} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

**3.** Suppose the writings of six authors—labeled A, B, C, D, E, and F—have been influenced by one another in the following ways:

A has been influenced by D and E.
B has been influenced by C and E.
C has been influenced by A.
D has been influenced by B, E, and F.
E has been influenced by B and C.
F has been influenced by D.

Draw the digraph that represents these relationships—drawing a directed edge from Author X to Author Y if X influences Y. What is its adjacency matrix? What would the transpose of this adjacency matrix represent?

**4.** Using the adjacency matrix for the graph in Fig. 8.6, find the following:

★ **(a)** The number of paths of length 3 from $P_2$ to $P_4$

**(b)** The number of paths of length 4 from $P_1$ to $P_5$

★ **(c)** The number of paths of length $\leq 4$ from $P_4$ to $P_2$

**(d)** The number of paths of length $\leq 4$ from $P_3$ to $P_1$

★ **(e)** The length of the shortest path from $P_4$ to $P_3$

**(f)** The length of the shortest path from $P_5$ to $P_2$

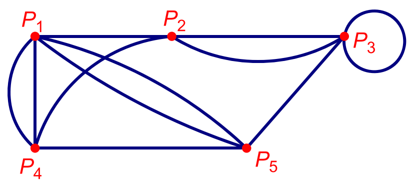


**FIGURE 8.6** Graph for Exercises 4 and 6

5. Using the adjacency matrix for the digraph in Fig. 8.7, find the following:
   ★ (a) The number of paths of length 3 from $P_2$ to $P_4$
   (b) The number of paths of length 4 from $P_1$ to $P_5$
   ★ (c) The number of paths of length $\leq 3$ from $P_3$ to $P_2$
   (d) The number of paths of length $\leq 4$ from $P_3$ to $P_1$
   ★ (e) The length of the shortest path from $P_3$ to $P_5$
   (f) The length of the shortest path from $P_5$ to $P_4$

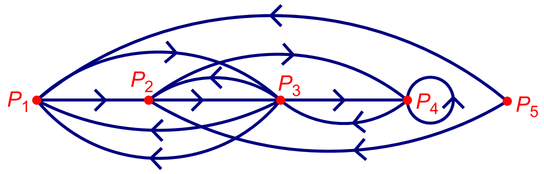


FIGURE 8.7 Digraph for Exercises 5, 7, and 13

6. A **cycle** in a graph or digraph is a path connecting a vertex to itself. For the graph in Fig. 8.6, find the following:
   ★ (a) The number of cycles of length 3 connecting $P_2$ to itself
   (b) The number of cycles of length 4 connecting $P_1$ to itself
   ★ (c) The number of cycles of length $\leq 4$ connecting $P_4$ to itself
7. Using the definition of a cycle in Exercise 6, find the following for the digraph in Fig. 8.7:
   ★ (a) The number of cycles of length 3 connecting $P_2$ to itself
   (b) The number of cycles of length 4 connecting $P_1$ to itself
   ★ (c) The number of cycles of length $\leq 4$ connecting $P_4$ to itself
8. This exercise involves isolated vertices in a graph or digraph.
   ★ (a) Suppose that there is one vertex that is not connected to any other in a graph. How will this situation be reflected in the adjacency matrix for the graph?
   (b) Suppose that there is one vertex that is not directed to any other in a digraph. How will this situation be reflected in the adjacency matrix for the digraph?
9. This exercise concerns the trace of an adjacency matrix (see Exercise 13 of Section 1.4) and its powers.
   ★ (a) What information does the trace of the adjacency matrix of a graph or digraph give?
   (b) Suppose $\mathbf{A}$ is the adjacency matrix of a graph or digraph, and $k > 0$. What information does the trace of $\mathbf{A}^k$ give? (Hint: See Exercise 6.)
10. In each part, use Theorem 8.3 to determine whether the graph associated with the given adjacency matrix is connected.

★ (a) $\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 2 & 0 & 2 & 0 \end{bmatrix}$

★ (b) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 2 & 0 & 0 & 3 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 2 & 0 & 0 & 1 & 0 \end{bmatrix}$

11. Suppose $G_1$ is a graph (with possible loops) having $n$ vertices, for $n \geq 2$, and $G_2$ is the graph obtained by adding a loop at every vertex of $G_1$. Let $\mathbf{A}$ be the adjacency matrix for $G_1$.
   ★ (a) Explain why $\mathbf{A} + \mathbf{I}_n$ is the adjacency matrix for $G_2$.
   ★ (b) Prove that $G_1$ is connected if and only if $G_2$ is connected.
   ★ (c) Prove that if there is a path of length $\leq k$ from $P_i$ to $P_j$ in $G_1$, then there is a path of length $k$ from $P_i$ to $P_j$ in $G_2$.
   (d) Prove that $G_2$ is connected if and only if, given any two distinct vertices, there is a path of length $n - 1$ connecting them.

**(e)** Prove that $G_1$ is connected if and only if $(\mathbf{A}+\mathbf{I}_n)^{n-1}$ has no zero entries. (Hint: Use parts (b) and (d).)

**(f)** Use the method suggested by part (e) to determine whether or not each of the graphs from Exercise 10 is connected.

**12.** Prove the assertion in the last paragraph of this section: If $\mathbf{A}$ is the adjacency matrix for a connected graph having $n$ vertices, with $n \geq 3$, then $\mathbf{A}^2$ has all nonzero entries on the main diagonal. (Note: This result is also true if $n=2$.)

**13.** This exercise investigates the concept of connectivity for digraphs.

★ **(a)** A **strongly connected digraph** is a digraph in which, given any pair of distinct vertices, there is a directed path (of some length) from each of these two vertices to the other. Determine whether the digraph $D_2$ in Fig. 8.2 and the digraph in Fig. 8.7 are strongly connected.

**(b)** Prove that a digraph with $n$ vertices having adjacency matrix $\mathbf{A}$ is strongly connected if and only if $\mathbf{A}+\mathbf{A}^2+\mathbf{A}^3+\cdots+\mathbf{A}^{n-1}$ has the property that all entries not on the main diagonal are nonzero.

**(c)** A **weakly connected digraph** is a digraph that would be connected as a graph once all of its directed edges are replaced with (non-directed) edges. Prove that a digraph with $n$ vertices having adjacency matrix $\mathbf{A}$ is weakly connected if and only if $\left(\mathbf{A}+\mathbf{A}^T\right)+\left(\mathbf{A}+\mathbf{A}^T\right)^2+\left(\mathbf{A}+\mathbf{A}^T\right)^3+\cdots+\left(\mathbf{A}+\mathbf{A}^T\right)^{n-1}$ has all entries nonzero off the main diagonal.

★ **(d)** Use the result in part (c) to determine whether the digraphs having the following adjacency matrices are weakly connected:

$$\mathbf{A}=\begin{bmatrix}0&0&0&2&0\\0&1&0&0&1\\0&0&0&1&0\\0&0&2&0&0\\0&0&0&0&1\end{bmatrix},\qquad \mathbf{B}=\begin{bmatrix}0&0&2&0&0\\0&0&0&1&0\\0&0&0&0&0\\2&0&0&0&0\\0&0&1&0&1\end{bmatrix}.$$

**14.** This exercise introduces dominance digraphs.

**(a)** A **dominance digraph** is one with no loops in which, for any two distinct vertices $P_i$ and $P_j$, there is either a single directed edge from $P_i$ to $P_j$, or a single directed edge from $P_j$ to $P_i$, but *not both*. (Dominance digraphs are useful in psychology, sociology, and communications.) Show that the following matrix is the adjacency matrix for a dominance digraph:

$$\begin{array}{c} \\ P_1\\P_2\\P_3\\P_4\end{array}\begin{array}{c}\begin{array}{cccc}P_1&P_2&P_3&P_4\end{array}\\ \begin{bmatrix}0&1&0&1\\0&0&1&0\\1&0&0&1\\0&1&0&0\end{bmatrix}\end{array}.$$

★ **(b)** Suppose six teams in a league play a tournament in which each team plays every other team exactly once (with no tie games possible). Consider a digraph representing the outcomes of such a tournament in which a directed edge is drawn from the vertex for Team A to the vertex for Team B if Team A defeats Team B. Is this a dominance digraph? Why or why not?

**(c)** Suppose that $\mathbf{A}$ is a square matrix with each entry equal to 0 or to 1. Show that $\mathbf{A}$ is the adjacency matrix for a dominance digraph if and only if $\mathbf{A}+\mathbf{A}^T$ has all main diagonal entries equal to 0, and all other entries equal to 1.

▶ **15.** Prove Theorem 8.1. (Hint: Use a proof by induction on the length of the path between vertices $P_i$ and $P_j$. In the Inductive Step, use the fact that the total number of paths from $P_i$ to $P_j$ of length $k+1$ is the sum of $n$ products, where each product is the number of paths of length $k$ from $P_i$ to some vertex $P_q$ $(1 \leq q \leq n)$ times the number of paths of length 1 from $P_q$ to $P_j$.)

★ **16.** True or False:

**(a)** The adjacency matrix of a graph must be symmetric.

**(b)** The adjacency matrix for a digraph must be skew-symmetric.

**(c)** If $\mathbf{A}$ is the adjacency matrix for a digraph and the $(1,2)$ entry of $\mathbf{A}^n$ is zero for all $n \geq 1$, then there is no path from vertex $P_1$ to $P_2$.

**(d)** The number of edges in any graph equals the sum of the entries in its adjacency matrix.

**(e)** The number of directed edges in any digraph equals the sum of the entries in its adjacency matrix.

**(f)** If a graph has a path of length $k$ from $P_1$ to $P_2$ and a path of length $j$ from $P_2$ to $P_3$, then it has a path of length $k + j$ from $P_1$ to $P_3$.

**(g)** The sum of the numbers in the $i$th column of the adjacency matrix for a graph gives the number of edges connected to $P_i$.

**(h)** The sum of the numbers in the $i$th column of the adjacency matrix for a digraph gives the number of directed edges connected to $P_i$.

**(i)** If $\mathbf{A}$ is the adjacency matrix for a graph or digraph, and the $(3, 4)$ entry of $\mathbf{A}^4$ is 5, then there are 4 paths of length 5 from $P_3$ to $P_4$.

**(j)** If $\mathbf{A}$ is the adjacency matrix for a graph or digraph, and the $(5, 2)$ entry of $\mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 + \mathbf{A}^4$ is 12, then the number of paths of length 4 from $P_5$ to $P_2$ is 12.

**(k)** If $\mathbf{A}$ is the adjacency matrix for a graph with 5 vertices, and the $(2, 3)$ entry of $\mathbf{A}^4$ is 0, then the graph is not connected.

**(l)** If $\mathbf{A}$ is the adjacency matrix for a graph with 5 vertices, and the $(2, 3)$ entry of $\mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3$ is 0, then the graph is not connected.

**(m)** If $\mathbf{A}$ is the adjacency matrix for a graph with 5 vertices, and the $(2, 3)$ entry of $\mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 + \mathbf{A}^4 + \mathbf{A}^5$ is 0, then the graph is not connected.

## 8.2 Ohm's Law

*Prerequisite: Section 2.2, Gauss-Jordan Row Reduction and Reduced Row Echelon Form*

In this section, we examine an important application of systems of linear equations to circuit theory in physics.

### Circuit Fundamentals and Ohm's Law

In a simple electrical circuit, such as the one in Fig. 8.8, **voltage sources** (for example, batteries) stimulate electric current to flow through the circuit. **Voltage** ($V$) is measured in **volts**, and **current** ($I$) is measured in **amperes**. The circuit in Fig. 8.8 has two voltage sources: $48V$ and $9V$. Current flows from the positive (+) end of the voltage source to the negative (−) end.

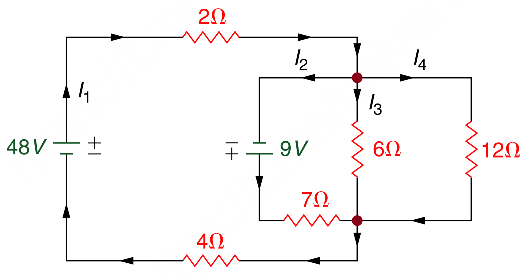


**FIGURE 8.8** Electrical circuit in Section 8.2

In contrast to voltage sources, there are **voltage drops**, or **sinks**, when **resistors** are present, because resistors impede the flow of current. In particular, the following principle holds:

**OHM'S LAW**
At any resistor, the amount of voltage $V$ dropped is proportional to the amount of current $I$ flowing through the resistor. That is, $V = IR$, where the proportionality constant $R$ is a measure of the resistance to the current.

**Resistance** ($R$) is measured in **ohms**, or volts/ampere. The Greek letter Ω is used to denote ohms.

Any point in the circuit where current-carrying branches meet is called a **junction**. Any path that the current takes along the branches of a circuit is called a **loop** if the path begins and ends at the same location. The following two principles involving junctions and loops are very important:

**KIRCHHOFF'S LAWS**

**First Law:** The sum of the currents flowing into a junction must equal the sum of the currents leaving that junction.

**Second Law:** The sum of the voltage sources and drops around any loop of a circuit is zero.

**Example 1**

Consider the electrical circuit in Fig. 8.8. We will use Kirchhoff's Laws and Ohm's Law to find the amount of current flowing through each branch of the circuit. We consider each of Kirchhoff's Laws in turn.

**Kirchhoff's First Law:** The circuit has the following two junctions: the first where current $I_1$ branches into the three currents $I_2$, $I_3$, and $I_4$ and the second where these last three currents merge again into $I_1$. By the First Law, both junctions produce the same equation: $I_1 = I_2 + I_3 + I_4$.

**Kirchhoff's Second Law:** All of the current runs through the $48V$ voltage source, and there are only three different loops that start and end at this voltage source:

$$\begin{array}{ll} (1) & I_1 \to I_2 \to I_1 \\ (2) & I_1 \to I_3 \to I_1 \\ (3) & I_1 \to I_4 \to I_1 \end{array}$$

The Ohm's Law equation for each of these loops is:

$$\begin{array}{ll} 48V + 9V - I_1(2\Omega) - I_2(7\Omega) - I_1(4\Omega) = 0 & \textbf{(loop 1)} \\ 48V - I_1(2\Omega) - I_3(6\Omega) - I_1(4\Omega) = 0 & \textbf{(loop 2)} \\ 48V - I_1(2\Omega) - I_4(12\Omega) - I_1(4\Omega) = 0 & \textbf{(loop 3)} \end{array}.$$

Thus, Kirchhoff's First and Second Laws together lead to the following system of four equations and four variables:

$$\begin{cases} -I_1 + I_2 + I_3 + I_4 = 0 \\ 6I_1 + 7I_2 = 57 \\ 6I_1 + 6I_3 = 48 \\ 6I_1 + 12I_4 = 48 \end{cases}.$$

After applying the Gauss-Jordan Method to the augmented matrix for this system, we obtain

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right].$$

Hence, $I_1 = 6$ amperes, $I_2 = 3$ amperes, $I_3 = 2$ amperes, and $I_4 = 1$ ampere.

## New Vocabulary

current (in amperes)
junction
Kirchhoff's First Law
Kirchhoff's Second Law
loop
Ohm's Law
resistance (in ohms)
voltage (in volts)
voltage drops
voltage sources

## Highlights

- Ohm's Law: At any resistor, $V = IR$ (voltage = current × resistance).
- Kirchhoff's First Law: The sum of the currents entering a junction equals the sum of the currents leaving the junction.
- Kirchhoff's Second Law: Around any circuit loop, the sum of the voltage sources and drops is zero.
- Kirchhoff's First and Second Laws are used together to find the current in each branch of a circuit when the voltage sources and drops are known along every possible loop in the circuit.

## Exercises for Section 8.2

**1.** Use Kirchhoff's Laws and Ohm's Law to find the current in each branch of the electrical circuits in Fig. 8.9, with the indicated voltage sources and resistances.

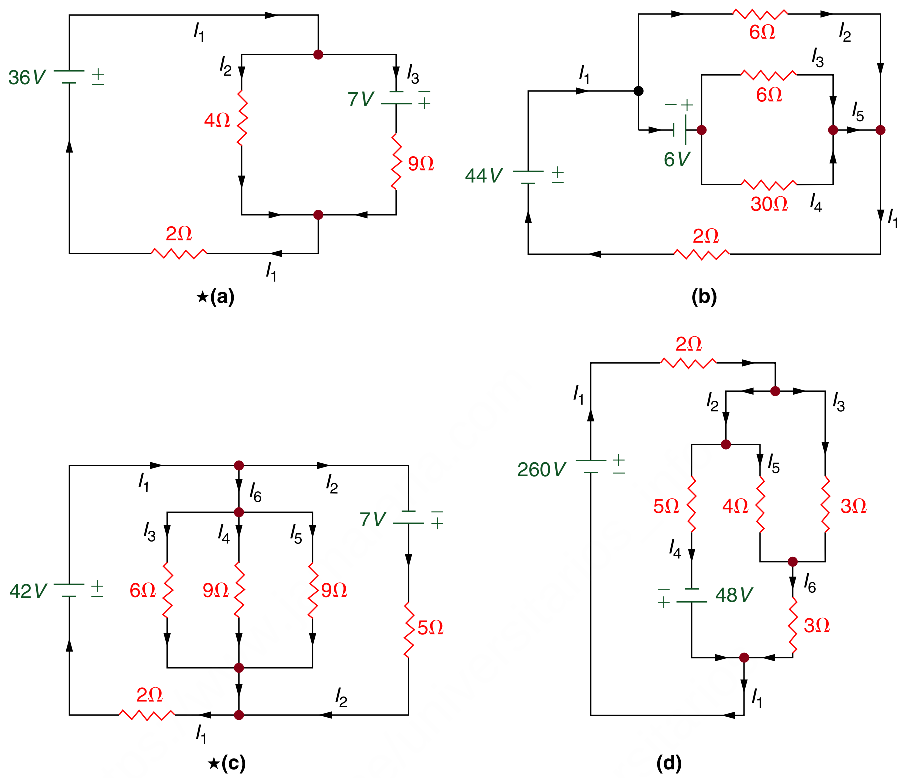


**FIGURE 8.9** Electrical circuits for Exercise 1

★ **2.** True or False:

**(a)** Kirchhoff's laws produce one equation for each junction and one equation for each loop.

**(b)** The resistance $R$ is the constant of proportionality in Ohm's Law relating the current $I$ and the voltage $V$.

# 8.3 Least-Squares Polynomials

*Prerequisite: Section 2.2, Gauss-Jordan Row Reduction and Reduced Row Echelon Form*

In this section, we present the least-squares technique for finding a polynomial "closest" to a given set of data points. You should have a calculator or computer handy as you work through some of the examples and exercises.

## Least-Squares Polynomials

In science and business, we often need to predict the relationship between two given variables. In many cases, we begin by performing an appropriate laboratory experiment or statistical analysis to obtain the necessary data. However, even if a simple law governs the behavior of the variables, this law may not be easy to find because of errors introduced in measuring or sampling. In practice, therefore, we are often content with a polynomial equation that provides a close approximation to the data.

Suppose we are given a set of data points $(a_1, b_1), (a_2, b_2), (a_3, b_3), \ldots, (a_n, b_n)$ that may have been obtained from an analysis or experiment. We want a technique for finding polynomial equations $y = f(x)$ to fit these points as "closely" as possible. One approach would be to minimize the sum of the vertical distances $|f(a_1) - b_1|, |f(a_2) - b_2|, \ldots, |f(a_n) - b_n|$ between the graph of $y = f(x)$ and the data points. These distances are the lengths of the line segments in Fig. 8.10. However, this is not the approach typically used. Instead, we will minimize the distance between the vectors $\mathbf{y} =$

$[f(a_1), \ldots, f(a_n)]$ and $\mathbf{b} = [b_1, \ldots, b_n]$, which equals $\|\mathbf{y} - \mathbf{b}\|$. This is equivalent to minimizing the sum of the *squares* of the vertical distances shown in Fig. 8.10.

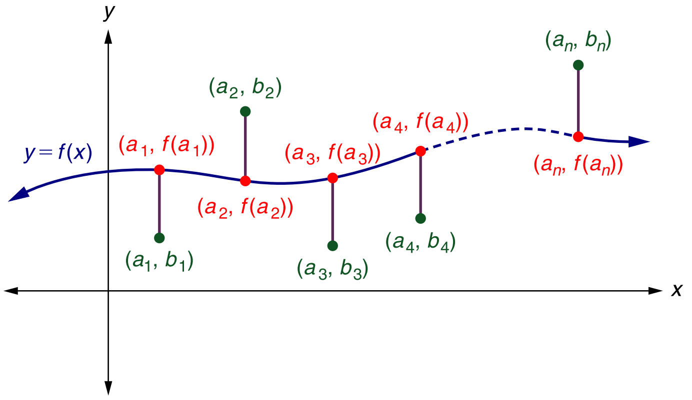


**FIGURE 8.10** Vertical distances from data points $(a_k, b_k)$ to $y = f(x)$, for $1 \le k \le n$

**Definition** A **degree** $t$ **least-squares polynomial** for the points $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$ is a polynomial $y = f(x) = c_t x^t + \cdots + c_2 x^2 + c_1 x + c_0$ for which the sum

$$S_f = (f(a_1) - b_1)^2 + (f(a_2) - b_2)^2 + (f(a_3) - b_3)^2 + \cdots + (f(a_n) - b_n)^2$$

of the squares of the vertical distances from each of the given points to the polynomial is less than or equal to the corresponding sum, $S_g$, for any other polynomial $g$ of degree $\le t$.

Note that it is possible for a "degree $t$ least-squares polynomial" to actually have a degree less than $t$ because there is no guarantee that its leading coefficient will be nonzero.

We will illustrate the computation of a least-squares line and a least-squares quadratic in the examples to follow. After these concrete examples, we state a general technique for calculating least-squares polynomials in Theorem 8.4.

## Least-Squares Lines

Suppose we are given a set of points $(a_1, b_1), (a_2, b_2), (a_3, b_3), \ldots, (a_n, b_n)$ and we want to find a degree 1 least-squares polynomial for these points. This will give us a straight line $y = c_1 x + c_0$ that fits these points as "closely" as possible. Such a least-squares line for a given set of data is often called a **line of best fit**, or a **linear regression**.

Let

$$\mathbf{A} = \begin{bmatrix} 1 & a_1 \\ 1 & a_2 \\ 1 & a_3 \\ \vdots & \vdots \\ 1 & a_n \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}.$$

We will see in Theorem 8.4 that the solutions $c_0$ and $c_1$ of the linear system $\mathbf{A}^T\mathbf{A}\begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \mathbf{A}^T\mathbf{B}$ give the coefficients of a least-squares line $y = c_1 x + c_0$.

**Example 1**

We will find a least-squares line $y = c_1 x + c_0$ for the points $(a_1, b_1) = (-4, 6)$, $(a_2, b_2) = (-2, 4)$, $(a_3, b_3) = (1, 1)$, $(a_4, b_4) = (2, -1)$, and $(a_5, b_5) = (4, -3)$.

We let

$$\mathbf{A} = \begin{bmatrix} 1 & a_1 \\ 1 & a_2 \\ 1 & a_3 \\ 1 & a_4 \\ 1 & a_5 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 1 & -2 \\ 1 & 1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 1 \\ -1 \\ -3 \end{bmatrix}.$$

Then $\mathbf{A}^T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -4 & -2 & 1 & 2 & 4 \end{bmatrix}$, and so $\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 5 & 1 \\ 1 & 41 \end{bmatrix}$ and $\mathbf{A}^T\mathbf{B} = \begin{bmatrix} 7 \\ -45 \end{bmatrix}$. Hence, the equation

$$\mathbf{A}^T\mathbf{A}\begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \mathbf{A}^T\mathbf{B} \qquad \text{becomes} \qquad \begin{bmatrix} 5 & 1 \\ 1 & 41 \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 7 \\ -45 \end{bmatrix}.$$

Row reducing the augmented matrix

$$\left[\begin{array}{cc|c} 5 & 1 & 7 \\ 1 & 41 & -45 \end{array}\right] \qquad \text{gives} \qquad \left[\begin{array}{cc|c} 1 & 0 & 1.63 \\ 0 & 1 & -1.14 \end{array}\right],$$

and so a least-squares line for the given data points is $y = c_1 x + c_0 = -1.14x + 1.63$ (see Fig. 8.11).

Notice that, in this example, for each given $a_i$ value, this line produces a value "close" to the given $b_i$ value. For example, when $x = a_1 = -4$, $y = -1.14(-4) + 1.63 = 6.19$, which is close to $b_1 = 6$.

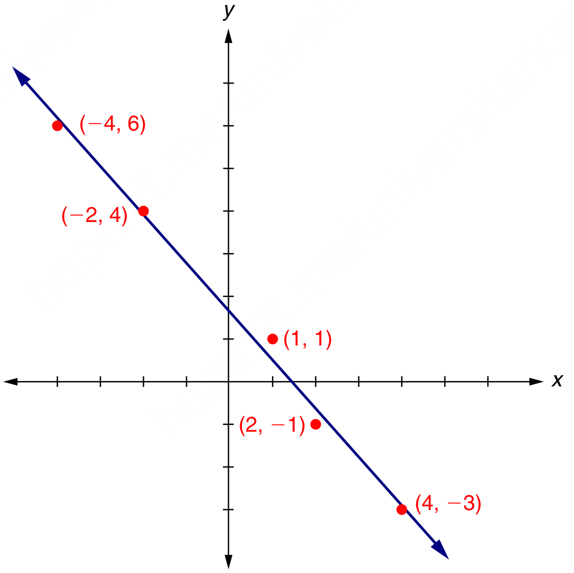


**FIGURE 8.11** Least-squares line for the data points in Example 1

Once we have calculated a least-squares polynomial for a given set of data points, we can use it to estimate the $y$-values for other nearby $x$-values. If a value of $x$ is chosen that is *outside* the range of the $x$-values for the given data points, an estimate for its corresponding $y$-value is called an **extrapolation**. For example, returning to Example 1, if we choose $x = 7$ (which is *not* between $x = -4$ and $x = 4$), the corresponding extrapolated value of $y$ is $-1.14(7) + 1.63 = -6.35$. Thus, we would expect the experiment that produced the original data to give a $y$-value close to $-6.35$ for an $x$-value of 7. On the other hand, when the chosen $x$-value is *within* the range of the $x$-values for the given data points, an estimate of the corresponding $y$-value is called an **interpolation**. For example, returning again to the data points in Example 1, if we choose $x = -1$ (which *is* between $x = -4$ and $x = 4$), the corresponding interpolated value of $y$ is $-1.14(-1) + 1.63 = 2.77$. That is, the experiment that produced the original data should give a $y$-value close to 2.77 for an $x$-value of $-1$.

## Least-Squares Quadratics

In the next example, we encounter data that suggest a parabolic rather than a linear shape. Here we find a second-degree least-squares polynomial to fit the data. The process is similar in spirit to that for least-squares lines. Let

$$\mathbf{A} = \begin{bmatrix} 1 & a_1 & a_1^2 \\ 1 & a_2 & a_2^2 \\ 1 & a_3 & a_3^2 \\ \vdots & \vdots & \vdots \\ 1 & a_n & a_n^2 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}.$$

The solutions $c_0$, $c_1$, and $c_2$ of the linear system $\mathbf{A}^T\mathbf{A}\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \mathbf{A}^T\mathbf{B}$ give the coefficients of a least-squares quadratic $y = c_2x^2 + c_1x + c_0$.

### Example 2

We will find a quadratic least-squares polynomial for the points $(-3, 7)$, $(-1, 4)$, $(2, 0)$, $(3, 1)$, and $(5, 6)$. We label these points $(a_1, b_1)$ through $(a_5, b_5)$, respectively. Let

$$\mathbf{A} = \begin{bmatrix} 1 & a_1 & a_1^2 \\ 1 & a_2 & a_2^2 \\ 1 & a_3 & a_3^2 \\ \vdots & \vdots & \vdots \\ 1 & a_n & a_n^2 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 9 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 5 & 25 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ 0 \\ 1 \\ 6 \end{bmatrix}.$$

Hence,

$$\mathbf{A}^T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -3 & -1 & 2 & 3 & 5 \\ 9 & 1 & 4 & 9 & 25 \end{bmatrix}, \text{ and so } \mathbf{A}^T\mathbf{A} = \begin{bmatrix} 5 & 6 & 48 \\ 6 & 48 & 132 \\ 48 & 132 & 804 \end{bmatrix} \text{ and } \mathbf{A}^T\mathbf{B} = \begin{bmatrix} 18 \\ 8 \\ 226 \end{bmatrix}.$$

Then the equation

$$\mathbf{A}^T\mathbf{A}\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \mathbf{A}^T\mathbf{B} \quad \text{becomes} \quad \begin{bmatrix} 5 & 6 & 48 \\ 6 & 48 & 132 \\ 48 & 132 & 804 \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 18 \\ 8 \\ 226 \end{bmatrix}.$$

Solving, we find $c_0 = 1.21$, $c_1 = -1.02$, and $c_2 = 0.38$. Hence, a least-squares quadratic polynomial is $y = c_2x^2 + c_1x + c_0 = 0.38x^2 - 1.02x + 1.21$ (see Fig. 8.12).

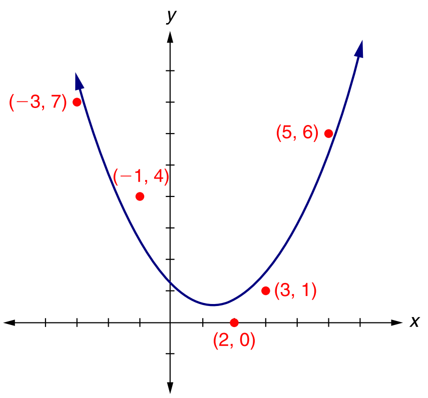


**FIGURE 8.12** Least-squares quadratic polynomial for the data points in Example 2

## Generalization of the Process

The process illustrated in Examples 1 and 2 is generalized in the following theorem, in which we are given $n$ data points and construct a least-squares polynomial of degree $t$ for the data. (This technique is usually used to find a least-squares polynomial whose degree $t$ is less than the given number $n$ of data points.)

**Theorem 8.4** *Let $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$ be $n$ points, and let*

$$\mathbf{A} = \begin{bmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^t \\ 1 & a_2 & a_2^2 & \cdots & a_2^t \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^t \end{bmatrix}, \text{ and } \mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

*($\mathbf{A}$ is an $n \times (t+1)$ matrix, and $\mathbf{B}$ is an $n \times 1$ matrix.) Then:*

(1) *A polynomial*

$$c_t x^t + \cdots + c_2 x^2 + c_1 x + c_0$$

*is a degree $t$ least-squares polynomial for the given points if and only if its coefficients $c_0, c_1, \ldots, c_t$ satisfy the linear system*

$$\mathbf{A}^T\mathbf{A} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_t \end{bmatrix} = \mathbf{A}^T\mathbf{B}.$$

(2) *The system $(\mathbf{A}^T\mathbf{A})\mathbf{X} = \mathbf{A}^T\mathbf{B}$ is always consistent, and so, for the given set of points, a degree $t$ least-squares polynomial exists.*

(3) *Furthermore, if $\mathbf{A}^T\mathbf{A}$ row reduces to $\mathbf{I}_{t+1}$, there is a unique degree $t$ least-squares polynomial for the given set of points.*

Notice from Theorem 8.4 that $\mathbf{A}^T$ is a $(t+1) \times n$ matrix. Thus, $\mathbf{A}^T\mathbf{A}$ is a $(t+1) \times (t+1)$ matrix, and so the matrix products in Theorem 8.4 make sense.

We do not prove Theorem 8.4 here. However, the theorem follows in a straightforward manner from Theorem 8.13 in Section 8.10. You may want to prove Theorem 8.4 later if you study Section 8.10.

## New Vocabulary

extrapolation
interpolation
least-squares line (= line of best fit = linear regression)
least-squares polynomial
least-squares quadratic

## Highlights

- A degree $t$ least-squares polynomial for the points $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$ is a polynomial $y = f(x) = c_t x^t + \cdots + c_2 x^2 + c_1 x + c_0$ (that is, a polynomial of degree $\leq t$) for which the sum $(f(a_1) - b_1)^2 + (f(a_2) - b_2)^2 + (f(a_3) - b_3)^2 + \cdots + (f(a_n) - b_n)^2$ of the squares of the vertical distances to these data points is a minimum.
- A degree $t$ least-squares polynomial for the points $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$ is a polynomial $c_t x^t + \cdots + c_2 x^2 + c_1 x + c_0$ whose corresponding vector of coefficients $\mathbf{X} = [c_0, c_1, \ldots, c_t]$ satisfies the linear system $\left(\mathbf{A}^T\mathbf{A}\right)\mathbf{X} = \mathbf{A}^T\mathbf{B}$, where $\mathbf{A} = \begin{bmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^t \\ 1 & a_2 & a_2^2 & \cdots & a_2^t \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^t \end{bmatrix}$, and $\mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$.
- Once a least-squares polynomial for a given set of data points has been calculated, it can be used to extrapolate or interpolate $y$-values for additional nearby $x$-values.

## Exercises for Section 8.3

**Note**: You should have a calculator or computer handy for the computations in many of these exercises.

1. For each of the following sets of points, find a line of best fit (that is, the least-squares line). In each case, extrapolate to find the approximate $y$-value when $x = 5$.
   ★ (a) $(3, -8), (1, -5), (0, -4), (2, -1)$
   (b) $(-6, -6), (-4, -3), (-1, 0), (1, 2)$
   ★ (c) $(-4, 10), (-3, 8), (-2, 7), (-1, 5), (0, 4)$
2. For each of the following sets of points, find a least-squares quadratic polynomial:
   ★ (a) $(-4, 8), (-2, 5), (0, 3), (2, 6)$
   (b) $(-1, -4), (0, -2), (2, -2), (3, -5)$
   ★ (c) $(-4, -3), (-3, -2), (-2, -1), (0, 0), (1, 1)$
3. For each of the following sets of points, find a least-squares cubic (degree 3) polynomial:
   ★ (a) $(-3, -3), (-2, -1), (-1, 0), (0, 1), (1, 4)$
   (b) $(-2, 5), (-1, 4), (0, 3), (1, 3), (2, 1)$
4. Use the points given for each function to find the desired approximation.
   ★ (a) Least-squares quadratic polynomial for $y = x^4$, using $x = -2, -1, 0, 1, 2$
   (b) Least-squares quadratic polynomial for $y = e^x$, using $x = -2, -1, 0, 1, 2$
   ★ (c) Least-squares quadratic polynomial for $y = \ln x$, using $x = 1, 2, 3, 4$
   (d) Least-squares cubic polynomial for $y = \sin x$, using $x = -\frac{\pi}{2}, -\frac{\pi}{4}, 0, \frac{\pi}{4}, \frac{\pi}{2}$
   ★ (e) Least-squares cubic polynomial for $y = \cos x$, using $x = -\frac{\pi}{2}, -\frac{\pi}{4}, 0, \frac{\pi}{4}, \frac{\pi}{2}$
5. An engineer is monitoring a leaning tower whose angle from the vertical over a period of months is given below.

| Month | 1 | 3 | 5 | 7 | 9 |
|---|---|---|---|---|---|
| Angle from vertical | 3° | 3.3° | 3.7° | 4.1° | 4.6° |

   ★ (a) Find a line of best fit for the given data. Use interpolation with this line to predict the tower's angle from the vertical in the eighth month.
   (b) Use extrapolation with the line of best fit in part (a) to predict the tower's angle from the vertical in the twelfth month.
   ★ (c) Find a least-squares quadratic approximation for the data. Use extrapolation with this quadratic approximation to predict the tower's angle from the vertical in the twelfth month.
   (d) Compare your answers to parts (b) and (c). Which approximation do you think is more accurate? Why?
   (e) Using the line of best fit from part (a), predict the month (with at least one decimal place of accuracy) in which the tower's angle will be 3.82° from the vertical.
   (f) Using the quadratic approximation from part (c), predict the month (with at least one decimal place of accuracy) in which the tower's angle will be 20° from the vertical.
6. The population of the United States (in millions), according to the Census Bureau, is given here.

| Year | 1970 | 1980 | 1990 | 2000 | 2010 | 2020 |
|---|---|---|---|---|---|---|
| Population | 203.3 | 226.5 | 248.7 | 281.4 | 308.7 | 331.1 |

   (a) Find a line of best fit for the data, and use interpolation to predict the population in 2005. (Hint: Renumber the given years as 1 through 6 to simplify the computation.)
   (b) Use extrapolation with the line of best fit in part (a) to predict the population in 2030.
   (c) Find a least-squares quadratic approximation for the data, and extrapolate to predict the population in 2030.

★ 7. Show that the least-squares technique gives the *exact* quadratic polynomial that goes through the points $(-2, 6)$, $(0, 2)$, and $(3, 8)$.

8. Show that the following system has the same solutions for $c_0$ and $c_1$ as the system in Theorem 8.4 when $t = 1$:

$$\begin{cases} nc_0 + \left(\sum_{i=1}^n a_i\right) c_1 = \sum_{i=1}^n b_i \\ \left(\sum_{i=1}^n a_i\right) c_0 + \left(\sum_{i=1}^n a_i^2\right) c_1 = \sum_{i=1}^n a_i b_i \end{cases}.$$

9. Although an inconsistent system $\mathbf{AX} = \mathbf{B}$ has no solutions, the least-squares technique is sometimes used to find values that come "close" to satisfying all the equations in the system. Solutions to the related system $\mathbf{A}^T\mathbf{AX} = \mathbf{A}^T\mathbf{B}$ (obtained by multiplying on the left by $\mathbf{A}^T$) are called **least-squares solutions** for the inconsistent system $\mathbf{AX} = \mathbf{B}$.

For each inconsistent system, find a least-squares solution, and check that it comes close to satisfying each equation in the system. (For more detailed information on least-squares solutions, see Section 8.10.)

★ (a) $\begin{cases} 4x_1 - 3x_2 = 12 \\ 2x_1 + 5x_2 = 32 \\ 3x_1 + \phantom{3}x_2 = 21 \end{cases}$

(b) $\begin{cases} 2x_1 - \phantom{3}x_2 + \phantom{2}x_3 = 11 \\ -x_1 + 3x_2 - \phantom{2}x_3 = -9 \\ \phantom{2}x_1 - 2x_2 + 3x_3 = 12 \\ 3x_1 - 4x_2 + 2x_3 = 21 \end{cases}$

★ **10.** True or False:

**(a)** If a set of data points all lie on the same line, then that line will be the line of best fit for the data.

**(b)** A degree 3 least-squares polynomial for a set of points must have degree 3.

**(c)** A line of best fit for a set of points must pass through at least one of the points.

**(d)** When finding a degree $t$ least-squares polynomial using Theorem 8.4, the product $\mathbf{A}^T\mathbf{A}$ is a $t \times t$ matrix.

## 8.4 Markov Chains

*Prerequisite: Section 2.2, Gauss-Jordan Row Reduction and Reduced Row Echelon Form*

In this section, we introduce Markov chains and demonstrate how they are used to predict the future states of an interdependent system. You should have a calculator or computer handy as you work through the examples and exercises.

### An Introductory Example

The following example will introduce many of the ideas associated with Markov chains:

#### Example 1

Suppose that three banks in a certain town are competing for investors. Currently, Bank A has 40% of the investors, Bank B has 10%, and Bank C has the remaining 50%. We can set up the following **probability** (or **state**) vector **p** to represent this distribution:

$$\mathbf{p} = \begin{bmatrix} .4 \\ .1 \\ .5 \end{bmatrix}.$$

Suppose the townsfolk are tempted by various promotional campaigns to switch banks. Records show that each year Bank A keeps half of its investors, with the remainder switching equally to Banks B and C. However, Bank B keeps two-thirds of its investors, with the remainder switching equally to Banks A and C. Finally, Bank C keeps half of its investors, with the remainder switching equally to Banks A and B. The following **transition matrix M** (rounded to three decimal places) keeps track of the changing investment patterns:

$$\mathbf{M} = \text{Next Year} \begin{matrix} \\ A \\ B \\ C \end{matrix} \overset{\text{Current Year}}{\begin{matrix} \quad A & \quad B & \quad C \\ \end{matrix}} \begin{bmatrix} .500 & .167 & .250 \\ .250 & .667 & .250 \\ .250 & .167 & .500 \end{bmatrix}.$$

The $(i, j)$ entry of **M** represents the fraction of current investors going *from* Bank $j$ *to* Bank $i$ next year.[2]

To find the distribution of investors after one year, consider

$$\mathbf{p}_1 = \mathbf{M}\mathbf{p} = \text{Next Year} \begin{matrix} A \\ B \\ C \end{matrix} \overset{\text{Current Year: } A \;\; B \;\; C}{\begin{bmatrix} .500 & .167 & .250 \\ .250 & .667 & .250 \\ .250 & .167 & .500 \end{bmatrix}} \begin{bmatrix} .4 \\ .1 \\ .5 \end{bmatrix} = \begin{bmatrix} .342 \\ .292 \\ .367 \end{bmatrix}.$$

The entries of $\mathbf{p}_1$ give the distribution of investors after one year. For example, the first entry of this product, .342, is obtained by taking the dot product of the first row of **M** with **p** as follows:

[2] It may seem more natural to let the $(i, j)$ entry of **M** represent the fraction going *from* Bank $i$ *to* Bank $j$. However, we arrange the matrix entries this way to facilitate matrix multiplication.

$$\underbrace{(.500)}_{\substack{\text{at Bank A,}\\\text{fraction of}\\\text{investors}\\\text{who stay at}\\\text{Bank A}}}\underbrace{(.4)}_{\substack{\text{fraction of}\\\text{investors}\\\text{currently at}\\\text{Bank A}}}+\underbrace{(.167)}_{\substack{\text{at Bank B,}\\\text{fraction of}\\\text{investors}\\\text{who switch}\\\text{to Bank A}}}\underbrace{(.1)}_{\substack{\text{fraction of}\\\text{investors}\\\text{currently at}\\\text{Bank B}}}+\underbrace{(.250)}_{\substack{\text{at Bank C,}\\\text{fraction of}\\\text{investors}\\\text{who switch}\\\text{to Bank A}}}\underbrace{(.5)}_{\substack{\text{fraction of}\\\text{investors}\\\text{currently at}\\\text{Bank C}}},$$

which gives .342, the total fraction of investors at Bank A after one year. We can continue this process for another year, as follows:

$$\mathbf{p}_2 = \mathbf{M}\mathbf{p}_1 = \begin{matrix} \\ \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \end{matrix} \begin{matrix} \begin{matrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{matrix} \\ \begin{bmatrix} .500 & .167 & .250 \\ .250 & .667 & .250 \\ .250 & .167 & .500 \end{bmatrix} \end{matrix} \begin{bmatrix} .342 \\ .292 \\ .367 \end{bmatrix} = \begin{bmatrix} .312 \\ .372 \\ .318 \end{bmatrix}.$$

Since multiplication by $\mathbf{M}$ gives the yearly change and the entries of $\mathbf{p}_1$ represent the distribution of investors at the end of the first year, we see that the entries of $\mathbf{p}_2$ represent the correct distribution of investors at the end of the second year. That is, after two years, 31.2% of the investors are at Bank A, 37.2% are at Bank B, and 31.8% are at Bank C. Notice that

$$\mathbf{p}_2 = \mathbf{M}\mathbf{p}_1 = \mathbf{M}(\mathbf{M}\mathbf{p}) = \mathbf{M}^2\mathbf{p}.$$

In other words, the matrix $\mathbf{M}^2$ takes us directly from $\mathbf{p}$ to $\mathbf{p}_2$. Similarly, if $\mathbf{p}_3$ is the distribution after three years, then

$$\mathbf{p}_3 = \mathbf{M}\mathbf{p}_2 = \mathbf{M}(\mathbf{M}^2\mathbf{p}) = \mathbf{M}^3\mathbf{p}.$$

A simple induction proof shows that, in general, if $\mathbf{p}_n$ represents the distribution after $n$ years, then $\mathbf{p}_n = \mathbf{M}^n\mathbf{p}$. We can use this formula to find the distribution of investors after 6 years. After tedious calculation (rounding to three decimal places at each step), we find

$$\mathbf{M}^6 = \begin{bmatrix} .288 & .285 & .288 \\ .427 & .432 & .427 \\ .288 & .285 & .288 \end{bmatrix}.$$

Then

$$\mathbf{p}_6 = \mathbf{M}^6\mathbf{p} = \begin{bmatrix} .288 & .285 & .288 \\ .427 & .432 & .427 \\ .288 & .285 & .288 \end{bmatrix} \begin{bmatrix} .4 \\ .1 \\ .5 \end{bmatrix} = \begin{bmatrix} .288 \\ .428 \\ .288 \end{bmatrix}.$$

## Formal Definitions

We now recap many of the ideas presented in Example 1 and give them a more formal treatment.

The notion of probability is important when discussing Markov chains. Probabilities of events are always given as values between $0 = 0\%$ and $1 = 100\%$, where a probability of 0 indicates no possibility, and a probability of 1 indicates certainty. For example, if we draw a random card from a standard deck of 52 playing cards, the probability that the card is an ace is $\frac{4}{52} = \frac{1}{13}$, because exactly 4 of the 52 cards are aces. The probability that the card is a red card is $\frac{26}{52} = \frac{1}{2}$, since there are 26 red cards in the deck. The probability that the card is both red and black (at the same time) is $\frac{0}{52} = 0$, since this event is impossible. Finally, the probability that the card is red or black is $\frac{52}{52} = 1$, since this event is certain.

Now consider a set of events that are completely "distinct" and "exhaustive" (that is, one and only one of them must occur at any time). The sum of all of their probabilities must total $100\% = 1$. For example, if we select a card at random, we have a $\frac{13}{52} = \frac{1}{4}$ chance each of choosing a club, diamond, heart, or spade. These represent the only distinct suit possibilities, and the sum of these four probabilities is 1.

Now recall that each column of the matrix $\mathbf{M}$ in Example 1 represents the probabilities that an investor switches assets to Bank A, B, or C. Since these are the only banks in town, the sum of the probabilities in each column of $\mathbf{M}$ must total 1, or Example 1 would not make sense as stated. Hence, $\mathbf{M}$ is a matrix of the following type:

**Definition** A **stochastic matrix** is a square matrix in which all entries are nonnegative and the entries of each column add up to 1.

A column matrix with all entries nonnegative and summing to 1 is often called a **stochastic vector**. The next theorem can be proven in a straightforward manner by induction (see Exercise 9):

**Theorem 8.5** *The product of any finite number of stochastic matrices is a stochastic matrix.*

Now we are ready to formally define a Markov chain.

**Definition** A **Markov chain** (or **Markov process**) is a system containing a finite number of distinct states $S_1, S_2, \ldots, S_n$ on which steps are performed such that

**(1)** At any time, each element of the system resides in exactly one of the states.
**(2)** At each step in the process, elements in the system can move from one state to another.
**(3)** The probabilities of moving from state to state are fixed—that is, they are the same at each step in the process.

In Example 1, the distinct states of the Markov chain are the three banks, A, B, and C, and the elements of the system are the investors, each one keeping money in only one of the three banks at any given time. Each new year represents another step in the process, during which time investors could switch banks or remain with their current bank. Finally, we have assumed that the probabilities of switching banks do not change from year to year.

**Definition** A **probability** (or **state**) **vector p** for a Markov chain is a stochastic vector whose $i$th entry is the probability that an element in the system is currently in state $S_i$. A **transition matrix M** for a Markov chain is a stochastic matrix whose $(i, j)$ entry is the probability that an element in state $S_j$ will move to state $S_i$ during the next step of the process.

The next theorem can be proven in a straightforward manner by induction (see Exercise 10).

**Theorem 8.6** *Let* $\mathbf{p}$ *be the (current) probability vector and* $\mathbf{M}$ *be the transition matrix for a Markov chain. After* $n$ *steps in the process, where* $n \geq 1$*, the (new) probability vector is given by* $\mathbf{p}_n = \mathbf{M}^n\mathbf{p}$.

Theorem 8.6 asserts that once the initial probability vector $\mathbf{p}$ and the transition matrix $\mathbf{M}$ for a Markov chain are known, all future steps of the Markov chain are determined.

## Limit Vectors and Fixed Points

A natural question to ask about a given Markov chain is whether we can discern any long-term trend.

### Example 2

Consider the Markov chain from Example 1, with transition matrix

$$\mathbf{M} = \begin{bmatrix} .500 & .167 & .250 \\ .250 & .667 & .250 \\ .250 & .167 & .500 \end{bmatrix}.$$

What happens in the long run? To discern this, we calculate $\mathbf{p}_k$ for large values of $k$. Starting with $\mathbf{p} = [.4, .1, .5]$ and computing $\mathbf{p}_k = \mathbf{M}^k\mathbf{p}$ for increasing values of $k$ (a calculator or computer is extremely useful here), we find that $\mathbf{p}_k$ approaches[3] the vector

$$\mathbf{p}_f = [.286,\ .429,\ .286],$$

where we are again rounding to three decimal places.[4]

[3] The intuitive concept of a sequence of vectors approaching a vector can be defined precisely using limits. We say that $\lim_{k\to\infty} \mathbf{p}_k = \mathbf{p}_f$ if and only if $\lim_{k\to\infty} \|\mathbf{p}_k - \mathbf{p}_f\| = 0$. It can be shown that this is equivalent to having the differences between the corresponding entries of $\mathbf{p}_k$ and $\mathbf{p}_f$ approach 0 as $k$ grows larger. A similar approach can be used with matrices, where we say that $\lim_{k\to\infty} \mathbf{M}^k = \mathbf{M}_f$ if the differences between corresponding entries of $\mathbf{M}^k$ and $\mathbf{M}_f$ approach 0 as $k$ grows larger.

[4] When raising matrices, such as $\mathbf{M}$, to high powers, roundoff error can quickly compound. Although we have printed $\mathbf{M}$ rounded to 3 significant digits, we actually performed the computations using $\mathbf{M}$ rounded to 12 digits of accuracy. In general, minimize your roundoff error by using as many digits as your calculator or software will provide.

Alternatively, to calculate $\mathbf{p}_f$, we could have first shown that as $k$ gets larger, $\mathbf{M}^k$ approaches the matrix

$$\mathbf{M}_f = \begin{bmatrix} .286 & .286 & .286 \\ .429 & .429 & .429 \\ .286 & .286 & .286 \end{bmatrix},$$

by multiplying out higher powers of $\mathbf{M}$ until successive powers agree to the desired number of decimal places. The probability vector $\mathbf{p}_f$ could then be found by

$$\mathbf{p}_f = \mathbf{M}_f\mathbf{p} = \begin{bmatrix} .286 & .286 & .286 \\ .429 & .429 & .429 \\ .286 & .286 & .286 \end{bmatrix}\begin{bmatrix} .4 \\ .1 \\ .5 \end{bmatrix} = \begin{bmatrix} .286 \\ .429 \\ .286 \end{bmatrix}.$$

Both techniques yield the same answer for $\mathbf{p}_f$. Ultimately, Banks A and C each capture 28.6%, or $\frac{2}{7}$, of the investors, and Bank B captures 42.9%, or $\frac{3}{7}$, of the investors. The vector $\mathbf{p}_f$ is called a **limit vector** of the Markov chain.

We now give a formal definition for a limit vector of a Markov chain.

**Definition** Let $\mathbf{M}$ be the transition matrix, and let $\mathbf{p}$ be the current probability vector for a Markov chain. Let $\mathbf{p}_k$ represent the probability vector after $k$ steps of the Markov chain. If the sequence $\mathbf{p}, \mathbf{p}_1, \mathbf{p}_2, \ldots$ of vectors approaches some vector $\mathbf{p}_f$, then $\mathbf{p}_f$ is called a **limit vector** for the Markov chain.

The computation of $\mathbf{p}_k$ for large $k$, or equivalently, the computation of large powers of the transition matrix $\mathbf{M}$, is not always an easy task, even with the use of a computer. We now show a quicker method to obtain the limit vector $\mathbf{p}_f$ for the Markov chain of Example 2. Notice that this vector $\mathbf{p}_f$ has the property that

$$\mathbf{M}\mathbf{p}_f = \begin{bmatrix} .500 & .167 & .250 \\ .250 & .667 & .250 \\ .250 & .167 & .500 \end{bmatrix}\begin{bmatrix} .286 \\ .429 \\ .286 \end{bmatrix} = \begin{bmatrix} .286 \\ .429 \\ .286 \end{bmatrix} = \mathbf{p}_f.$$

This remarkable property says that $\mathbf{p}_f$ is a vector that satisfies the equation $\mathbf{M}\mathbf{x} = \mathbf{x}$. Such a vector is called a **fixed point** for the Markov chain. Now, if we did not know $\mathbf{p}_f$, we could solve the equation

$$\mathbf{M}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

to find it. We can rewrite this as

$$\mathbf{M}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \text{or} \quad (\mathbf{M} - \mathbf{I}_3)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The augmented matrix for this system is

$$\left[\begin{array}{ccc|c} .500-1 & .167 & .250 & 0 \\ .250 & .667-1 & .250 & 0 \\ .250 & .167 & .500-1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} -.500 & .167 & .250 & 0 \\ .250 & -.333 & .250 & 0 \\ .250 & .167 & -.500 & 0 \end{array}\right].$$

We can also add another condition, since we know that $x_1 + x_2 + x_3 = 1$. Thus, the augmented matrix gets a fourth row as follows:

$$\left[\begin{array}{ccc|c} -.500 & .167 & .250 & 0 \\ .250 & -.333 & .250 & 0 \\ .250 & .167 & -.500 & 0 \\ 1.000 & 1.000 & 1.000 & 1 \end{array}\right].$$

After row reduction, we find that the solution set is $x_1 = .286$, $x_2 = .429$, and $x_3 = .286$, as expected. Thus, the fixed point solution to $\mathbf{Mx} = \mathbf{x}$ equals the limit vector $\mathbf{p}_f$ we computed previously.

In general, if a limit vector $\mathbf{p}_f$ exists, it is a fixed point, and so this technique for finding the limit vector is especially useful where there is a unique fixed point. However, we must be careful because a given state vector for a Markov chain does not necessarily converge to a limit vector, as the next example shows.

**Example 3**

Suppose that W, X, Y, and Z represent four train stations linked as shown in Fig. 8.13. Suppose that twelve trains shuttle between these stations. Currently, there are six trains at station W, three trains at station X, two trains at station Y, and one train at station Z. The probability that a randomly chosen train is at each station is given by the probability vector

$$\mathbf{p} = \begin{matrix} \mathbf{W} \\ \mathbf{X} \\ \mathbf{Y} \\ \mathbf{Z} \end{matrix} \begin{bmatrix} .500 \\ .250 \\ .167 \\ .083 \end{bmatrix}.$$

Suppose that during every hour, each train moves to the next station in Fig. 8.13. Then we have a Markov chain whose transition matrix is

$$\mathbf{M} = \text{Next State} \begin{matrix} \\ \mathbf{W} \\ \mathbf{X} \\ \mathbf{Y} \\ \mathbf{Z} \end{matrix} \begin{matrix} \text{Current State} \\ \begin{matrix} \mathbf{W} & \mathbf{X} & \mathbf{Y} & \mathbf{Z} \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{matrix}.$$

Intuitively, we can see there is no limit vector for this system, since the number of trains in each station never settles down to a fixed number but keeps rising and falling as the trains go around the "loop." This notion is borne out when we consider that the first few powers of the transition matrix are

$$\mathbf{M}^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{M}^3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{M}^4 = \mathbf{I}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since $\mathbf{M}^4 = \mathbf{I}_4$, all higher powers of $\mathbf{M}$ are equal to $\mathbf{M}$, $\mathbf{M}^2$, $\mathbf{M}^3$, or $\mathbf{I}_4$. (Why?) Therefore, the only probability vectors produced by this Markov chain are $\mathbf{p}$,

$$\mathbf{p}_1 = \mathbf{Mp} = \begin{bmatrix} .083 \\ .500 \\ .250 \\ .167 \end{bmatrix}, \quad \mathbf{p}_2 = \mathbf{M}^2\mathbf{p} = \begin{bmatrix} .167 \\ .083 \\ .500 \\ .250 \end{bmatrix}, \quad \text{and} \quad \mathbf{p}_3 = \mathbf{M}^3\mathbf{p} = \begin{bmatrix} .250 \\ .167 \\ .083 \\ .500 \end{bmatrix}$$

because $\mathbf{p}_4 = \mathbf{M}^4\mathbf{p} = \mathbf{I}_4\mathbf{p} = \mathbf{p}$ again. Since $\mathbf{p}_k$ keeps changing to one of four distinct vectors, the initial state vector $\mathbf{p}$ does not converge to a limit vector.

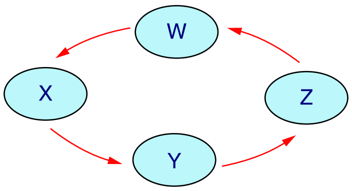


FIGURE 8.13 Four train stations

## Regular Transition Matrices

**Definition** A square matrix $\mathbf{R}$ is **regular** if and only if $\mathbf{R}$ is a stochastic matrix and some power $\mathbf{R}^k$, for $k \geq 1$, has all entries nonzero.

**Example 4**

The transition matrix $\mathbf{M}$ in Example 1 is a regular matrix, since $\mathbf{M}^1 = \mathbf{M}$ is a stochastic matrix with all entries nonzero. However, the transition matrix $\mathbf{M}$ in Example 3 is not regular because, as we saw in that example, all positive powers of $\mathbf{M}$ are equal to one of four matrices, each containing zero entries. Finally,

$$\mathbf{R} = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

is regular since it is stochastic and

$$\mathbf{R}^4 = \left(\mathbf{R}^2\right)^2 = \left(\begin{bmatrix} \frac{1}{4} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{4} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}\right)^2 = \begin{bmatrix} \frac{3}{16} & \frac{1}{8} & \frac{3}{8} \\ \frac{1}{2} & \frac{9}{16} & \frac{1}{4} \\ \frac{5}{16} & \frac{5}{16} & \frac{3}{8} \end{bmatrix},$$

which has all entries nonzero.

The next theorem, stated without proof, shows that Markov chains with regular transition matrices always have a limit vector $\mathbf{p}_f$ for *every* choice of an initial probability vector $\mathbf{p}$.

**Theorem 8.7** *If* $\mathbf{R}$ *is a regular* $n \times n$ *transition matrix for a Markov chain, then*

**(1)** $\mathbf{R}_f = \lim_{k\to\infty} \mathbf{R}^k$ *exists.*

**(2)** $\mathbf{R}_f$ *has all entries positive, and every column of* $\mathbf{R}_f$ *is identical.*

**(3)** *For all initial probability vectors* $\mathbf{p}$, *the Markov chain has a limit vector* $\mathbf{p}_f$. *Also, the limit vector* $\mathbf{p}_f$ *is the same for all* $\mathbf{p}$.

**(4)** $\mathbf{p}_f$ *is equal to any of the identical columns of* $\mathbf{R}_f$.

**(5)** $\mathbf{p}_f$ *is the unique stochastic* $n$*-vector such that* $\mathbf{R}\mathbf{p}_f = \mathbf{p}_f$. *That is,* $\mathbf{p}_f$ *is also the unique fixed point of the Markov chain.*

When the matrix for a Markov chain is regular, Theorem 8.7 shows that the Markov chain has a unique fixed point, and that it agrees with the limit vector $\mathbf{p}_f$ for any initial state. When the transition matrix is regular, this unique vector $\mathbf{p}_f$ is called the **steady-state vector** for the Markov chain.

**Example 5**

Consider a school of fish hunting for food in three adjoining lakes $L_1$, $L_2$, and $L_3$. Each day, the fish select a different lake to hunt in than the previous day, with probabilities given in the transition matrix below.

$$\mathbf{M} = \begin{array}{c} \\ \text{Next Day} \end{array} \begin{array}{c} \\ L_1 \\ L_2 \\ L_3 \end{array} \overset{\text{Current Day}}{\overset{\begin{array}{ccc} L_1 & L_2 & L_3 \end{array}}{\begin{bmatrix} 0 & .5 & 0 \\ .5 & 0 & 1 \\ .5 & .5 & 0 \end{bmatrix}}}.$$

Can we determine what percentage of time the fish will spend in each lake in the long run? Notice that $\mathbf{M}$ is equal to the matrix $\mathbf{R}$ in Example 4, and so $\mathbf{M}$ is regular. Theorem 8.7 asserts that the associated Markov chain has a steady-state vector. To find this vector, we solve the system

$$(\mathbf{M} - \mathbf{I}_3)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 & .5 & 0 \\ .5 & -1 & 1 \\ .5 & .5 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

to find a fixed point for the Markov chain, under the extra condition that $x_1 + x_2 + x_3 = 1$. The solution is $x_1 = .222$, $x_2 = .444$, and $x_3 = .333$; that is, $\mathbf{p}_f = [.222, .444, .333]$. Therefore, in the long run, the fish will hunt $22.2\% = \frac{2}{9}$ of the time in lake $L_1$, $44.4\% = \frac{4}{9}$ of the time in lake $L_2$, and $33.3\% = \frac{1}{3}$ of the time in lake $L_3$.

Notice in Example 5 that the initial probability state vector **p** was unneeded to find $\mathbf{p}_f$. The steady-state vector could also have been found by calculating larger and larger powers of **M** to see that they converge to the matrix

$$\mathbf{M}_f = \begin{bmatrix} .222 & .222 & .222 \\ .444 & .444 & .444 \\ .333 & .333 & .333 \end{bmatrix}.$$

Each of the identical columns of $\mathbf{M}_f$ is the steady-state vector for this Markov chain.

## New Vocabulary

fixed point (of a Markov chain)
limit vector (of a Markov chain)
Markov chain (process)
probability (state) vector (for a Markov chain)
regular matrix
steady-state vector (of a Markov chain)
stochastic matrix (or vector)
transition matrix (for a Markov chain)

## Highlights

- A square matrix **M** is stochastic if and only if all entries of **M** are nonnegative, and each column of **M** sums to 1.
- If **M** and **N** are stochastic matrices of the same size, then **MN** is stochastic.
- At each step in a Markov chain process, the probability that an element moves from a certain state to another is fixed. In particular, if **M** is the transition matrix for a Markov chain, then **M** is a stochastic matrix whose $(i, j)$ entry gives the (fixed) probability that an element moves from the $j$th state to the $i$th state during any step of the process.
- If **M** is the transition matrix for a Markov chain, and **p** is the initial probability vector (for an element to reside in each state of the Markov chain), then the probability vector after $n$ steps is $\mathbf{M}^n\mathbf{p}$.
- A fixed point for a Markov chain with transition matrix **M** is a vector **x** such that $\mathbf{Mx} = \mathbf{x}$.
- If **p** is the initial probability vector for a Markov chain, if $\mathbf{p}_k$ represents the probability vector after $k$ steps, and if the sequence $\mathbf{p}, \mathbf{p}_1, \mathbf{p}_2, \ldots$ approaches some vector $\mathbf{p}_f$, then $\mathbf{p}_f$ is a limit vector for the Markov chain. A limit vector is always a fixed point.
- A stochastic square matrix **M** is regular if some positive power of **M** has all entries nonzero.
- If the transition matrix **M** for a Markov chain is regular, then the Markov chain has a unique steady-state vector $\mathbf{p}_f$, which is its unique limit vector (regardless of the choice of the initial probability vector **p**).
- If the transition matrix **M** for a Markov chain is regular, the positive powers of **M** approach a limit (matrix) $\mathbf{M}_f$, and all columns of $\mathbf{M}_f$ equal the steady-state vector $\mathbf{p}_f$ of the Markov chain.

# Exercises for Section 8.4

**Note**: You should have a calculator or computer handy for many of these exercises.

★ **1.** Which of the following are stochastic matrices? Which are regular? Why?

$$\mathbf{A} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ \frac{1}{4} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} .2 & .4 & .5 \\ .5 & .1 & .4 \\ .3 & .4 & .1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} \frac{1}{5} & \frac{2}{3} \\ \frac{4}{5} & \frac{1}{3} \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{E} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix} \quad \mathbf{F} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 1 \\ 0 & 0 & 0 \\ \frac{2}{3} & \frac{2}{3} & 0 \end{bmatrix} \quad \mathbf{G} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \quad \mathbf{H} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

**2.** Suppose that each of the following represents the transition matrix **M** and the initial probability vector **p** for a Markov chain. Find the probability vectors $\mathbf{p}_1$ (after one step of the process) and $\mathbf{p}_2$ (after two steps).

★ **(a)** $\mathbf{M} = \begin{bmatrix} \frac{1}{4} & \frac{1}{3} \\ \frac{3}{4} & \frac{2}{3} \end{bmatrix}$, $\mathbf{p} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$

**(b)** $\mathbf{M} = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & 0 \\ 0 & \frac{2}{3} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$, $\mathbf{p} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{6} \\ \frac{1}{2} \end{bmatrix}$

★ **(c)** $\mathbf{M} = \begin{bmatrix} \frac{1}{4} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$, $\mathbf{p} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ \frac{1}{4} \end{bmatrix}$

**3.** Suppose that each of the following regular matrices represents the transition matrix **M** for a Markov chain. Find the steady-state vector for the Markov chain by solving an appropriate system of linear equations.

★ **(a)** $\begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{2}{3} \end{bmatrix}$

**(b)** $\begin{bmatrix} \frac{1}{3} & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{3} \end{bmatrix}$

**(c)** $\begin{bmatrix} \frac{1}{5} & \frac{1}{2} & 0 & \frac{1}{3} \\ \frac{3}{5} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{5} & 0 & 0 & \frac{2}{3} \end{bmatrix}$

**4.** Find the steady-state vector for the Markov chains in parts (a) and (b) of Exercise 3 by calculating large powers of the transition matrix (using a computer or calculator).

★ **5.** Suppose that the citizens in a certain community tend to switch their votes among political parties, as shown in the following transition matrix:

| | | Current Election | | | |
|---|---|---|---|---|---|
| | | **Party A** | **Party B** | **Party C** | **Nonvoting** |
| **Next Election** | **Party A** | .7 | .2 | .2 | .1 |
| | **Party B** | .1 | .6 | .1 | .1 |
| | **Party C** | .1 | .2 | .6 | .1 |
| | **Nonvoting** | .1 | 0 | .1 | .7 |

**(a)** Suppose that in the last election 30% of the citizens voted for Party A, 15% voted for Party B, and 45% voted for Party C. What is the likely outcome of the next election? What is the likely outcome of the election after that?

**(b)** If current trends continue, what percentage of the citizens will vote for Party A one century from now? Party C?

★ **6.** In a psychology experiment, a rat wanders in the maze in Fig. 8.14. During each time interval, the rat is allowed to pass through exactly one doorway. Assume there is a 50% probability that the rat will switch rooms during each interval. If it does switch rooms, assume that it has an equally likely chance of using any doorway out of its current room.

**(a)** What is the transition matrix for the associated Markov chain?

**(b)** Show that the transition matrix from part (a) is regular.

**(c)** If the rat is known to be in room C, what is the probability it will be in room D after two time intervals have passed?

**(d)** What is the steady-state vector for this Markov chain? Over time, which room does the rat frequent the least? Which room does the rat frequent the most?

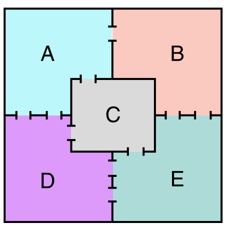


**FIGURE 8.14** Maze with five rooms

**7.** Show that the converse to part (3) of Theorem 8.7 is not true by demonstrating that the transition matrix

$$\mathbf{M} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

has the same limit vector for any initial input but is not regular. Does this Markov chain have a unique fixed point?

**8.** This exercise finds the steady-state vector for a certain transition matrix.

**(a)** Show that the transition matrix $\begin{bmatrix} 1-a & b \\ a & 1-b \end{bmatrix}$ has $\left(\frac{1}{a+b}\right)\begin{bmatrix} b \\ a \end{bmatrix}$ as a steady-state vector if $a$ and $b$ are not both 0.

**(b)** Use the result in part (a) to check that your answer for Exercise 3(a) is correct.

▶ **9.** Prove Theorem 8.5.

▶ **10.** Prove Theorem 8.6.

★ **11.** True or False:

**(a)** The transpose of a stochastic matrix is stochastic.

**(b)** For $n > 1$, no upper triangular $n \times n$ matrix is regular.

**(c)** If $\mathbf{M}$ is a regular $n \times n$ stochastic matrix, then there is a probability vector $\mathbf{p}$ such that $(\mathbf{M} - \mathbf{I}_n)\mathbf{p} = \mathbf{0}$.

**(d)** If $\mathbf{M}$ is a stochastic matrix and $\mathbf{p}$ and $\mathbf{q}$ are distinct probability vectors such that $\mathbf{Mp} = \mathbf{q}$ and $\mathbf{Mq} = \mathbf{p}$, then $\mathbf{M}$ is not regular.

**(e)** The entries of a transition matrix $\mathbf{M}$ give the probabilities of a Markov process being in each of its states.

## 8.5 Hill Substitution: An Introduction to Coding Theory

*Prerequisite: Section 2.4, Inverses of Matrices*

In this section, we show how matrix inverses can be used in a simple manner to encode and decode textual information.

### Substitution Ciphers

The coding and decoding of secret messages has been important in times of warfare, of course, but it is also quite valuable in peacetime for keeping government and business secrets under tight security. Throughout history, many ingenious coding mechanisms have been proposed. One of the simplest is the **substitution cipher**, in which an array of symbols is used to assign each character of a given text (**plaintext**) to a corresponding character in coded text (**ciphertext**). For example, consider the **cipher array** in Fig. 8.15. A message can be encoded by replacing every instance of the $k$th letter of the alphabet with the $k$th character in the cipher array. For example, the message

LINEAR ALGEBRA IS EXCITING

is encoded as

FXUSRI RFTSWIR XG SNEXVXUT.

This type of substitution can be extended to other characters, such as punctuation symbols and blanks.

| | | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| R | W | E | H | S | K | T | Q | X | B | Z | F | P | U | M | C | Y | I | G | V | O | D | A | N | L | J |
| [1] | [2] | [3] | [4] | [5] | [6] | [7] | [8] | [9] | [10] | [11] | [12] | [13] | [14] | [15] | [16] | [17] | [18] | [19] | [20] | [21] | [22] | [23] | [24] | [25] | [26] |
| ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ |
| A | B | C | D | E | F | G | H | I | J | K | L | M | N | O | P | Q | R | S | T | U | V | W | X | Y | Z |

**FIGURE 8.15** A cipher array

Messages can be decoded by reversing the process. In fact, we can create an "inverse" array, or **decipher array**, as in Fig. 8.16, to restore the symbols of FXUSRI RFTSWIR XG SNEXVXUT back to LINEAR ALGEBRA IS EXCITING.

| W | J | P | V | C | L | S | D | R | Z | F | Y | O | X | U | M | H | A | E | G | N | T | B | I | Q | K |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| [1] | [2] | [3] | [4] | [5] | [6] | [7] | [8] | [9] | [10] | [11] | [12] | [13] | [14] | [15] | [16] | [17] | [18] | [19] | [20] | [21] | [22] | [23] | [24] | [25] | [26] |
| ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ |
| A | B | C | D | E | F | G | H | I | J | K | L | M | N | O | P | Q | R | S | T | U | V | W | X | Y | Z |

**FIGURE 8.16** A decipher array

Cryptograms, a standard feature in newspapers and puzzle magazines, are substitution ciphers. However, these ciphers are relatively easy to "crack" because the relative frequencies (occurrences per length of text) of the letters of the English alphabet have been studied extensively.[5]

## Hill Substitution

We now illustrate a method that uses matrices to create codes that are harder to break. This technique is known as **Hill substitution** after the mathematician Lester Hill, who developed it between the world wars. To begin, we choose any nonsingular $n \times n$ matrix $\mathbf{A}$. (Usually $\mathbf{A}$ is chosen with integer entries.) We split the message into blocks of $n$ symbols each and replace each symbol with an integer value. To simplify things, we replace each letter by its position in the alphabet. The last block may have to be "padded" with random values to ensure that each block contains exactly $n$ integers. In effect, we are creating a set of $n$-vectors that we can label as $\mathbf{x}_1$, $\mathbf{x}_2$, and so on. We then multiply the matrix $\mathbf{A}$ by each of these vectors in turn to produce the following new set of $n$-vectors: $\mathbf{A}\mathbf{x}_1$, $\mathbf{A}\mathbf{x}_2$, and so on. When these vectors are concatenated together, they form the coded message. The matrix $\mathbf{A}$ used in the process is often called the **key matrix**, or **encoding matrix**.

### Example 1

Suppose we wish to encode the message LINEAR ALGEBRA IS EXCITING using the key matrix

$$\mathbf{A} = \begin{bmatrix} -7 & 5 & 3 \\ 3 & -2 & -2 \\ 3 & -2 & -1 \end{bmatrix}.$$

Since we are using a $3 \times 3$ matrix, we break the characters of the message into blocks of length 3 and replace each character by its position in the alphabet. This procedure gives

$$\underbrace{\begin{matrix} \text{L} \\ \text{I} \\ \text{N} \end{matrix}\begin{bmatrix} 12 \\ 9 \\ 14 \end{bmatrix}}_{\mathbf{x}_1}, \quad \underbrace{\begin{matrix} \text{E} \\ \text{A} \\ \text{R} \end{matrix}\begin{bmatrix} 5 \\ 1 \\ 18 \end{bmatrix}}_{\mathbf{x}_2}, \quad \underbrace{\begin{matrix} \text{A} \\ \text{L} \\ \text{G} \end{matrix}\begin{bmatrix} 1 \\ 12 \\ 7 \end{bmatrix}}_{\mathbf{x}_3}, \quad \underbrace{\begin{matrix} \text{E} \\ \text{B} \\ \text{R} \end{matrix}\begin{bmatrix} 5 \\ 2 \\ 18 \end{bmatrix}}_{\mathbf{x}_4}, \quad \underbrace{\begin{matrix} \text{A} \\ \text{I} \\ \text{S} \end{matrix}\begin{bmatrix} 1 \\ 9 \\ 19 \end{bmatrix}}_{\mathbf{x}_5}, \quad \underbrace{\begin{matrix} \text{E} \\ \text{X} \\ \text{C} \end{matrix}\begin{bmatrix} 5 \\ 24 \\ 3 \end{bmatrix}}_{\mathbf{x}_6}, \quad \underbrace{\begin{matrix} \text{I} \\ \text{T} \\ \text{I} \end{matrix}\begin{bmatrix} 9 \\ 20 \\ 9 \end{bmatrix}}_{\mathbf{x}_7}, \quad \underbrace{\begin{matrix} \text{N} \\ \text{G} \\ \text{-} \end{matrix}\begin{bmatrix} 14 \\ 7 \\ 27 \end{bmatrix}}_{\mathbf{x}_8},$$

where the last entry of the last vector was chosen outside the range from 1 to 26. Now, forming the products with $\mathbf{A}$, we have

$$\mathbf{A}\mathbf{x}_1 = \begin{bmatrix} -7 & 5 & 3 \\ 3 & -2 & -2 \\ 3 & -2 & -1 \end{bmatrix}\begin{bmatrix} 12 \\ 9 \\ 14 \end{bmatrix} = \begin{bmatrix} 3 \\ -10 \\ 4 \end{bmatrix},$$

$$\mathbf{A}\mathbf{x}_2 = \begin{bmatrix} -7 & 5 & 3 \\ 3 & -2 & -2 \\ 3 & -2 & -1 \end{bmatrix}\begin{bmatrix} 5 \\ 1 \\ 18 \end{bmatrix} = \begin{bmatrix} 24 \\ -23 \\ -5 \end{bmatrix}, \quad \text{and so on.}$$

The final encoded text is

$$\begin{matrix} 3 & -10 & 4 & 24 & -23 & -5 & 74 & -35 & -28 & 29 & -25 & -7 \\ 95 & -53 & -34 & 94 & -39 & -36 & 64 & -31 & -22 & 18 & -26 & 1. \end{matrix}$$

---

[5] The longer the enciphered text is, the easier it is to decode by comparing the number of times each letter appears. The actual frequency of the letters depends on the type of text, but the letters $E$, $T$, $A$, $O$, $I$, $N$, $S$, $H$, and $R$ typically appear most often (about 70% of the time), with $E$ usually the most common (about 12–13% of the time). Once a few letters have been deciphered, the rest of the text is usually easy to determine. Sample frequency tables can be found on p. 219 of *Cryptanalysis* by Gaines (published by Dover, 1989) and on p. 16 of *Cryptography: A Primer* by Konheim (published by Wiley, 1981).

The code produced by a Hill substitution is much harder to break than a simple substitution cipher, since the coding of a given letter depends not only on the way the text is broken into blocks, but also on the letters adjacent to it. (Nevertheless, there are techniques to decode Hill substitutions using high-speed computers.) However, a Hill substitution is easy to decode if you know the inverse of the key matrix. In Example 6 of Section 2.4, we noted that

$$\mathbf{A}^{-1} = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 2 & 5 \\ 0 & -1 & 1 \end{bmatrix}.$$

Breaking the encoded text back into 3-vectors and multiplying $\mathbf{A}^{-1}$ by each of these vectors in turn restores the original message. For example,

$$\mathbf{A}^{-1}(\mathbf{A}\mathbf{x}_1) = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 2 & 5 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -10 \\ 4 \end{bmatrix} = \begin{bmatrix} 12 \\ 9 \\ 14 \end{bmatrix} = \mathbf{x}_1,$$

which represents the first three letters LIN.

## New Vocabulary

cipher array
ciphertext
decipher array
encoding matrix
Hill substitution
key matrix
plaintext
substitution cipher

## Highlights

- Hill substitution is a coding technique in which the plaintext to be encoded is converted into numerical form, then split into equal-length blocks $\mathbf{x}_1, \mathbf{x}_2, \ldots$, etc. Each block is multiplied by the same (nonsingular) key matrix $\mathbf{A}$ to obtain new blocks $\mathbf{A}\mathbf{x}_1, \mathbf{A}\mathbf{x}_2, \ldots$, etc. whose entries form the ciphertext.
- In Hill substitution, the ciphertext is determined not only by the selection of a key matrix and the choice of a block size, but also by the particular letters that happen to be adjacent to each letter in the given plaintext.
- The decoding process for a ciphertext of equal-length blocks $\mathbf{x}_1, \mathbf{x}_2, \ldots$, etc. is similar to the encoding process except that the key matrix $\mathbf{A}$ is replaced by its inverse $\mathbf{A}^{-1}$ (to produce $\mathbf{A}^{-1}\mathbf{x}_1, \mathbf{A}^{-1}\mathbf{x}_2, \ldots$, etc.).

## Exercises for Section 8.5

1. Encode each message with the given key matrix.

★ (a) PROOF BY INDUCTION using the matrix $\begin{bmatrix} 3 & -4 \\ 5 & -7 \end{bmatrix}$

(b) CONTACT HEADQUARTERS using the matrix $\begin{bmatrix} 4 & 1 & 5 \\ 7 & 2 & 9 \\ 6 & 2 & 7 \end{bmatrix}$

2. Each of the following coded messages was produced with the key matrix shown. In each case, find the inverse of the key matrix, and use it to decode the message.

★ (a)

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 162 | 108 | 23 | 303 | 206 | 33 | 276 | 186 | 33 | 170 | 116 | 21 |
| 281 | 191 | 36 | 576 | 395 | 67 | 430 | 292 | 51 | 340 | 232 | 45 |

with key matrix $\begin{bmatrix} -10 & 19 & 16 \\ -7 & 13 & 11 \\ -1 & 2 & 2 \end{bmatrix}$

**(b)**

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 69 | 44 | −28 | −43 | 104 | 53 | −38 | −25 |
| 71 | 38 | −3 | −7 | 58 | 32 | −11 | −14 |

$$\text{with key matrix} \quad \begin{bmatrix} 1 & 2 & 5 & 1 \\ 0 & 1 & 3 & 1 \\ -2 & 0 & 0 & -1 \\ 0 & 0 & -1 & -2 \end{bmatrix}$$

**(c)**

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 188 | 408 | 348 | 345 | 115 | 244 | 224 | 235 | 235 | 545 | 432 | 403 | 227 | 521 | 417 | 392 |
| 177 | 403 | 334 | 325 | 202 | 489 | 375 | 339 | 75 | 167 | 140 | 137 | 323 | 769 | 599 | 550 |

$$\text{with key matrix} \quad \begin{bmatrix} 3 & 4 & 5 & 3 \\ 4 & 11 & 12 & 7 \\ 6 & 7 & 9 & 6 \\ 8 & 5 & 8 & 6 \end{bmatrix}$$

**(d)**

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 19 | 43 | 33 | 76 | 64 | 37 | 27 | 36 | 66 | 71 | 71 | 47 | 51 | 119 | 89 |
| 135 | 65 | 87 | 152 | 169 | 72 | 54 | 37 | 105 | 57 | 77 | 24 | 52 | 74 | 106 |

$$\text{with key matrix} \quad \begin{bmatrix} -2 & 3 & 3 & 2 & 1 \\ 2 & -2 & 2 & 2 & -1 \\ -1 & 3 & 1 & 1 & 1 \\ 2 & -1 & 4 & 2 & 1 \\ -3 & 8 & 1 & 2 & 2 \end{bmatrix}$$

(Note: The answer describes a different kind of “Hill substitution.”)

★ **3.** True or False:

**(a)** Text encoded with a Hill substitution is more difficult to decipher than text encoded with a substitution cipher.

**(b)** The encoding matrix for a Hill substitution should not be singular.

**(c)** To encode a message using Hill substitution that is $n$ characters long, an $n \times n$ matrix is always used.

## 8.6 Linear Recurrence Relations and the Fibonacci Sequence

*Prerequisite: Section 3.4, Eigenvalues and Diagonalization*

In this section, we introduce linear recurrence relations for sequences and illustrate how a formula for any term in the sequence can be found using eigenvalues and eigenvectors. We also examine a linear recurrence relation for the Fibonacci sequence to obtain a formula for the $n$th term of that sequence.

### Linear Recurrence Relations

In what follows, suppose that $a_1, a_2, a_3, \ldots, a_n, \ldots$ is a sequence of real numbers, where the first few values of the sequence are known, and the general term $a_n$ is defined as a particular linear combination of previous values. The equation giving the linear combination for $a_n$ is called a **linear recurrence relation** or a **linear difference equation**.

#### Example 1

Consider the sequence having $a_1 = 2$, $a_2 = 5$, and with $a_n$ defined (for integers $n > 2$) by the linear recurrence relation $a_n = 3a_{n-1} - 2a_{n-2}$. It is easy to calculate by hand that

$$a_3 = 3a_2 - 2a_1 = 3(5) - 2(2) = 11,$$
$$a_4 = 3a_3 - 2a_2 = 3(11) - 2(5) = 23,$$
$$a_5 = 3a_4 - 2a_3 = 3(23) - 2(11) = 47,$$

etc. Hence, the sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ has the values $2, 5, 11, 23, 47, \ldots$

The particular equation $a_n = 3a_{n-1} - 2a_{n-2}$ given in Example 1 is considered a **linear recurrence relation of order** 2 since the linear combination for $a_n$ involves the 2 previous terms in the sequence. In general, a **linear recurrence relation of order** $k$ is one for which the linear combination for $a_n$ involves the previous $k$ terms of the sequence. For instance,

$$a_n = 4a_{n-1} - a_{n-2} + 2a_{n-3}$$

is a linear recurrence relation of order 3, while

$$a_n = 4a_{n-1} - a_{n-2} + 2a_{n-4} \quad (= 4a_{n-1} - a_{n-2} + 0a_{n-3} + 2a_{n-4})$$

is a linear recurrence relation of order 4.

## Using Diagonalization to Solve a Linear Recurrence Relation of Order 2

Suppose we have a linear recurrence relation and want to obtain the values for its corresponding sequence. Rather than tediously working out each member of the sequence in turn by hand, we can use eigenvalues and diagonalization to get a general formula (that is, a solution) for the $n$th term of the sequence. This process is illustrated in the next example.

### Example 2

Consider the linear recurrence relation $a_n = 3a_{n-1} - 2a_{n-2}$ of order 2 from Example 1, with $a_1 = 2$ and $a_2 = 5$. Increasing all of the subscripts by 2, we can restate the linear recurrence relation as $a_{n+2} = 3a_{n+1} - 2a_n$, and in matrix form as

$$\begin{bmatrix} a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}.$$

Notice that this matrix equation gives precisely the same information as the original linear recurrence relation. Let

$$\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}. \quad \text{Then,} \quad \begin{bmatrix} a_{n+2} \\ a_{n+1} \end{bmatrix} = \mathbf{A} \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}.$$

The characteristic polynomial for $\mathbf{A}$ is $|x\mathbf{I}_2 - \mathbf{A}| = \begin{vmatrix} x-3 & 2 \\ -1 & x-0 \end{vmatrix} = x^2 - 3x + 2 = (x-1)(x-2)$. Therefore, the eigenvalues of $\mathbf{A}$ are $\lambda_1 = 1$ and $\lambda_2 = 2$. We next compute the eigenspaces for these two eigenvalues.

Now, $E_{\lambda_1}$ is the solution set of $(1\mathbf{I}_2 - \mathbf{A})\mathbf{X} = \mathbf{0}$. We solve this by row reducing

$$\left[\begin{array}{cc|c} -2 & 2 & 0 \\ -1 & 1 & 0 \end{array}\right] \text{ to } \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Hence, a fundamental eigenvector for $\lambda_1$ is $[1, 1]$. Therefore, $E_{\lambda_1} = E_1 = \{b[1, 1]\}$.

Similarly, $E_{\lambda_2}$ is the solution set of $(2\mathbf{I}_2 - \mathbf{A})\mathbf{X} = \mathbf{0}$. We solve this by row reducing

$$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ -1 & 2 & 0 \end{array}\right] \text{ to } \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Hence, a fundamental eigenvector for $\lambda_2$ is $[2, 1]$. Therefore, $E_{\lambda_2} = E_2 = \{b[2, 1]\}$.

Since there are two fundamental eigenvectors for $\mathbf{A}$ here, we let $\mathbf{P}$ represent the matrix whose columns are these fundamental eigenvectors. That is,

$$\mathbf{P} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \text{ whose inverse is } \mathbf{P}^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}.$$

Then,

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \mathbf{D},$$

where $\mathbf{D}$ is a diagonal matrix whose main diagonal entries are the eigenvalues of $\mathbf{A}$. Therefore, analogous to Example 12 of Section 3.4, we have

$$\begin{aligned} \mathbf{A}^n &= \mathbf{P}\mathbf{D}^n\mathbf{P}^{-1} \\ &= \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1^n & 0 \\ 0 & 2^n \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} 1 & 2^{n+1} \\ 1 & 2^n \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 2^{n+1}-1 & -2^{n+1}+2 \\ 2^n-1 & -2^n+2 \end{bmatrix}.$$

But then,

$$\begin{bmatrix} a_{n+2} \\ a_{n+1} \end{bmatrix} = \mathbf{A} \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix} = \mathbf{A}^2 \begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix} = \cdots = \mathbf{A}^n \begin{bmatrix} a_2 \\ a_1 \end{bmatrix}$$
$$= \begin{bmatrix} 2^{n+1}-1 & -2^{n+1}+2 \\ 2^n-1 & -2^n+2 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} 5(2^{n+1})-5-2^{n+2}+4 \\ 5(2^n)-5-2^{n+1}+4 \end{bmatrix} = \begin{bmatrix} 5(2^{n+1})-2^{n+2}-1 \\ 5(2^n)-2^{n+1}-1 \end{bmatrix}.$$

The last entry[6] of the final matrix gives us a formula for the $(n+1)$st term of the sequence:

$$a_{n+1} = 5(2^n) - 2^{n+1} - 1 = 5(2^n) - 2(2^n) - 1 = 3(2^n) - 1,$$

or, equivalently,

$$a_n = 3(2^{n-1}) - 1.$$

We verify this formula is valid for $a_1$ through $a_5$ here:

$$a_1 = 3(2^0) - 1 = 2,$$
$$a_2 = 3(2^1) - 1 = 5,$$
$$a_3 = 3(2^2) - 1 = 11,$$
$$a_4 = 3(2^3) - 1 = 23, \quad \text{and}$$
$$a_5 = 3(2^4) - 1 = 47.$$

## The Fibonacci Sequence

The Fibonacci sequence is a particularly famous sequence $F_1, F_2, F_3, \ldots, F_n, \ldots$ of integers, where $F_1 = F_2 = 1$, and with its $n$th term (for integers $n > 2$) defined by the linear recurrence relation $F_n = F_{n-1} + F_{n-2}$ of order 2. That is, each value from $F_3$ onward is found by taking the sum of the previous two values. The first few values of the Fibonacci sequence are:

$$F_1 = 1 \qquad F_6 = F_4 + F_5 = 3 + 5 = 8$$
$$F_2 = 1 \qquad F_7 = F_5 + F_6 = 5 + 8 = 13$$
$$F_3 = F_1 + F_2 = 1 + 1 = 2 \qquad F_8 = F_6 + F_7 = 8 + 13 = 21$$
$$F_4 = F_2 + F_3 = 1 + 2 = 3 \qquad F_9 = F_7 + F_8 = 13 + 21 = 34$$
$$F_5 = F_3 + F_4 = 2 + 3 = 5 \qquad F_{10} = F_8 + F_9 = 21 + 34 = 55$$

We can apply a method similar to that used in Example 2 to get a general formula for the $n$th term of the Fibonacci sequence. Increasing all of the subscripts by 2, we can restate the linear recurrence relation as $F_{n+2} = F_{n+1} + F_n$, which is equivalent to

$$\begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}.$$

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}. \quad \text{Then,} \quad \begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix} = \mathbf{A} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}.$$

[6] It was really only necessary earlier to compute the final row of $\mathbf{A}^n$, since only the last entry of the final product $\mathbf{A}^n \begin{bmatrix} a_2 \\ a_1 \end{bmatrix}$ is needed. However, computing the entire final product can be a helpful check on your work, since its entries give related answers for $a_{n+2}$ and $a_{n+1}$.

The characteristic polynomial for $\mathbf{A}$ is $|x\mathbf{I}_n - \mathbf{A}| = \begin{vmatrix} x-1 & -1 \\ -1 & x-0 \end{vmatrix} = x^2 - x - 1$. The eigenvalues of $\mathbf{A}$ are easily seen to be $\lambda_1 = \frac{1+\sqrt{5}}{2}$ and $\lambda_2 = \frac{1-\sqrt{5}}{2}$. We next compute the eigenspaces for these two eigenvalues.

Now, $E_{\lambda_1}$ is the solution set of $\left(\left(\frac{1+\sqrt{5}}{2}\right)\mathbf{I}_2 - \mathbf{A}\right)\mathbf{X} = \mathbf{0}$. The corresponding augmented matrix is

$$\left[\begin{array}{cc|c} \left(\frac{1+\sqrt{5}}{2}\right)-1 & -1 & 0 \\ -1 & \frac{1+\sqrt{5}}{2} & 0 \end{array}\right] = \left[\begin{array}{cc|c} \frac{-1+\sqrt{5}}{2} & -1 & 0 \\ -1 & \frac{1+\sqrt{5}}{2} & 0 \end{array}\right].$$

In Exercise 2(a), you are asked to verify that this row reduces to

$$\left[\begin{array}{cc|c} 1 & -\left(\frac{1+\sqrt{5}}{2}\right) & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Hence, a fundamental eigenvector for $\lambda_1$ is $\left[\frac{1+\sqrt{5}}{2}, 1\right]$. Therefore, $E_{\lambda_1} = E_{\frac{1+\sqrt{5}}{2}} = \left\{b\left[\left(\frac{1+\sqrt{5}}{2}\right),\ 1\right]\right\}$, which can also be expressed as $\left\{b\left[1+\sqrt{5},\ 2\right]\right\}$.

Similarly, $E_{\lambda_2}$ is the solution set of $\left(\left(\frac{1-\sqrt{5}}{2}\right)\mathbf{I}_2 - \mathbf{A}\right)\mathbf{X} = \mathbf{0}$. The corresponding augmented matrix is

$$\left[\begin{array}{cc|c} \left(\frac{1-\sqrt{5}}{2}\right)-1 & -1 & 0 \\ -1 & \frac{1-\sqrt{5}}{2} & 0 \end{array}\right] = \left[\begin{array}{cc|c} \frac{-1-\sqrt{5}}{2} & -1 & 0 \\ -1 & \frac{1-\sqrt{5}}{2} & 0 \end{array}\right].$$

In Exercise 2(b), you are asked to verify that this row reduces to

$$\left[\begin{array}{cc|c} 1 & -\left(\frac{1-\sqrt{5}}{2}\right) & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Hence, a fundamental eigenvector for $\lambda_2$ is $\left[\frac{1-\sqrt{5}}{2}, 1\right]$. Therefore, $E_{\lambda_2} = E_{\frac{1-\sqrt{5}}{2}} = \left\{b\left[\left(\frac{1-\sqrt{5}}{2}\right),\ 1\right]\right\}$, which can also be expressed as $\left\{b\left[1-\sqrt{5}, 2\right]\right\}$.

Since there are two fundamental eigenvectors for $\mathbf{A}$ here, we let $\mathbf{P}$ represent the matrix whose columns are these fundamental eigenvectors. That is,

$$\mathbf{P} = \begin{bmatrix} 1+\sqrt{5} & 1-\sqrt{5} \\ 2 & 2 \end{bmatrix}.$$

It can be readily shown (see Exercise 2(c)) that

$$\mathbf{P}^{-1} = \frac{1}{4\sqrt{5}}\begin{bmatrix} 2 & -1+\sqrt{5} \\ -2 & 1+\sqrt{5} \end{bmatrix}.$$

Then,

$$\begin{aligned} \mathbf{P}^{-1}\mathbf{A}\mathbf{P} &= \frac{1}{4\sqrt{5}}\begin{bmatrix} 2 & -1+\sqrt{5} \\ -2 & 1+\sqrt{5} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1+\sqrt{5} & 1-\sqrt{5} \\ 2 & 2 \end{bmatrix} \\ &= \frac{1}{4\sqrt{5}}\begin{bmatrix} 1+\sqrt{5} & 2 \\ -1+\sqrt{5} & -2 \end{bmatrix}\begin{bmatrix} 1+\sqrt{5} & 1-\sqrt{5} \\ 2 & 2 \end{bmatrix} \end{aligned}$$

$$= \frac{\sqrt{5}}{20}\begin{bmatrix} 10+2\sqrt{5} & 0 \\ 0 & -10+2\sqrt{5} \end{bmatrix} = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix} = \mathbf{D},$$

a diagonal matrix with the eigenvalues of **A** along its main diagonal. Therefore,

$$\begin{aligned} \mathbf{A}^n &= \mathbf{P}\mathbf{D}^n\mathbf{P}^{-1} \\ &= \begin{bmatrix} 1+\sqrt{5} & 1-\sqrt{5} \\ 2 & 2 \end{bmatrix}\begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{bmatrix}\frac{1}{4\sqrt{5}}\begin{bmatrix} 2 & -1+\sqrt{5} \\ -2 & 1+\sqrt{5} \end{bmatrix}. \end{aligned}$$

In Exercise 2(d), you are asked to verify that this product equals

$$\frac{1}{\sqrt{5}}\begin{bmatrix} \frac{\left(1+\sqrt{5}\right)^{n+1}}{2^{n+1}} - \frac{\left(1-\sqrt{5}\right)^{n+1}}{2^{n+1}} & \frac{\left(1+\sqrt{5}\right)^{n}}{2^{n}} - \frac{\left(1-\sqrt{5}\right)^{n}}{2^{n}} \\ \frac{\left(1+\sqrt{5}\right)^{n}}{2^{n}} - \frac{\left(1-\sqrt{5}\right)^{n}}{2^{n}} & \frac{\left(1+\sqrt{5}\right)^{n-1}}{2^{n-1}} - \frac{\left(1-\sqrt{5}\right)^{n-1}}{2^{n-1}} \end{bmatrix}.$$

Then, using the same reasoning as in Example 2, we have

$$\begin{aligned} \begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix} &= \mathbf{A}^n\begin{bmatrix} F_2 \\ F_1 \end{bmatrix} \\ &= \frac{1}{\sqrt{5}}\begin{bmatrix} \frac{\left(1+\sqrt{5}\right)^{n+1}}{2^{n+1}} - \frac{\left(1-\sqrt{5}\right)^{n+1}}{2^{n+1}} & \frac{\left(1+\sqrt{5}\right)^{n}}{2^{n}} - \frac{\left(1-\sqrt{5}\right)^{n}}{2^{n}} \\ \frac{\left(1+\sqrt{5}\right)^{n}}{2^{n}} - \frac{\left(1-\sqrt{5}\right)^{n}}{2^{n}} & \frac{\left(1+\sqrt{5}\right)^{n-1}}{2^{n-1}} - \frac{\left(1-\sqrt{5}\right)^{n-1}}{2^{n-1}} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \frac{1}{\sqrt{5}}\begin{bmatrix} \frac{\left(1+\sqrt{5}\right)^{n+1}-\left(1-\sqrt{5}\right)^{n+1}}{2^{n+1}} + \frac{\left(1+\sqrt{5}\right)^{n}-\left(1-\sqrt{5}\right)^{n}}{2^{n}} \\ \frac{\left(1+\sqrt{5}\right)^{n}-\left(1-\sqrt{5}\right)^{n}}{2^{n}} + \frac{\left(1+\sqrt{5}\right)^{n-1}-\left(1-\sqrt{5}\right)^{n-1}}{2^{n-1}} \end{bmatrix}. \end{aligned}$$

The last entry of this matrix gives a formula for $F_{n+1}$. Adjusting the subscripts down by 1, we obtain the following formula:

$$\begin{aligned} F_n &= \frac{1}{\sqrt{5}}\left(\frac{\left(1+\sqrt{5}\right)^{n-1}-\left(1-\sqrt{5}\right)^{n-1}}{2^{n-1}} + \frac{\left(1+\sqrt{5}\right)^{n-2}-\left(1-\sqrt{5}\right)^{n-2}}{2^{n-2}}\right) \\ &= \frac{1}{\sqrt{5}}\left(\frac{1}{2^{n-1}}\right)\left(\left(3+\sqrt{5}\right)\left(1+\sqrt{5}\right)^{n-2} - \left(3-\sqrt{5}\right)\left(1-\sqrt{5}\right)^{n-2}\right). \end{aligned}$$

In Exercise 3(a), you are asked to prove the identities $\left(1+\sqrt{5}\right)^2 = 2(3+\sqrt{5})$ and $\left(1-\sqrt{5}\right)^2 = 2(3-\sqrt{5})$. Using these, we obtain

$$\begin{aligned} F_n &= \frac{1}{\sqrt{5}}\left(\frac{1}{2^n}\right)\left(\left(1+\sqrt{5}\right)^2\left(1+\sqrt{5}\right)^{n-2} - \left(1-\sqrt{5}\right)^2\left(1-\sqrt{5}\right)^{n-2}\right) \\ &= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right). \end{aligned}$$

The last expression for the $n$th term of the Fibonacci sequence is commonly known as **Binet's Formula**. We can verify that this formula holds for $F_1$ through $F_3$ as follows:

$$F_1 = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1\right) = \frac{1}{\sqrt{5}}\left(\frac{2\sqrt{5}}{2}\right) = 1,$$

$$F_2 = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2\right)$$
$$= \frac{1}{\sqrt{5}}\left(\left(\frac{6+2\sqrt{5}}{4}\right) - \left(\frac{6-2\sqrt{5}}{4}\right)\right) = \frac{1}{\sqrt{5}}\left(\frac{4\sqrt{5}}{4}\right) = 1, \quad \text{and}$$
$$F_3 = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^3 - \left(\frac{1-\sqrt{5}}{2}\right)^3\right)$$
$$= \frac{1}{\sqrt{5}}\left(\left(\frac{6+2\sqrt{5}}{4}\right)\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{6-2\sqrt{5}}{4}\right)\left(\frac{1-\sqrt{5}}{2}\right)\right)$$
$$= \frac{1}{\sqrt{5}}\left(\left(\frac{11+8\sqrt{5}}{8}\right) - \left(\frac{11-8\sqrt{5}}{8}\right)\right) = \frac{1}{\sqrt{5}}\left(\frac{16\sqrt{5}}{8}\right) = 2.$$

The symbol $\phi$ is frequently used to represent the value $\frac{1+\sqrt{5}}{2}$, which is commonly referred to as the **golden ratio**. It is easy to show that $\phi^2 = \phi + 1$. (See Exercise 3(b).) Interestingly, the value $\frac{1-\sqrt{5}}{2}$ is actually the negative reciprocal of the golden ratio (See Exercise 3(c).) This implies that $\frac{1-\sqrt{5}}{2} = -\frac{1}{\phi}$, from which it follows that $\phi - 1 = \frac{1}{\phi}$. (See Exercise 3(d).) Therefore, we can express Binet's Formula more succinctly as follows:

$$F_n = \frac{1}{\sqrt{5}}\left(\phi^n - \left(-\frac{1}{\phi}\right)^n\right) = \frac{1}{\sqrt{5}}\left(\phi^n + \frac{(-1)^{n+1}}{\phi^n}\right).$$

## A Linear Recurrence Relation of Order 3

Linear recurrence relations of order 3 and higher can be approached in a manner similar to that of Example 2.

### Example 3

Consider the linear recurrence relation $a_n = 6a_{n-1} - 5a_{n-2} - 12a_{n-3}$, with $a_1 = 2$, $a_2 = -3$, and $a_3 = 1$. Increasing all of the subscripts by 3, we restate the relation as $a_{n+3} = 6a_{n+2} - 5a_{n+1} - 12a_n$, and in matrix form as

$$\begin{bmatrix} a_{n+3} \\ a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 6 & -5 & -12 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n+2} \\ a_{n+1} \\ a_n \end{bmatrix}.$$

Notice that this matrix equation gives precisely the same information as the original linear recurrence relation. Let

$$\mathbf{A} = \begin{bmatrix} 6 & -5 & -12 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \quad \text{Then,} \quad \begin{bmatrix} a_{n+3} \\ a_{n+2} \\ a_{n+1} \end{bmatrix} = \mathbf{A} \begin{bmatrix} a_{n+2} \\ a_{n+1} \\ a_n \end{bmatrix}.$$

The characteristic polynomial for $\mathbf{A}$ is $|x\mathbf{I}_3 - \mathbf{A}| = \begin{vmatrix} x-6 & 5 & 12 \\ -1 & x-0 & 0 \\ 0 & -1 & x-0 \end{vmatrix} = x^3 - 6x^2 + 5x + 12 = (x+1)(x-3)(x-4)$. Therefore, the eigenvalues of $\mathbf{A}$ are $\lambda_1 = -1$, $\lambda_2 = 3$, and $\lambda_3 = 4$.

Now, $E_{\lambda_1}$ is the solution set of $(-1\mathbf{I}_2 - \mathbf{A})\mathbf{X} = \mathbf{0}$. We solve this by row reducing

$$\left[\begin{array}{rrr|r} -7 & 5 & 12 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right] \quad \text{to} \quad \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Hence, a fundamental eigenvector for $\lambda_1$ is $[1, -1, 1]$. Therefore, $E_{\lambda_1} = E_{-1} = \{c[1, -1, 1]\}$.

Similarly, $E_{\lambda_2}$ is the solution set of $(3\mathbf{I}_2 - \mathbf{A})\mathbf{X} = \mathbf{0}$. We solve this by row reducing

$$\left[\begin{array}{rrr|r} -3 & 5 & 12 & 0 \\ -1 & 3 & 0 & 0 \\ 0 & -1 & 3 & 0 \end{array}\right] \text{ to } \left[\begin{array}{rrr|r} 1 & 0 & -9 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Hence, a fundamental eigenvector for $\lambda_2$ is $[9, 3, 1]$. Therefore, $E_{\lambda_2} = E_3 = \{c[9, 3, 1]\}$.

Similarly, $E_{\lambda_3}$ is the solution set of $(4\mathbf{I}_2 - \mathbf{A})\mathbf{X} = \mathbf{0}$. We solve this by row reducing

$$\left[\begin{array}{rrr|r} -2 & 5 & 12 & 0 \\ -1 & 4 & 0 & 0 \\ 0 & -1 & 4 & 0 \end{array}\right] \text{ to } \left[\begin{array}{rrr|r} 1 & 0 & -16 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Hence, a fundamental eigenvector for $\lambda_3$ is $[16, 4, 1]$. Therefore, $E_{\lambda_3} = E_4 = \{c[16, 4, 1]\}$.

Since there are three fundamental eigenvectors for $\mathbf{A}$ here, we let $\mathbf{P}$ represent the matrix having these as columns. That is,

$$\mathbf{P} = \begin{bmatrix} 1 & 9 & 16 \\ -1 & 3 & 4 \\ 1 & 1 & 1 \end{bmatrix}, \text{ whose inverse is } \mathbf{P}^{-1} = \left(\frac{1}{20}\right) \begin{bmatrix} 1 & -7 & 12 \\ -5 & 15 & 20 \\ 4 & -8 & -12 \end{bmatrix}.$$

Then,

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \mathbf{D},$$

where $\mathbf{D}$ is a diagonal matrix whose main diagonal entries are the eigenvalues of $\mathbf{A}$. Then,

$$\begin{aligned} \mathbf{A}^n &= \mathbf{P}\mathbf{D}^n\mathbf{P}^{-1} \\ &= \begin{bmatrix} 1 & 9 & 16 \\ -1 & 3 & 4 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} (-1)^n & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 4^n \end{bmatrix} \left(\frac{1}{20}\right) \begin{bmatrix} 1 & -7 & 12 \\ -5 & 15 & 20 \\ 4 & -8 & -12 \end{bmatrix}. \end{aligned}$$

It can be shown (see Exercise 4(a)) that this reduces to

$$\left(\frac{1}{20}\right) \begin{bmatrix} (-1)^n - 5(3^{n+2}) + 4(4^{n+2}) & -7(-1)^n + 15(3^{n+2}) - 8(4^{n+2}) & 12(-1)^n + 20(3^{n+2}) - 12(4^{n+2}) \\ (-1)^{n+1} - 5(3^{n+1}) + 4(4^{n+1}) & -7(-1)^{n+1} + 15(3^{n+1}) - 8(4^{n+1}) & 12(-1)^{n+1} + 20(3^{n+1}) - 12(4^{n+1}) \\ (-1)^n - 5(3^n) + 4(4^n) & -7(-1)^n + 15(3^n) - 8(4^n) & 12(-1)^n + 20(3^n) - 12(4^n) \end{bmatrix}.$$

But then,

$$\begin{bmatrix} a_{n+3} \\ a_{n+2} \\ a_{n+1} \end{bmatrix} = \mathbf{A} \begin{bmatrix} a_{n+2} \\ a_{n+1} \\ a_n \end{bmatrix} = \mathbf{A}^2 \begin{bmatrix} a_{n+1} \\ a_n \\ a_{n-1} \end{bmatrix} = \cdots = \mathbf{A}^n \begin{bmatrix} a_3 \\ a_2 \\ a_1 \end{bmatrix} = \mathbf{A}^n \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}.$$

It can be shown (see Exercise 4(b)) that the last product reduces to

$$\left(\frac{1}{20}\right) \begin{bmatrix} 46(-1)^n - 10(3^{n+2}) + 4(4^{n+2}) \\ 46(-1)^{n+1} - 10(3^{n+1}) + 4(4^{n+1}) \\ 46(-1)^n - 10(3^n) + 4(4^n) \end{bmatrix}.$$

The last entry of the final matrix gives us a formula for the $(n+1)$st term of the sequence:

$$a_{n+1} = \frac{46(-1)^n - 10(3^n) + 4(4^n)}{20},$$

or, equivalently,

$$a_n = \frac{46(-1)^{n-1} - 10(3^{n-1}) + 4^n}{20}.$$

We verify this formula is valid for $a_1$ through $a_5$ here:

$$a_1 = \frac{46(-1)^0 - 10(3^0) + 4^1}{20} = \frac{46 - 10 + 4}{20} = \frac{40}{20} = 2,$$

$$a_2 = \frac{46(-1)^1 - 10(3^1) + 4^2}{20} = \frac{-46 - 30 + 16}{20} = -\frac{60}{20} = -3,$$
$$a_3 = \frac{46(-1)^2 - 10(3^2) + 4^3}{20} = \frac{46 - 90 + 64}{20} = \frac{20}{20} = 1,$$
$$a_4 = \frac{46(-1)^3 - 10(3^3) + 4^4}{20} = \frac{-46 - 270 + 256}{20} = -\frac{60}{20} = -3,$$
while $6a_3 - 5a_2 - 12a_1 = 6(1) - 5(-3) - 12(2) = 6 + 15 - 24 = -3$, and
$$a_5 = \frac{46(-1)^4 - 10(3^4) + 4^5}{20} = \frac{46 - 810 + 1024}{20} = \frac{260}{20} = 13,$$
while $6a_4 - 5a_3 - 12a_2 = 6(-3) - 5(1) - 12(-3) = -18 - 5 + 36 = 13$.

The examples given in this section have only included cases where the $n \times n$ matrix $\mathbf{A}$ (representing the linear combination for the recurrence relation) has $n$ distinct (real) eigenvalues, and is therefore diagonalizable. Exercises 8 and 9 below provide some shortcuts for finding the characteristic polynomial, eigenvalues and fundamental eigenvectors for linear recurrence relations.

If the characteristic polynomial for $\mathbf{A}$ has complex roots, then the matrix could be diagonalizable using complex matrices, complex eigenvalues, and complex eigenvectors, which are covered in Chapter 7. Cases involving fewer than $n$ eigenvalues require more sophisticated techniques involving the Jordan Canonical Form of a matrix, which is introduced in one of the web sections for this textbook. Additional methods are given in many discrete mathematics textbooks for such cases.

Solving linear recurrence relations is analogous to solving linear homogeneous differential equations, a topic covered in Section 8.9. (This section relies on material from Chapter 4, and some terminology from Chapter 5.) The solution of certain linear homogeneous differential equations also requires the use of Jordan Canonical Form.

## New Vocabulary

Binet's Formula
Fibonacci sequence
golden ratio
linear difference equation
linear recurrence relation
linear recurrence relation of order $k$

## Highlights

- For a sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ of real numbers in which the general term $a_n$ is defined as a particular linear combination of previous values, this linear combination is a **linear recurrence relation**.
- A **linear recurrence relation of order** $k$ is a linear recurrence relation in which the particular linear combination for $a_n$ involves a linear combination of the previous $k$ terms in the sequence.
- A linear recurrence relation of order 2 for the sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$, in which $a_n$ is defined as a linear combination of $a_{n-1}$ and $a_{n-2}$, can be expressed in the form $\begin{bmatrix} a_{n+2} \\ a_{n+1} \end{bmatrix} = \mathbf{A}\begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$, where the entries of $\mathbf{A}$ are determined from the linear combination in this recurrence relation. A general formula for $a_n$ can be derived by calculating the eigenvalues of $\mathbf{A}$, creating a matrix $\mathbf{P}$ having distinct fundamental eigenvectors of $\mathbf{A}$ as columns, calculating the diagonalization $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$, and using $\mathbf{A}^n = \mathbf{PD}^n\mathbf{P}^{-1}$ to solve $\begin{bmatrix} a_{n+2} \\ a_{n+1} \end{bmatrix} = \mathbf{A}^n \begin{bmatrix} a_2 \\ a_1 \end{bmatrix}$.
- The Fibonacci sequence $F_1, F_2, F_3, \ldots, F_n, \ldots$ is defined by setting $F_1 = F_2 = 1$, with general term given by the linear recurrence relation $F_n = F_{n-1} + F_{n-2}$.
- The $n$th term of the Fibonacci Sequence is given by Binet's Formula: $F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right) = \frac{1}{\sqrt{5}}\left(\phi^n + \frac{(-1)^{n+1}}{\phi^n}\right)$.

## Exercises for Section 8.6

1. Use the method of Example 2 to find a formula for the general term $a_n$ for each of the following sequences with given initial values and given linear recurrence relation, and verify that the formula is correct for the first five values of the sequence.

★ **(a)** $a_1 = 3$, $a_2 = 7$, and for $n > 2$, $a_n = 5a_{n-1} - 6a_{n-2}$

**(b)** $a_1 = 6$, $a_2 = -3$, and for $n > 2$, $a_n = 9a_{n-1} - 20a_{n-2}$

**(c)** $a_1 = 4$, $a_2 = 3$, and for $n > 2$, $a_n = -3a_{n-1} + 10a_{n-2}$

★ **(d)** $a_1 = 2$, $a_2 = -1$, $a_3 = 5$, and for $n > 3$, $a_n = -6a_{n-1} - 5a_{n-2} + 12a_{n-3}$

**(e)** $a_1 = 5$, $a_2 = 1$, $a_3 = 4$, and for $n > 3$, $a_n = a_{n-1} + 9a_{n-2} - 9a_{n-3}$

**2.** Verify the following assertions made in this section. (Be sure to use *exact* values throughout rather than decimal approximations.)

**(a)** Show that

$$\left[\begin{array}{cc|c} \frac{-1+\sqrt{5}}{2} & -1 & 0 \\ -1 & \frac{1+\sqrt{5}}{2} & 0 \end{array}\right] \text{ row reduces to } \left[\begin{array}{cc|c} 1 & -\left(\frac{1+\sqrt{5}}{2}\right) & 0 \\ 0 & 0 & 0 \end{array}\right].$$

**(b)** Show that

$$\left[\begin{array}{cc|c} \frac{-1-\sqrt{5}}{2} & -1 & 0 \\ -1 & \frac{1-\sqrt{5}}{2} & 0 \end{array}\right] \text{ row reduces to } \left[\begin{array}{cc|c} 1 & -\left(\frac{1-\sqrt{5}}{2}\right) & 0 \\ 0 & 0 & 0 \end{array}\right].$$

**(c)** Show that the inverse of

$$\mathbf{P} = \begin{bmatrix} 1+\sqrt{5} & 1-\sqrt{5} \\ 2 & 2 \end{bmatrix} \text{ is given by } \frac{1}{4\sqrt{5}}\begin{bmatrix} 2 & -1+\sqrt{5} \\ -2 & 1+\sqrt{5} \end{bmatrix}.$$

**(d)** Show that

$$\begin{bmatrix} 1+\sqrt{5} & 1-\sqrt{5} \\ 2 & 2 \end{bmatrix}\begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{bmatrix}\frac{1}{4\sqrt{5}}\begin{bmatrix} 2 & -1+\sqrt{5} \\ -2 & 1+\sqrt{5} \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}}\begin{bmatrix} \frac{\left(1+\sqrt{5}\right)^{n+1}}{2^{n+1}} - \frac{\left(1-\sqrt{5}\right)^{n+1}}{2^{n+1}} & \frac{\left(1+\sqrt{5}\right)^{n}}{2^{n}} - \frac{\left(1-\sqrt{5}\right)^{n}}{2^{n}} \\ \frac{\left(1+\sqrt{5}\right)^{n}}{2^{n}} - \frac{\left(1-\sqrt{5}\right)^{n}}{2^{n}} & \frac{\left(1+\sqrt{5}\right)^{n-1}}{2^{n-1}} - \frac{\left(1-\sqrt{5}\right)^{n-1}}{2^{n-1}} \end{bmatrix}.$$

**3.** Verify the following identities used in this section:

**(a)** $\left(1+\sqrt{5}\right)^2 = 2(3+\sqrt{5})$ and $\left(1-\sqrt{5}\right)^2 = 2(3-\sqrt{5})$.

**(b)** $\phi^2 = \phi + 1$.

**(c)** $\frac{1-\sqrt{5}}{2}$ is the negative reciprocal of the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$.

**(d)** $1 - \phi = \frac{1-\sqrt{5}}{2} = -\frac{1}{\phi}$, or equivalently, $\phi - 1 = \frac{1}{\phi}$.

**4.** Verify that the values computed in Example 3 for the following expressions are correct:

**(a)** $\mathbf{A}^n$

**(b)** $\mathbf{A}^n\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$

**5.** Use Binet's Formula and a calculator to determine $F_{30}$, the 30th Fibonacci number. (Round the final decimal answer obtained from the calculator to the nearest integer.)

▶ **6.** Give an alternate proof of Binet's Formula for the $n$th term $F_n$ of the Fibonacci sequence using induction. (Hint: For the Base Step, we need to verify the formula for both $n = 1$ and $n = 2$, but this was already done in the section. For the Inductive Step, assume Binet's Formula is true for both $n$ and $n+1$, and then prove that it holds for $n+2$. This version of induction in which we assume the desired result is true for additional previous cases is called **strong induction**.)

**7.** Prove that a formula for the $n$th term $L_n$ of the **Lucas sequence** $1, 3, 4, 7, 11, \ldots$, where every term after the first two is the sum of the previous two terms (that is, $L_n = L_{n-1} + L_{n-2}$) is given by $L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n = \phi^n + \frac{(-1)^n}{\phi^n}$. Verify that this formula is correct for $L_1$ through $L_4$.

**8.** This exercise gives formulas for the characteristic polynomial of the matrix associated with a linear recurrence relation.

**(a)** Suppose that a linear recurrence relation of order 2 for the sequence $a_1, a_2, a_3, \ldots$ has the associated linear combination $a_n = c_{n-1}a_{n-1} + c_{n-2}a_{n-2}$, with corresponding matrix

$$\mathbf{A} = \begin{bmatrix} c_{n-1} & c_{n-2} \\ 1 & 0 \end{bmatrix}.$$

Show that the characteristic polynomial of $\mathbf{A}$ is $x^2 - c_{n-1}x - c_{n-2}$.

**(b)** Suppose that a linear recurrence relation of order 3 for the sequence $a_1, a_2, a_3, \ldots$ has the associated linear combination $a_n = c_{n-1}a_{n-1} + c_{n-2}a_{n-2} + c_{n-3}a_{n-3}$, with corresponding matrix

$$\mathbf{A} = \begin{bmatrix} c_{n-1} & c_{n-2} & c_{n-3} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Show that the characteristic polynomial of $\mathbf{A}$ is

$$x^3 - c_{n-1}x^2 - c_{n-2}x - c_{n-3}.$$

**(c)** Suppose that a linear recurrence relation of order $k$ (with $k \geq 2$) for the sequence $a_1, a_2, a_3, \ldots$ has the associated linear combination $a_n = c_{n-1}a_{n-1} + c_{n-2}a_{n-2} + \cdots + c_{n-k}a_{n-k}$, with corresponding matrix

$$\mathbf{A} = \begin{bmatrix} c_{n-1} & c_{n-2} & c_{n-3} & \cdots & c_{n-(k-1)} & c_{n-k} \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Prove by induction on $k$ (using part (a) of this exercise as the Base Step) that the characteristic polynomial of $\mathbf{A}$ is

$$x^k - c_{n-1}x^{k-1} - c_{n-2}x^{k-2} - \cdots - c_{n-(k-1)}x - c_{n-k}.$$

**9.** This exercise gives formulas for the fundamental eigenvectors for the matrix associated with a linear recurrence relation.

**(a)** For the matrix $\mathbf{A}$ in part (a) of Exercise 8, show that if $\lambda$ is an eigenvalue for $\mathbf{A}$, and $[a, b]$ is a corresponding eigenvector, then $[a, b] = b[\lambda, 1]$.

**(b)** Verify that the result of part (a) holds for the linear recurrence relation in Example 2 as well as for the Fibonacci linear recurrence relation.

**(c)** For the matrix $\mathbf{A}$ in part (c) of Exercise 8, show that if $\lambda$ is an eigenvalue for $\mathbf{A}$, and $\mathbf{v}$ is an eigenvector for $\mathbf{A}$ corresponding to $\lambda$, then $\mathbf{v} = c[\lambda^{k-1}, \lambda^{k-2}, \ldots, \lambda, 1]$ for some scalar $c$. (Hint: Assume $[a_1, a_2, \ldots, a_k]$ is an eigenvector for $\lambda$, and show that $a_{k-1} = \lambda a_k$, $a_{k-2} = \lambda a_{k-1}$, etc.)

★ **10.** True or False:

**(a)** If the $n$th term for a linear recurrence relation is defined by the formula $a_n = 3a_{n-1} + 2a_{n-3}$, then the recurrence relation has order 2.

**(b)** If a linear recurrence relation of order 2 has both coefficients in its recurrence formula positive, and the first two terms of the associated sequence are positive, then all terms in the sequence will be positive.

**(c)** If a linear recurrence of order 2 has both coefficients in its recurrence formula positive, then the eigenvalues for the matrix for the recurrence formula are all positive.

**(d)** The matrix for a linear recurrence relation of order 2, and hence the eigenvalues of that matrix, are independent of the values of the initial 2 terms chosen for the corresponding sequence.

**(e)** The Golden Ratio is an eigenvalue for the matrix for the Fibonacci sequence.

**(f)** If the sequence for a linear recurrence relation of order $k$ has 0 for its first $k$ terms, then every term of the sequence will be 0.

**(g)** If the matrix for a linear recurrence relation has an eigenvalue with algebraic multiplicity greater than 1, then solving for a formula for the $n$th term of the sequence for the relation using linear algebra involves more sophisticated techniques than those covered in this section.

## 8.7 Rotation of Axes for Conic Sections

*Prerequisite: Section 4.7, Coordinatization*

In this section, we show how to use a rotation of the plane to find the center or vertex of a given conic section (ellipse, parabola, or hyperbola) along with all of its axes of symmetry. The circle, a special case of the ellipse, has an axis of symmetry in every direction. However, both a non-circular ellipse and a hyperbola each have two (perpendicular) axes of symmetry, which meet at the center of the figure. A parabola has only one axis of symmetry, which intersects the figure at the vertex. (See Fig. 8.17.)

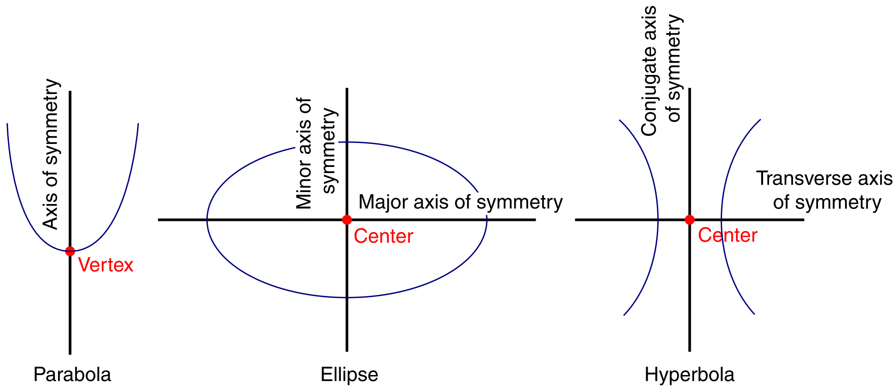


**FIGURE 8.17** Axis of symmetry and vertex for a parabola; axes of symmetry and center for an ellipse and a hyperbola

### Simplifying the Equation of a Conic Section

The general form of the equation of a conic section in the $xy$-plane is

$$ax^2 + by^2 + cxy + dx + ey + f = 0.$$

If the conic is not a circle, and if $c \neq 0$, the term $cxy$ in this equation causes all axes of symmetry of the conic to be on a slant, rather than horizontal or vertical.[7] In this section, we show how to express the equation of a non-circular conic using a different set of coordinates in the plane so that such a term does not appear. This new coordinate system makes it easier to determine the center or vertex of the conic section, as well as any axes of symmetry.

Our goal is to find an angle $\theta$ between the positive $x$-axis and an axis of symmetry of the conic section. Once $\theta$ is known, we rotate all points in the plane *clockwise* about the origin through the angle $\theta$. In particular, the original graph of the conic will move *clockwise* about the origin through the angle $\theta$, so that all of its axes of symmetry are now horizontal and/or vertical. Since this rotation has moved the original $x$- and $y$-axes out of their customary positions, we establish a new coordinate system to replace the original one. If we think of the horizontal direction after rotation as the $u$-axis, and the vertical direction after rotation as the $v$-axis, then we have created a new $uv$-coordinate system for the plane, in which all axes of symmetry of the conic section lie on the new $u$- and/or $v$-axes. Thus, in this new coordinate system, the equation for the conic section will not have a $uv$ term. This process is illustrated in Fig. 8.18 for the hyperbola $xy = 1$.

Before the rotation occurs, each point in the plane has a set of coordinates $(x, y)$ in the original $xy$-coordinate system (with the $x$- and $y$-axes in their customary positions), and after that point has been rotated, it has a new set of coordinates $(u, v)$ relative to the $u$- and $v$-axes in the $uv$-coordinate system. A similar statement is true for vectors. From Fig. 8.19, we

[7] The equation of a circle never contains a non-trivial $xy$ term.

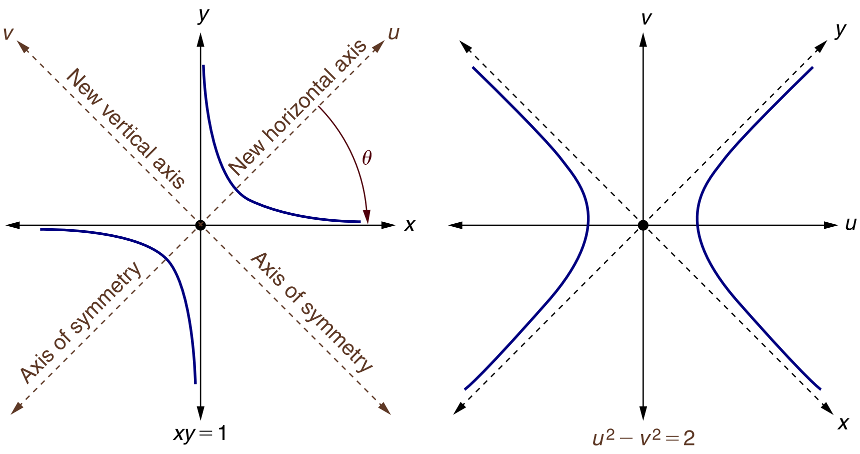


**FIGURE 8.18** Clockwise rotation of the hyperbola $xy = 1$ through angle $\theta$

see that, in particular, the vectors $[\cos\theta, \sin\theta]$ and $[-\sin\theta, \cos\theta]$ in original $xy$-coordinates before the rotation, correspond, respectively, to the unit vectors $[1, 0]$ and $[0, 1]$ in $uv$-coordinates after the rotation (see Fig. 8.19).

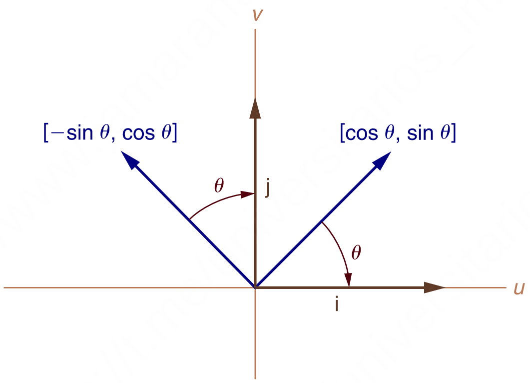


**FIGURE 8.19** Vectors that map to the standard basis vectors in $\mathbb{R}^2$ after a clockwise rotation through the angle $\theta$

Let $B$ and $C$ be the standard (ordered) bases, respectively, for the original $xy$-coordinates and the new $uv$-coordinates. The transition matrix $\mathbf{P}$ from $C$ ($uv$-coordinates) to $B$ ($xy$-coordinates) is the $2 \times 2$ matrix whose columns are the basis vectors of $C$ expressed in $B$-coordinates. We have just seen that the unit vectors $[1, 0]$ and $[0, 1]$ in $C$-coordinates correspond, respectively, to $[\cos\theta, \sin\theta]$ and $[-\sin\theta, \cos\theta]$ in $B$-coordinates. Hence,

$$\mathbf{P} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

is the transition matrix from $C$ to $B$. Thus, we can convert points in $C$-coordinates ($uv$-coordinates) to points in $B$-coordinates ($xy$-coordinates) using the equation

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}, \quad \text{or,} \quad \begin{cases} x = u\cos\theta - v\sin\theta \\ y = u\sin\theta + v\cos\theta \end{cases}.$$

We now substitute these expressions for $x$ and $y$ into the original equation for the conic section to obtain an equivalent equation in $u$ and $v$:

$$a\,(u\cos\theta - v\sin\theta)^2 + b\,(u\sin\theta + v\cos\theta)^2 + c\,(u\cos\theta - v\sin\theta)\,(u\sin\theta + v\cos\theta)$$
$$+\, d(u\cos\theta - v\sin\theta) + e(u\sin\theta + v\cos\theta) + f = 0.$$

After expanding, we find that the $uv$ term is

$$\left(2\sin\theta\cos\theta(b-a)+(\cos^2\theta-\sin^2\theta)c\right)uv=((\sin 2\theta)(b-a)+(\cos 2\theta)c)\,uv.$$

In order to ensure the coefficient of $uv$ is equal to zero in this expression, we set $(\sin 2\theta)(b-a)=-(\cos 2\theta)c$, which, if $a \neq b$, leads to $\tan 2\theta=\frac{c}{a-b}$. Thus, we choose the clockwise angle $\theta$ of rotation to be

$$\theta=\begin{cases}\frac{1}{2}\arctan\left(\frac{c}{a-b}\right) & \text{if } a\neq b\\ \frac{\pi}{4} & \text{if } a=b\end{cases}.$$

(Adding multiples of $\pi/2$ to this solution yields other solutions for $\theta$.)

**Example 1**

Consider the ellipse having equation

$$5x^2+7y^2-10xy-3x+2y-8=0.$$

We will find the center $(x_0, y_0)$ of this ellipse and the angle of inclination, $\theta$, of an axis of symmetry (with respect to the positive $x$-axis). We first find a simpler equation for the ellipse in the $uv$-coordinate system; that is, an equation that will have no $uv$-term. From the preceding formula, the appropriate[8] clockwise angle of rotation is $\theta=\frac{1}{2}\arctan(\frac{-10}{-2})\approx 39.35°$ ($\approx 0.6867$ radians). Now, $\cos\theta\approx 0.7733$ and $\sin\theta\approx 0.6340$. Hence, the expressions for $x$ and $y$ in terms of $u$ and $v$ are

$$\begin{cases}x=0.7733u-0.6340v\\ y=0.6340u+0.7733v\end{cases}.$$

Substituting these formulas for $x$ and $y$ into the equation for the ellipse, and simplifying, yields

$$0.9010u^2+11.10v^2-1.052u+3.449v-8=0.$$

Completing the squares gives

$$0.9010(u-0.5838)^2+11.10(v+0.1554)^2=8.575,$$

or

$$\frac{(u-0.5838)^2}{(3.085)^2}+\frac{(v+0.1554)^2}{(0.8790)^2}=1.$$

The graph of this equation in the $uv$-plane is an ellipse centered at $(u_0, v_0)=(0.5838, -0.1554)$, with axes of symmetry $u=0.5838$ and $v=-0.1554$, parallel to the $u$- and $v$-axes, respectively, as depicted in Fig. 8.20. In this case, the major axis is parallel to the $u$-axis, since the denominator of the $u$-term is larger.

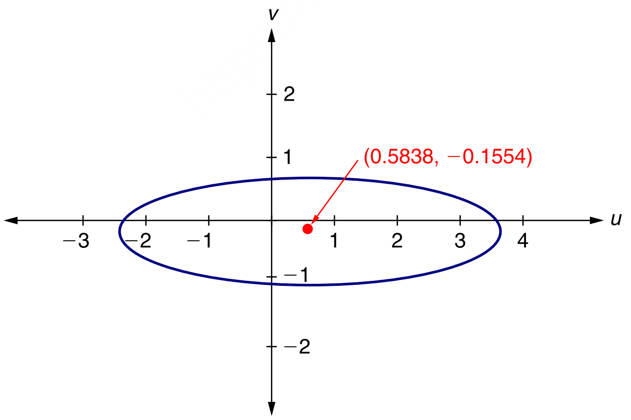


**FIGURE 8.20** The ellipse $\frac{(u-0.5838)^2}{(3.085)^2}+\frac{(v+0.1554)^2}{(0.8790)^2}=1$

[8] All computations in this example were done on a calculator rounding to 12 significant digits. However, we have printed only 4 significant digits in the text.

Now, the center $(x_0, y_0)$ of the original ellipse can be found by converting $(u_0, v_0)$ into $xy$-coordinates via the transition matrix

$$\mathbf{P} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \approx \begin{bmatrix} 0.7733 & -0.6340 \\ 0.6340 & 0.7733 \end{bmatrix}.$$

That is, the center of $5x^2 + 7y^2 - 10xy - 3x + 2y - 8 = 0$ is

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 0.7733 & -0.6340 \\ 0.6340 & 0.7733 \end{bmatrix} \begin{bmatrix} 0.5838 \\ -0.1554 \end{bmatrix} \approx \begin{bmatrix} 0.5500 \\ 0.2500 \end{bmatrix}.$$

Multiplication on the left by $\mathbf{P}$ can be thought of as rotating *counterclockwise* so that $uv$-coordinates are restored to the original $xy$-coordinates.

The original graph of the ellipse in $xy$-coordinates can therefore be obtained by rotating all of the points in $uv$-coordinates *counterclockwise* through the angle $\theta \approx 39.35°$. (See Fig. 8.21.) That is, the major axis of the original ellipse goes through $\approx (0.55, 0.25)$ at an angle of inclination $\theta \approx 39.35°$ with the positive $x$-axis.

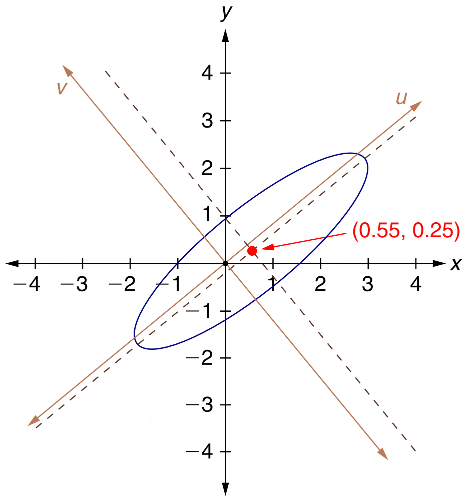


**FIGURE 8.21** The ellipse $5x^2 + 7y^2 - 10xy - 3x + 2y - 8 = 0$ with center and axes of symmetry indicated

Since we can convert directly from $uv$-coordinates to $xy$-coordinates using the transition matrix

$$\mathbf{P} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \text{ it follows that } \mathbf{P}^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

provides the means for converting from $xy$-coordinates to $uv$-coordinates. For example, with the angle $\theta \approx 39.35°$ in Example 1, the point $(-1, 0)$ on the ellipse in $xy$-coordinates corresponds to the point

$$\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} \approx \begin{bmatrix} 0.7733 & 0.6340 \\ -0.6340 & 0.7733 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} \approx \begin{bmatrix} -0.7733 \\ 0.6340 \end{bmatrix}$$

in $uv$-coordinates. Multiplication on the left by $\mathbf{P}^{-1}$ can be thought of as rotating *clockwise* so that $xy$-coordinates convert to $uv$-coordinates.

The material in this section is revisited in a more general, abstract manner in Section 8.11, "Quadratic Forms."

## New Vocabulary

axes of symmetry for a conic section
center of an ellipse or hyperbola
transition matrix from $xy$-coordinates to $uv$-coordinates
vertex of a parabola

## Highlights

- For a given (non-circular) conic $ax^2 + by^2 + cxy + dx + ey + f = 0$, with $c \neq 0$, the angle $\theta = \frac{1}{2}\arctan\left(\frac{c}{a-b}\right)$ (or, $\theta = \frac{\pi}{4}$ if $a = b$) represents the angle of inclination of an axis of the conic in $xy$-coordinates with respect to the positive $x$-axis.
- A *clockwise* rotation of a (non-circular) conic through the angle $\theta$ (as defined above) establishes a new $uv$-coordinate system for the conic in which all of its axes of symmetry are parallel to the $u$- or $v$-axes, and the corresponding equation for the conic in $uv$-coordinates has no $uv$ term. The transition matrix $\mathbf{P} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ restores $uv$-coordinates of points in the plane to $xy$-coordinates, while $\mathbf{P}^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ converts $xy$-coordinates to $uv$-coordinates.
- To find the center (or vertex) $(x_0, y_0)$ of a conic section in $xy$-coordinates, first convert the conic to $uv$-coordinates (in which the center $(u_0, v_0)$ is apparent after completing the square), and then multiply $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix}$ on the left by the transition matrix $\mathbf{P}$.

## Exercises for Section 8.7

1. For each of the given conic sections, perform the following steps:
   (i) Find an appropriate angle $\theta$ through which to rotate clockwise from $xy$-coordinates into $uv$-coordinates so that the resulting conic has no $uv$ term.
   (ii) Calculate the transition matrix $\mathbf{P}$ from $uv$-coordinates to $xy$-coordinates.
   (iii) Solve for the equation of the conic in $uv$-coordinates.
   (iv) Determine the center of the conic in $uv$-coordinates if it is an ellipse or hyperbola, or the vertex in $uv$-coordinates if it is a parabola. Graph the conic in $uv$-coordinates.
   (v) Use the transition matrix $\mathbf{P}$ to solve for the center or vertex of the conic in $xy$-coordinates. Draw the graph of the conic in $xy$-coordinates.

   (a) $3x^2 - 3y^2 - 2\sqrt{3}(xy) - 4\sqrt{3} = 0$ (hyperbola)
   (b) $13x^2 + 13y^2 - 10xy - 8\sqrt{2}x - 8\sqrt{2}y - 64 = 0$ (ellipse)
   ★ (c) $3x^2 + y^2 - 2\sqrt{3}xy - (1 + 12\sqrt{3})x + (12 - \sqrt{3})y + 26 = 0$ (parabola)
   ★ (d) $29x^2 + 36y^2 - 24xy - 118x + 24y - 55 = 0$ (ellipse)
   (e) $-16x^2 - 9y^2 + 24xy - 60x + 420y = 0$ (parabola)
   ★ (f) $8x^2 - 5y^2 + 16xy - 37 = 0$ (hyperbola)

★ 2. True or False:
   (a) The conic section $x^2 + xy + y^2 = 12$ has an axis of symmetry that makes a 45° angle with the positive $x$-axis.
   (b) The coordinates of the center of a hyperbola always stay fixed when changing from $xy$-coordinates to $uv$-coordinates.
   (c) If $\mathbf{P}$ is the transition matrix that converts from $uv$-coordinates to $xy$-coordinates, then $\mathbf{P}^{-1}$ is the matrix that converts from $xy$-coordinates to $uv$-coordinates.
   (d) The equation of a conic section with no $xy$ term has a graph in $xy$-coordinates that is symmetric with respect to the $x$-axis.

## 8.8 Computer Graphics

### *Prerequisite: Section 5.2, the Matrix of a Linear Transformation*

In this section, we give some insight into how linear algebra is used to manipulate objects on a computer screen. We will see that, in many cases, shifting the position or size of objects can be accomplished using matrix multiplication. However, to represent all possible movements by matrix multiplication, we will find it necessary to work in higher dimensions and use a somewhat different method of coordinatizing vectors, known as "homogeneous coordinates."

## Introduction to Computer Graphics

Computer screens consist of **pixels**, tiny areas of the screen arranged in rows and columns. Pixels are turned "off" and "on" to create patterns on the screen.[9] A typical $1024 \times 768$ screen, for example, would have 1024 pixels in each row (labeled "0" through "1023") and 768 pixels in each column (labeled "0" through "767"). (See Fig. 8.22.) We can think of the screen pixels as forming a lattice (grid), with a single pixel at the intersection of each row and column.

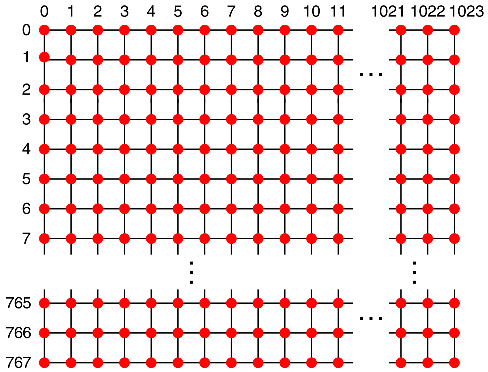


**FIGURE 8.22** A typical $1024 \times 768$ computer screen, with labeled pixels

Notice that pixels are normally labeled so that the $y$-coordinates increase as one proceeds down a computer screen. In other words, the positive $y$-axis points "downward" instead of pointing upward as it is conventionally depicted. However, to simplify our study of transformations conceptually, throughout this section we will continue to draw our $xy$-coordinate systems in the usual manner—that is, with the positive $y$-axis pointing "upward." Essentially, then, all of the figures depicted in this section should be envisioned as vertically inverted versions of actual figures on computer screens.

Today, the most common computer graphics technique is **raster graphics**, in which the current screen content (text, figures, icons, etc.) is stored in the memory of the computer and updated and displayed whenever a change of screen contents is necessary. In this system, algorithms have been created to draw fundamental geometric figures at specified areas on the screen. For example, given two different points (pixels), we can display the line connecting them by calling an algorithm to turn on the appropriate pixels. Similarly, given the points that represent the vertices of a triangle (or any polygon), we can have the computer connect them to form the appropriate screen figure.

In this system, we can represent a polygon algebraically by storing its $n$ vertices as columns in a $2 \times n$ matrix, as in the next example.

### Example 1

The polygon in Fig. 8.23 (a "Knee") can be associated with the $2 \times 6$ matrix

$$\begin{bmatrix} 8 & 8 & 6 & 8 & 10 & 10 \\ 6 & 8 & 10 & 12 & 10 & 6 \end{bmatrix}.$$

Each column lists a pair of $x$- and $y$-coordinates representing a different vertex of the figure.

The "edges" of a polygonal figure could also be represented in computer memory. For example, we could represent the "edges" with a $6 \times 6$ matrix, with $(i, j)$ entry equal to 1 if the $i$th and $j$th vertices are connected by an edge, and 0 otherwise. However, we will focus on the vertices only in this section.

Whenever we rotate a given figure on the screen, each computed vertex for the new figure may not land "exactly" on a single pixel, since the new $x$- and $y$-coordinates may not be integers. For simplicity, we assume that whenever a figure is manipulated, we round off each computation of a pixel coordinate to the nearest integer. Also, a figure must be "clipped" whenever portions of the figure extend beyond the current screen window. Powerful algorithms have been developed to address such problems, but these and many similar issues are beyond the scope of this text.

[9] When a pixel is "on," commands can be given that adjust its brightness and color to produce a desired effect. However, to avoid complications, we will ignore brightness and color in what follows, and simply consider a pixel to be "off" or "on."

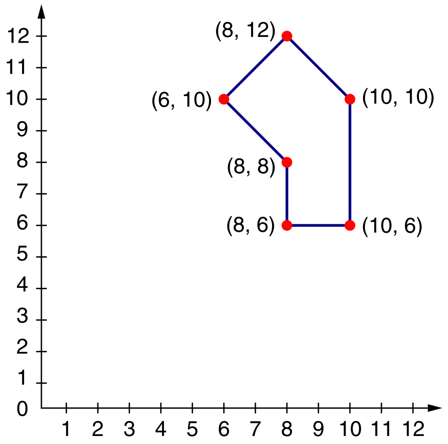


**FIGURE 8.23** Graphic with 6 vertices and 6 edges

In this section, we will illustrate how to manipulate two-dimensional figures on the screen. Similar methods are used to manipulate three-dimensional figures, although we will not consider them here. For further details, consult *Computer Graphics: Principles and Practice*, 3rd ed., by Hughes, vanDam, et al., published by Addison-Wesley, 2014.

## Fundamental Movements in the Plane

A **similarity** is a mapping of the plane to itself so that every figure in the plane and its image are similar in shape and related by the same ratio of sizes. Geometric arguments can be given to show that any similarity can be accomplished by composing one or more of the following mappings[10]:

**(1)** **Translation:** shifting all points of a figure along a fixed vector.

**(2)** **Rotation:** rotating all points of a figure about a given center point, through a given angle $\theta$. We will assume that all rotations are in a *counterclockwise* direction in the plane unless otherwise specified.

**(3)** **Reflection:** reflecting all points of a figure about a given line.

Finally, we also consider a fourth type of movement, which can change the size of a figure.

**(4)** **Scaling:** dilating/contracting the distance of all points in the figure from a given center point.

Each of these first three fundamental movements is actually an **isometry**; it maps a given figure to a *congruent* figure.

We consider each movement briefly in turn. As we will see, all translations are straightforward, but we begin with only the simplest possible type of rotation (about the origin), reflection (about a line through the origin), and scaling (with the origin as center point).

**(1)** **Translation:** To perform a translation of a vertex along a vector $\begin{bmatrix} a \\ b \end{bmatrix}$, we simply add $\begin{bmatrix} a \\ b \end{bmatrix}$ to the vertex.

**(2)** **Rotation about the origin:** In Section 5.1, we saw that multiplying on the left by the matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

rotates a vertex through an angle $\theta$ about the origin.

**(3)** **Reflection about a line through the origin:** In Exercise 21 of Section 5.2, we found that multiplying on the left by the matrix

$$\frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}$$

[10] In fact, it can be shown that any translation or rotation can be expressed as the composition of appropriate reflections. However, translations and rotations are used so often in computer graphics that it is useful to consider these mappings separately.

reflects a vertex about the line $y = mx$. In the special case where the line of reflection is the $y$-axis, the reflection matrix is simply

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}. \quad \text{(Why?)}$$

**(4)** **Scaling from the origin:** For a similarity, the scale factors in both the $x$- and $y$-directions need to be the same, but in what follows, we will, in fact, allow different scale factors in each direction since it is easy to do so. We multiply distances from the center point by $c$ in the $x$-direction and $d$ in the $y$-direction. With the origin as center point, we can achieve the desired scaling of a vertex simply by multiplying the vertex by the matrix

$$\begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}.$$

We have seen that the last three types of mappings (rotation about the origin, reflection about a line through the origin, scaling with the origin as center) can all be performed using matrix multiplication. Of course, by Example 10 in Section 5.1, these are linear transformations. However, (nontrivial) translations are not linear transformations, and neither are rotations, reflections, or scaling if they are not centered at the origin. Nevertheless, there is a way to represent all of these movements using matrix multiplication in a different type of coordinate system taken from projective geometry, "homogeneous coordinates."

## Homogeneous Coordinates

Our goal is to create a useful coordinate representation for the points in two-dimensional space by "going up" one dimension. We define any three-dimensional "point" of the form $(tx, ty, t) = t(x, y, 1)$, where $t \neq 0$, to be **equivalent** to the ordinary two-dimensional point $(x, y)$. That is, as far as we are concerned, the points $(3, 4, 1)$, $(6, 8, 2) = 2(3, 4, 1)$ and $(9, 12, 3) = 3(3, 4, 1)$ are all equivalent to $(3, 4)$. Similarly, the point $(2, -5)$ has three-dimensional representations, such as $(2, -5, 1)$, $(4, -10, 2)$, and $(-8, 20, -4)$. This three-dimensional coordinate system gives each two-dimensional point a corresponding set of **homogeneous coordinates**. Notice that there is an infinite set of homogeneous coordinates for each two-dimensional point. However, by dividing all three coordinates of a triple by its last coordinate, any point in homogeneous coordinates can be **normalized** so that its last coordinate equals 1. Each two-dimensional point thus has a unique triple of normalized homogeneous coordinates, which is said to be its **standard form**. Thus, $(5/2, -3/2, 1)$ is the standard form for the equivalent triples $(15, -9, 6)$ and $(10, -6, 4)$.

## Representing Movements With Matrix Multiplication in Homogeneous Coordinates

**Translation:** To translate vertex $(x, y)$ along a given vector $[a, b]$, we first convert $(x, y)$ to homogeneous coordinates. The simplest way to do this is to replace $(x, y)$ with the equivalent vector $[x, y, 1]$. Then, multiplication on the left by the matrix

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \quad \text{gives} \quad \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \\ 1 \end{bmatrix},$$

which is equivalent to the two-dimensional point $(x + a, y + b)$, the desired result.

**Rotation, Reflection, Scaling:** You can verify that multiplying $[x, y, 1]$ on the left by the following matrices performs, respectively, a rotation of $(x, y)$ about the origin through angle $\theta$, a reflection of $(x, y)$ about the line $y = mx$, and a scaling of $(x, y)$ about the origin by a factor of $c$ in the $x$-direction and $d$ in the $y$-direction.

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \left(\frac{1}{1+m^2}\right)\begin{bmatrix} 1-m^2 & 2m & 0 \\ 2m & m^2-1 & 0 \\ 0 & 0 & 1+m^2 \end{bmatrix}, \quad \begin{bmatrix} c & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Finally, the special case of a reflection about the $y$-axis can be accomplished by multiplying on the left by the matrix

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Recall that for any matrix $\mathbf{A}$ and vector $\mathbf{v}$ (of compatible sizes) and any scalar $t$, we have $\mathbf{A}(t\mathbf{v}) = t(\mathbf{A}\mathbf{v})$. Hence, multiplying a $3 \times 3$ matrix $\mathbf{A}$ by any two vectors of the form $t[x, y, 1] = [tx, ty, t]$ equivalent to $(x, y)$ always produces two results that are equivalent in homogeneous coordinates.

## Movements Not Centered at the Origin

Our next goal is to determine the matrices for rotations, reflections, and scaling that are not centered about the origin. This can be done by combining appropriate translation matrices with the matrices for origin-centered rotations, reflections, and scaling.

**Method for Performing Geometric Transformations Not Centered at the Origin (Similarity Method)**
**Step 1:** Use a translation to move the figure so that the rotation, reflection, or scaling to be performed is "about the origin." (This means moving the figure so that the center of rotation/scaling is the origin, or so that the line of reflection goes through the origin.)

**Step 2:** Perform the desired rotation, reflection, or scaling "about the origin."

**Step 3:** Translate the altered figure back to the position of the original figure by reversing the translation in Step 1.

The Similarity Method requires the composition of three movements. Theorem 5.7 shows that the matrix for a composition is the product of the corresponding matrices for the individual mappings in *reverse* order, as we will illustrate in Examples 2, 3, and 4. A little thought will convince you that the Similarity Method also has the overall effect of multiplying a vertex in homogeneous coordinates by a matrix *similar* to the matrix for the movement in Step 2 (see Exercise 10).

We will demonstrate the Similarity Method for each type of movement in turn.

### Example 2

**Rotation:** Suppose we rotate the vertices of the "Knee" from Example 1 through an angle of $\theta = 90°$ about the point $(r, s) = (12, 6)$. We first replace each $(x, y)$ with its vector $[x, y, 1]$ in homogeneous coordinates and follow the Similarity Method. In Step 1, we translate from $(12, 6)$ to $(0, 0)$ in order to establish the origin as center. In Step 2, we perform a rotation through angle $\theta = 90°$ about the origin. Finally, in Step 3, we translate from $(0, 0)$ back to $(12, 6)$. The net effect of these three operations is to rotate each vertex about $(12, 6)$. (Why?) The combined result of these operations is

$$\underbrace{\begin{bmatrix} 1 & 0 & 12 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix}}_{\substack{\text{translate from} \\ (0,0)\text{ back to }(12,6)}} \underbrace{\begin{bmatrix} \cos 90° & -\sin 90° & 0 \\ \sin 90° & \cos 90° & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\substack{\text{rotate about }(0,0) \\ \text{through angle }90°}} \underbrace{\begin{bmatrix} 1 & 0 & -12 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix}}_{\substack{\text{translate from} \\ (12,6)\text{ to }(0,0)}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}.$$

This reduces to

$$\begin{bmatrix} 0 & -1 & 18 \\ 1 & 0 & -6 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}. \text{ (Verify!)}$$

Therefore, performing the rotation on all vertices of the figure simultaneously, we obtain

$$\begin{bmatrix} 0 & -1 & 18 \\ 1 & 0 & -6 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 8 & 8 & 6 & 8 & 10 & 10 \\ 6 & 8 & 10 & 12 & 10 & 6 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 12 & 10 & 8 & 6 & 8 & 12 \\ 2 & 2 & 0 & 2 & 4 & 4 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

The columns of the final matrix (ignoring the last row entries) give the vertices of the rotated figure, as illustrated in Fig. 8.24(a).

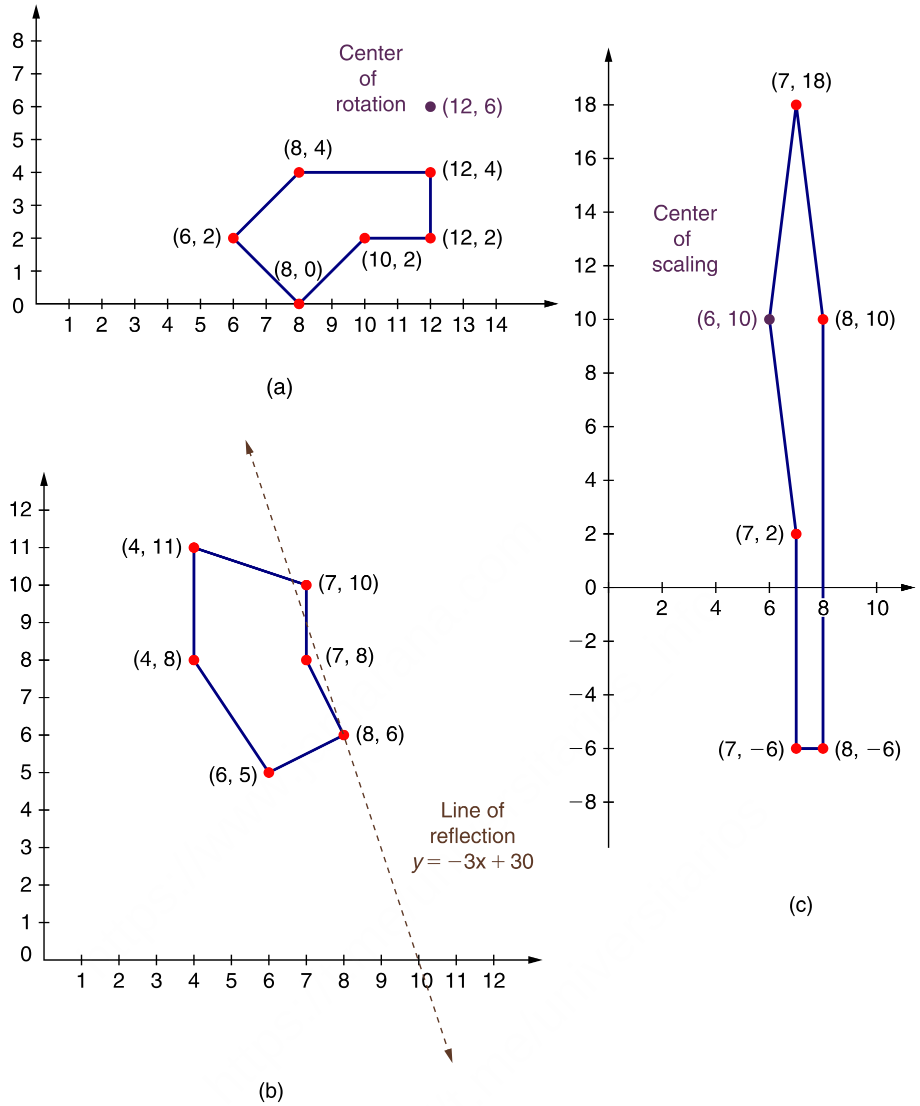


**FIGURE 8.24** Movements of "Knee": (a) rotation through 90° about $(12, 6)$; (b) reflection about line $y = -3x + 30$; (c) scaling with $c = 1/2$, $d = 4$ about $(6, 10)$

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## Example 3

**Reflection:** Suppose we reflect the vertices of the "Knee" in Example 1 about the line $y = -3x + 30$. In this case, $m = -3$ and $b = 30$. As before, we replace $(x, y)$ with its equivalent vector $[x, y, 1]$, and follow the Similarity Method. In Step 1, we translate from $(0, 30)$ to $(0, 0)$ in order to "drop" the line 30 units vertically so that it passes through the origin. In Step 2, we perform a reflection about the corresponding line $y = -3x$. Finally, in Step 3, we translate from $(0, 0)$ back to $(0, 30)$. The net effect of these three operations is to reflect each vertex about the line $y = -3x + 30$. (Why?) The combined result of these operations is

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 30 \\ 0 & 0 & 1 \end{bmatrix}}_{\substack{\text{translate from} \\ (0,0)\text{ back} \\ \text{to } (0,30)}} \underbrace{\left(\frac{1}{1+(-3)^2}\right) \begin{bmatrix} 1-(-3)^2 & 2(-3) & 0 \\ 2(-3) & (-3)^2-1 & 0 \\ 0 & 0 & 1+(-3)^2 \end{bmatrix}}_{\substack{\text{reflect about} \\ \text{the line } y=-3x}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -30 \\ 0 & 0 & 1 \end{bmatrix}}_{\substack{\text{translate from} \\ (0,30)\text{ to }(0,0)}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}.$$

This reduces to

$$\left(\frac{1}{10}\right)\begin{bmatrix}-8 & -6 & 180\\ -6 & 8 & 60\\ 0 & 0 & 10\end{bmatrix}\begin{bmatrix}x\\ y\\ 1\end{bmatrix}.$$

Performing the reflection on all vertices of the figure simultaneously, we obtain

$$\frac{1}{10}\begin{bmatrix}-8 & -6 & 180\\ -6 & 8 & 60\\ 0 & 0 & 10\end{bmatrix}\begin{bmatrix}8 & 8 & 6 & 8 & 10 & 10\\ 6 & 8 & 10 & 12 & 10 & 6\\ 1 & 1 & 1 & 1 & 1 & 1\end{bmatrix}\approx\begin{bmatrix}8 & 7 & 7 & 4 & 4 & 6\\ 6 & 8 & 10 & 11 & 8 & 5\\ 1 & 1 & 1 & 1 & 1 & 1\end{bmatrix},$$

after rounding the results for each vertex to the nearest integer. The columns of the final matrix (ignoring the last row entries) give the vertices of the reflected figure, as illustrated in Fig. 8.24(b). Notice that the reflected figure is slightly distorted because of the rounding involved. For simplicity in this example, small pixel values were used, but a larger figure on the screen would probably undergo less distortion after such a reflection.

The special case of a reflection about a line parallel to the $y$-axis is treated in Exercise 8.

### Example 4

**Scaling:** Suppose we scale the vertices of the "Knee" in Example 1 about the point $(r, s) = (6, 10)$ with a factor of $c = 1/2$ in the $x$-direction and $d = 4$ in the $y$-direction. In a manner similar to Examples 2 and 3 we obtain

$$\underbrace{\begin{bmatrix}1 & 0 & 6\\ 0 & 1 & 10\\ 0 & 0 & 1\end{bmatrix}}_{\substack{\text{translate from}\\ (0,0)\text{ back}\\ \text{to }(6,10)}}\underbrace{\begin{bmatrix}\frac{1}{2} & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 1\end{bmatrix}}_{\substack{\text{scale about }(0,0)\\ \text{using scale factors }\frac{1}{2}\\ \text{and 4, respectively}}}\underbrace{\begin{bmatrix}1 & 0 & -6\\ 0 & 1 & -10\\ 0 & 0 & 1\end{bmatrix}}_{\substack{\text{translate from}\\ (6,10)\text{ to }(0,0)}}\begin{bmatrix}x\\ y\\ 1\end{bmatrix}=\begin{bmatrix}\frac{1}{2} & 0 & 3\\ 0 & 4 & -30\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ 1\end{bmatrix}.$$

Therefore, scaling all vertices of the figure simultaneously, we obtain

$$\begin{bmatrix}\frac{1}{2} & 0 & 3\\ 0 & 4 & -30\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}8 & 8 & 6 & 8 & 10 & 10\\ 6 & 8 & 10 & 12 & 10 & 6\\ 1 & 1 & 1 & 1 & 1 & 1\end{bmatrix}=\begin{bmatrix}7 & 7 & 6 & 7 & 8 & 8\\ -6 & 2 & 10 & 18 & 10 & -6\\ 1 & 1 & 1 & 1 & 1 & 1\end{bmatrix},$$

as illustrated in Fig. 8.24(c). Two of the scaled vertices have negative $y$-values, and so would not be displayed on the computer screen.

## Composition of Movements

Now that we have established that all translations, rotations, reflections, and scaling operations can be performed by appropriate matrix multiplications in homogeneous coordinates, we can find the matrix for a composition of such movements.

### Example 5

Suppose we rotate the "Knee" in Example 1 through an angle of 300° about the point (8, 10), and then reflect the resulting figure about the line $y = -(1/2)x + 20$. With $\theta = 300°$, $m = -1/2$, and $b = 20$, the matrix for this composition is the product of the following six matrices:

$$\underbrace{\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 20\\ 0 & 0 & 1\end{bmatrix}}_{\substack{\text{translate from}\\ (0,0)\text{ back}\\ \text{to }(0,20)}}\underbrace{\left(\frac{1}{1+(-\frac{1}{2})^2}\right)\begin{bmatrix}1-(-\frac{1}{2})^2 & 2(-\frac{1}{2}) & 0\\ 2(-\frac{1}{2}) & (-\frac{1}{2})^2-1 & 0\\ 0 & 0 & 1+(-\frac{1}{2})^2\end{bmatrix}}_{\substack{\text{reflect about}\\ \text{the line }y=-\frac{1}{2}x}}\underbrace{\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -20\\ 0 & 0 & 1\end{bmatrix}}_{\substack{\text{translate from}\\ (0,20)\text{ to }(0,0)}}$$

$$\underbrace{\begin{bmatrix}1 & 0 & 8\\ 0 & 1 & 10\\ 0 & 0 & 1\end{bmatrix}}_{\substack{\text{translate from}\\ (0,0)\text{ back to }(8,10)}}\underbrace{\begin{bmatrix}\cos 300° & -\sin 300° & 0\\ \sin 300° & \cos 300° & 0\\ 0 & 0 & 1\end{bmatrix}}_{\substack{\text{rotate about }(0,0)\\ \text{through angle }300°}}\underbrace{\begin{bmatrix}1 & 0 & -8\\ 0 & 1 & -10\\ 0 & 0 & 1\end{bmatrix}}_{\substack{\text{translate from}\\ (8,10)\text{ to }(0,0)}}.$$

This reduces to (approximately)

$$\begin{bmatrix} 0.9928 & 0.1196 & 3.6613 \\ 0.1196 & -0.9928 & 28.5713 \\ 0 & 0 & 1 \end{bmatrix}.$$

Multiplying this matrix by all vertices of the figure simultaneously and rounding the results for each vertex to the nearest integer, we have

$$\begin{bmatrix} 0.9928 & 0.1196 & 3.6613 \\ 0.1196 & -0.9928 & 28.5713 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 8 & 8 & 6 & 8 & 10 & 10 \\ 6 & 8 & 10 & 12 & 10 & 6 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \approx \begin{bmatrix} 12 & 13 & 11 & 13 & 15 & 14 \\ 24 & 22 & 19 & 18 & 20 & 24 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

The columns of the final matrix (ignoring the last row entries) give the vertices of the final figure after the indicated rotation and reflection. These are illustrated in Fig. 8.25.

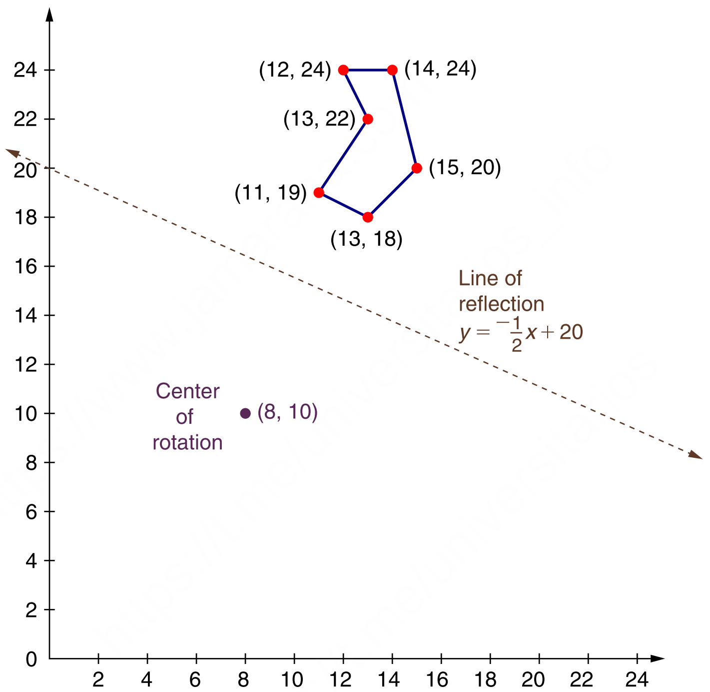


**FIGURE 8.25** Movement of "Knee" after rotation through an angle of 300° about the point $(8, 10)$, followed by reflection about the line $y = -(1/2)x + 20$

## New Vocabulary

homogeneous coordinates
isometry
normalized homogeneous coordinates
pixel
reflection (of a figure) about a line
rotation (of a figure) about a point
scaling (of a figure)
Similarity Method
similarity of figures in the plane
standard form (for homogeneous coordinates)
translation (of a figure)

## Highlights

- Any similarity of the plane is the composition of one or more of the following: translation, rotation (about a fixed point), reflection (about a fixed line), and scaling.
- Nontrivial translations are *not* linear transformations, so homogeneous coordinates are used so that translations, along with rotations, reflections, and scaling, can all be expressed using matrix multiplication.

- Homogeneous coordinates $(a, b, c)$ and $(x, y, z)$ in $\mathbb{R}^3$ represent the same point if $(a, b, c) = t(x, y, z)$ for some $t \neq 0$.
- Each point $(x, y)$ in $\mathbb{R}^2$ is equivalent to the set of homogeneous coordinates $(tx, ty, t)$ $(t \neq 0)$ in $\mathbb{R}^3$, and corresponds to a unique triple $(x, y, 1)$ of normalized homogeneous coordinates.
- In homogeneous coordinates, the result after translation of a point $(x, y)$ along the vector $[a, b]$ is

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \\ 1 \end{bmatrix}.$$

- In homogeneous coordinates, the result after rotation of a point $(x, y)$ about the origin counterclockwise through angle $\theta$ is $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$.
- In homogeneous coordinates, the result after reflection of a point $(x, y)$ about the line $y = mx$ is

$$\left(\frac{1}{1+m^2}\right) \begin{bmatrix} 1-m^2 & 2m & 0 \\ 2m & m^2-1 & 0 \\ 0 & 0 & 1+m^2 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}.$$

- In homogeneous coordinates, the result after scaling of a point $(x, y)$ about the origin by a factor of $c$ in the $x$-direction and a factor of $d$ in the $y$-direction is $\begin{bmatrix} c & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$.
- The Similarity Method performs a rotation or a scaling about a center $(x, y) \neq (0, 0)$ by first applying the translation that takes $(x, y)$ to $(0, 0)$, then carrying out the intended rotation or scaling about $(0, 0)$, and finally applying the reverse translation.
- The Similarity Method performs a reflection about the line $y = mx + b$ by first vertically translating the plane down $b$ units, then performing a reflection through the line $y = mx$, and finally applying the reverse translation.
- The matrix for any composition of translations, rotations, reflections, and scaling is obtained by multiplying the matrices representing the respective mappings in *reverse* order.

## Exercises for Section 8.8

Round all calculations of pixel coordinates to the nearest integer. Some of the resulting coordinate values may be "outside" a typical pixel configuration.

**1.** For the graphic in Fig. 8.26(a), use ordinary coordinates in $\mathbb{R}^2$ to find the new vertices after performing each indicated operation:

★ **(a)** translation along the vector $[4, -2]$

**(b)** rotation about the origin through $\theta = 30°$

★ **(c)** reflection about the line $y = 3x$

**(d)** scaling about the origin with scale factors of 4 in the $x$-direction and 2 in the $y$-direction

**2.** For the graphic in Fig. 8.26(b), use ordinary coordinates in $\mathbb{R}^2$ to find the new vertices after performing each indicated operation. Then sketch the figure that would result from this movement:

**(a)** translation along the vector $[-3, 5]$

★ **(b)** rotation about the origin through $\theta = 120°$

**(c)** reflection about the line $y = \frac{1}{2}x$

★ **(d)** scaling about the origin with scale factors of $\frac{1}{2}$ in the $x$-direction and 3 in the $y$-direction

**3.** For the graphic in Fig. 8.26(c), use homogeneous coordinates to find the new vertices after performing each indicated sequence of operations:

★ **(a)** rotation about the origin through $\theta = 45°$, followed by a reflection about the line $y = \frac{1}{2}x$

**(b)** reflection about the line $y = \frac{1}{2}x$, followed by a rotation about the origin through $\theta = 45°$

★ **(c)** scaling about the origin with scale factors of 3 in the $x$-direction and $\frac{1}{2}$ in the $y$-direction, followed by a reflection about the line $y = 2x$

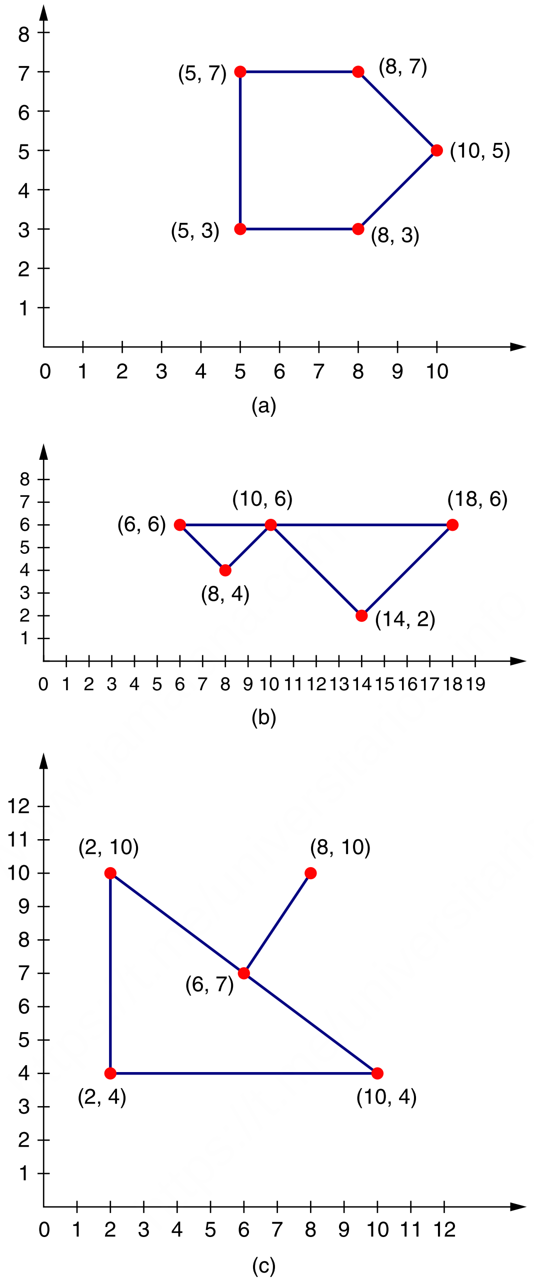


**FIGURE 8.26** (a) Figure for Exercise 1; (b) figure for Exercise 2; (c) figure for Exercise 3

**(d)** translation along the vector $[-2, 3]$, followed by a rotation about the origin through $\theta = 300°$

**4.** For the graphic in Fig. 8.27(a), use homogeneous coordinates to find the new vertices after performing each indicated operation:

★ **(a)** rotation about $(8, 9)$ through $\theta = 120°$

**(b)** reflection about the line $y = 2 - x$

★ **(c)** scaling about $(8, 4)$ with scale factors of 2 in the $x$-direction and $\frac{1}{3}$ in the $y$-direction

**5.** For the graphic in Fig. 8.27(b), use homogeneous coordinates to find the new vertices after performing each indicated operation:

**(a)** rotation about $(10, 8)$ through $\theta = 315°$

★ **(b)** reflection about the line $y = 4x - 10$

**(c)** scaling about $(7, 3)$ with scale factors of $\frac{1}{2}$ in the $x$-direction and 3 in the $y$-direction

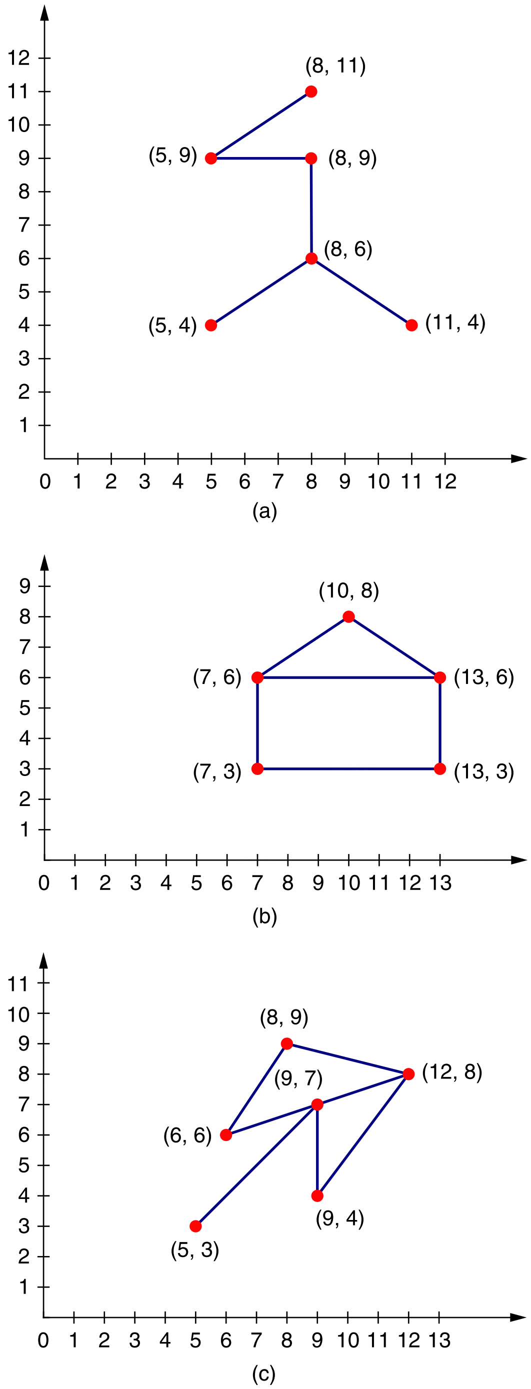


**FIGURE 8.27** (a) Figure for Exercise 4; (b) figure for Exercise 5; (c) figure for Exercise 6

**6.** For the graphic in Fig. 8.27(c), use homogeneous coordinates to find the new vertices after performing each indicated sequence of operations. Then sketch the final figure that would result from these movements:

★ **(a)** rotation about (12, 8) through $\theta = 60°$, followed by a reflection about the line $y = \frac{1}{2}x + 6$

**(b)** reflection about the line $y = 2x - 1$, followed by a rotation about (10, 10) through $\theta = 210°$

★ **(c)** scaling about (4, 9) with scale factors $\frac{3}{4}$ in the $x$-direction and $\frac{3}{2}$ in the $y$-direction, followed by a rotation about (2, 5) through $\theta = 50°$

(d) reflection about the line $y = 3x - 2$, followed by scaling about $(8, 6)$ using scale factors of 3 in the $x$-direction and $\frac{1}{2}$ in the $y$-direction

7. Use the Similarity Method to verify each of the following assertions:

(a) A rotation about $(r, s)$ through angle $\theta$ is represented by the matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta & r(1-\cos\theta) + s(\sin\theta) \\ \sin\theta & \cos\theta & s(1-\cos\theta) - r(\sin\theta) \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) A reflection about the line $y = mx + b$ is represented by the matrix

$$\left(\frac{1}{1+m^2}\right)\begin{bmatrix} 1-m^2 & 2m & -2mb \\ 2m & m^2-1 & 2b \\ 0 & 0 & 1+m^2 \end{bmatrix}.$$

(c) A scaling about $(r, s)$ with scale factors $c$ in the $x$-direction and $d$ in the $y$-direction is represented by the matrix

$$\begin{bmatrix} c & 0 & r(1-c) \\ 0 & d & s(1-d) \\ 0 & 0 & 1 \end{bmatrix}.$$

8. Show that a reflection about the line $x = k$ is represented by the matrix

$$\begin{bmatrix} -1 & 0 & 2k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(Hint: First, translate from $(k, 0)$ to $(0, 0)$, then, reflect about the $y$-axis, and finally, translate from $(0, 0)$ back to $(k, 0)$.)

9. Redo each part of Exercise 5 with a single matrix multiplication by using an appropriate matrix from Exercise 7 in each case.

10. This exercise concerns the inverse of the translation matrix.

(a) Verify computationally that the translation matrices

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & -a \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{bmatrix}$$

are inverses of each other.

(b) Explain geometrically why it makes sense that the translation matrices from part (a) are inverses.

(c) Explain why the matrices for a rotation about the origin through a given angle $\theta$ and a rotation about any other point $(r, s)$ through the same angle $\theta$ must be similar. (Hint: Use part (a).)

11. This exercise investigates whether certain similarities commute.

(a) Let $L_1$ be a scaling about the point $(r, s)$ with equal scale factors in the $x$- and $y$-directions, and let $L_2$ be a rotation about the point $(r, s)$ through angle $\theta$. Show that $L_1$ and $L_2$ commute. (That is, show $L_1 \circ L_2 = L_2 \circ L_1$.)

★ (b) Give a counterexample to show that, in general, a reflection and a rotation do not commute.

(c) Give a counterexample to show that, in general, a scaling and a reflection do not commute.

12. An $n \times n$ matrix $\mathbf{A}$ is an **orthogonal matrix** if and only if $\mathbf{A}\mathbf{A}^T = \mathbf{I}_n$.

(a) Show that the $2 \times 2$ matrix for rotation about the origin through an angle $\theta$, and its $3 \times 3$ counterpart in homogeneous coordinates (as given in this section), are both orthogonal matrices.

(b) Show that the single matrix for the rotation of the plane through an angle of 90° about the point $(12, 6)$ given in Example 2 is *not* an orthogonal matrix.

**(c)** Is either the $2 \times 2$ matrix for a reflection about a line through the origin, or its $3 \times 3$ counterpart in homogeneous coordinates (as given in this section), an orthogonal matrix? Why? (Hint: Let $\mathbf{A}$ be either matrix. Note that $\mathbf{A}^2 = \mathbf{I}$.)

★ **13.** True or False:

**(a)** We may use vectors in homogeneous coordinates having third coordinate 0 to represent pixels on the screen.
**(b)** Every pixel on the screen has a unique representation in homogeneous coordinates.
**(c)** Every rotation has a unique $3 \times 3$ matrix representing it in homogeneous coordinates.
**(d)** Every isometry in the plane can be expressed using the basic motions of rotation, reflection, and translation.
**(e)** Non-identity translations are not linear transformations.
**(f)** All rotations and reflections in the plane are linear transformations.

## 8.9 Differential Equations

*Prerequisite: Section 5.6, Diagonalization of Linear Operators*

In this section, we use the diagonalization process to solve certain first-order linear homogeneous systems of differential equations. We then adjust this technique to solve higher-order homogeneous differential equations as well.

### First-Order Linear Homogeneous Systems

**Definition** Let

$$\mathbf{F}(t) = \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix}$$

represent an $n \times 1$ matrix whose entries are real-valued functions, and let $\mathbf{A}$ be an $n \times n$ matrix of real numbers. Then the equation $\mathbf{F}'(t) - \mathbf{A}\mathbf{F}(t) = \mathbf{0}$, or $\mathbf{F}'(t) = \mathbf{A}\mathbf{F}(t)$, is called a **first-order linear homogeneous system of differential equations**. A **solution** for such a system is a particular function $\mathbf{F}(t)$ that satisfies the equation for all values of $t$.

For brevity, in the remainder of this section we will refer to an equation of the form $\mathbf{F}'(t) = \mathbf{A}\mathbf{F}(t)$ as a **first-order system**.

**Example 1**

Let $\mathbf{F} = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}$ and $\mathbf{A} = \begin{bmatrix} 13 & -45 \\ 6 & -20 \end{bmatrix}$, and consider the first-order system $\mathbf{F}'(t) = \mathbf{A}\mathbf{F}(t)$, or

$$\begin{bmatrix} f_1'(t) \\ f_2'(t) \end{bmatrix} = \begin{bmatrix} 13 & -45 \\ 6 & -20 \end{bmatrix} \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}.$$

Multiplying yields

$$\begin{cases} f_1'(t) = 13f_1(t) - 45f_2(t) \\ f_2'(t) = 6f_1(t) - 20f_2(t) \end{cases}.$$

A solution for this system consists of a pair of functions, $f_1(t)$ and $f_2(t)$, that satisfy both of these differential equations. One such solution is

$$\mathbf{F}(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} = \begin{bmatrix} 5e^{-5t} \\ 2e^{-5t} \end{bmatrix}.$$

(Verify.) We will see how to obtain such solutions later in this section.

In what follows, we concern ourselves only with solutions that are *continuously differentiable* (that is, solutions having continuous derivatives). First, we state, without proof, a well-known result from the theory of differential equations about solutions of a single first-order equation.

**Lemma 8.8** *A real-valued continuously differentiable function $f(t)$ is a solution to the differential equation $f'(t) = af(t)$ if and only if $f(t) = be^{at}$ for some real number $b$.*

A first-order system of the form $\mathbf{F}'(t) = \mathbf{AF}(t)$ is more complicated than the differential equation in Lemma 8.8, since it involves a matrix $\mathbf{A}$ instead of a real number $a$. However, in the special case when $\mathbf{A}$ is a diagonal matrix, the system $\mathbf{F}'(t) = \mathbf{AF}(t)$ can be written as

$$\begin{cases} f_1'(t) = a_{11}\, f_1(t) \\ f_2'(t) = a_{22}\, f_2(t) \\ \quad\vdots \\ f_n'(t) = a_{nn}\, f_n(t) \end{cases}.$$

Each of the differential equations in this system can be solved separately using Lemma 8.8. Hence, when $\mathbf{A}$ is diagonal, the general solution has the form

$$\mathbf{F}(t) = [b_1 e^{a_{11}t}, b_2 e^{a_{22}t}, \ldots, b_n e^{a_{nn}t}],$$

for some $b_1, \ldots, b_n \in \mathbb{R}$.

**Example 2**

Consider the first-order system $\mathbf{F}'(t) = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \mathbf{F}(t)$, whose matrix is diagonal. This system is equivalent to

$$\begin{cases} f_1'(t) = \phantom{-}3\, f_1(t) \\ f_2'(t) = -2\, f_2(t) \end{cases}.$$

Using Lemma 8.8, we see that the solutions are all functions of the form

$$\mathbf{F}(t) = [f_1(t), f_2(t)] = [b_1 e^{3t}, b_2 e^{-2t}].$$

Since first-order systems $\mathbf{F}'(t) = \mathbf{AF}(t)$ are easily solved when the matrix $\mathbf{A}$ is diagonal, it is natural to consider the case when $\mathbf{A}$ is diagonalizable. Thus, suppose $\mathbf{A}$ is a diagonalizable $n \times n$ matrix with (not necessarily distinct) eigenvalues $\lambda_1, \ldots, \lambda_n$ corresponding to the eigenvectors in the ordered basis $B = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ for $\mathbb{R}^n$. The matrix $\mathbf{P}$ having columns $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is the transition matrix from $B$ to standard coordinates, and $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$, the diagonal matrix having eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ along its main diagonal. Hence,

$$\begin{aligned} \mathbf{F}'(t) = \mathbf{AF}(t) &\Longleftrightarrow \mathbf{F}'(t) = (\mathbf{PP}^{-1}\mathbf{APP}^{-1})\mathbf{F}(t) \\ &\Longleftrightarrow \mathbf{F}'(t) = \mathbf{PDP}^{-1}\mathbf{F}(t) \\ &\Longleftrightarrow \mathbf{P}^{-1}\mathbf{F}'(t) = \mathbf{DP}^{-1}\mathbf{F}(t). \end{aligned}$$

Letting $\mathbf{G}(t) = \mathbf{P}^{-1}\mathbf{F}(t)$, we see that the original system $\mathbf{F}'(t) = \mathbf{AF}(t)$ is equivalent to the system $\mathbf{G}'(t) = \mathbf{DG}(t)$. Since $\mathbf{D}$ is diagonal, with diagonal entries $\lambda_1, \ldots, \lambda_n$, the latter system is solved as follows:

$$\mathbf{G}(t) = \left[b_1 e^{\lambda_1 t}, b_2 e^{\lambda_2 t}, \ldots, b_n e^{\lambda_n t}\right].$$

But, $\mathbf{F}(t) = \mathbf{PG}(t)$. Since the columns of $\mathbf{P}$ are the eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, we obtain

$$\mathbf{F}(t) = b_1 e^{\lambda_1 t}\mathbf{v}_1 + b_2 e^{\lambda_2 t}\mathbf{v}_2 + \cdots + b_n e^{\lambda_n t}\mathbf{v}_n.$$

Thus, we have proved the following:

**Theorem 8.9** *Let* $\mathbf{A}$ *be a diagonalizable* $n \times n$ *matrix and let* $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ *be an ordered basis for* $\mathbb{R}^n$ *consisting of eigenvectors for* $\mathbf{A}$ *corresponding to the (not necessarily distinct) eigenvalues* $\lambda_1, \ldots, \lambda_n$*. Then the continuously differentiable solutions for the first-order system* $\mathbf{F}'(t) = \mathbf{AF}(t)$ *are all functions of the form*

$$\mathbf{F}(t) = b_1 e^{\lambda_1 t}\mathbf{v}_1 + b_2 e^{\lambda_2 t}\mathbf{v}_2 + \cdots + b_n e^{\lambda_n t}\mathbf{v}_n,$$

*where* $b_1, \ldots, b_n \in \mathbb{R}$.

**Example 3**

We will solve the first-order system $\mathbf{F}'(t) = \mathbf{AF}(t)$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -2 & 6 \\ 4 & -1 & -4 & 12 \\ -32 & 9 & 40 & -114 \\ -11 & 3 & 14 & -40 \end{bmatrix}.$$

Following Steps 1 through 3 of the Diagonalization Method in Section 3.4, we find that $\mathbf{A}$ has the following fundamental eigenvectors and corresponding eigenvalues:

$$\begin{array}{ll} \mathbf{v}_1 = [-2, -4, 5, 2] & \text{corresponding to } \lambda_1 = 0 \\ \mathbf{v}_2 = [-3, 2, 0, 1] & \text{corresponding to } \lambda_2 = -1 \\ \mathbf{v}_3 = [1, -1, 1, 0] & \text{corresponding to } \lambda_3 = -1 \\ \mathbf{v}_4 = [0, 0, 3, 1] & \text{corresponding to } \lambda_4 = 2. \end{array}$$

(Notice that $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly independent eigenvectors for the eigenvalue $-1$, so that $\{\mathbf{v}_2, \mathbf{v}_3\}$ forms a basis for $E_{-1}$.) Therefore, Theorem 8.9 tells us that the continuously differentiable solutions to the first-order system $\mathbf{F}'(t) = \mathbf{AF}(t)$ consist precisely of all functions of the form

$$\begin{aligned} \mathbf{F}(t) &= [f_1(t),\ f_2(t),\ f_3(t),\ f_4(t)] \\ &= b_1[-2, -4, 5, 2] + b_2 e^{-t}[-3, 2, 0, 1] + b_3 e^{-t}[1, -1, 1, 0] + b_4 e^{2t}[0, 0, 3, 1] \\ &= [-2b_1 - 3b_2 e^{-t} + b_3 e^{-t},\ -4b_1 + 2b_2 e^{-t} - b_3 e^{-t},\ 5b_1 + b_3 e^{-t} + 3b_4 e^{2t},\ 2b_1 + b_2 e^{-t} + b_4 e^{2t}]. \end{aligned}$$

Notice that in order to use Theorem 8.9 to solve a first-order system $\mathbf{F}'(t) = \mathbf{AF}(t)$, $\mathbf{A}$ must be a diagonalizable matrix. If it is not, you can still find some of the solutions to the system using an analogous process. If $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a linearly independent set of eigenvectors for $\mathbf{A}$ corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_k$, then functions of the form

$$\mathbf{F}(t) = b_1 e^{\lambda_1 t}\mathbf{v}_1 + b_2 e^{\lambda_2 t}\mathbf{v}_2 + \cdots + b_k e^{\lambda_k t}\mathbf{v}_k$$

are solutions (see Exercise 3). However, these are not all the possible solutions for the system. To find all the solutions, you must use complex eigenvalues and eigenvectors, as well as *generalized eigenvectors*. Complex eigenvalues are studied in Section 7.2; generalized eigenvectors are covered in "Jordan Canonical Form," a companion web section for this text. The details of how these techniques can be applied to first-order systems can be found in the web companion section "Solving First-Order Systems of Linear Homogeneous Differential Equations."

## Higher-Order Homogeneous Differential Equations

Our next goal is to solve higher-order homogeneous differential equations of the form

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0.$$

**Example 4**

Consider the differential equation $y''' - 6y'' + 3y' + 10y = 0$. To find solutions for this equation, we define the functions $f_1(t)$, $f_2(t)$, and $f_3(t)$ as follows: $f_1 = y$, $f_2 = y'$, and $f_3 = y''$. We then have the system

$$\begin{cases} f_1' = f_2 \\ f_2' = f_3 \\ f_3' = -10f_1 - 3f_2 + 6f_3 \end{cases}.$$

The first two equations in this system come directly from the definitions of $f_1$, $f_2$, and $f_3$. The third equation is obtained from the original differential equation by moving all terms except $y'''$ to the right side. But this system can be expressed as

$$\begin{bmatrix} f_1'(t) \\ f_2'(t) \\ f_3'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -3 & 6 \end{bmatrix} \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{bmatrix};$$

that is, as $\mathbf{F}'(t) = \mathbf{AF}(t)$, with

$$\mathbf{F}(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{bmatrix} \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -3 & 6 \end{bmatrix}.$$

We now use the technique in Theorem 8.9 to solve this first-order system.

A quick calculation yields $p_{\mathbf{A}}(x) = x^3 - 6x^2 + 3x + 10 = (x+1)(x-2)(x-5)$, giving the eigenvalues $\lambda_1 = -1$, $\lambda_2 = 2$, and $\lambda_3 = 5$. Solving for fundamental eigenvectors for each of these eigenvalues, we obtain

$$\begin{aligned} \mathbf{v}_1 &= [1, -1, 1] && \text{corresponding to } \lambda_1 = -1 \\ \mathbf{v}_2 &= [1, 2, 4] && \text{corresponding to } \lambda_2 = 2 \\ \mathbf{v}_3 &= [1, 5, 25]. && \text{corresponding to } \lambda_3 = 5 \end{aligned}$$

Hence, Theorem 8.9 gives us the general solution

$$\begin{aligned} \mathbf{F}(t) &= b_1 e^{-t}[1, -1, 1] + b_2 e^{2t}[1, 2, 4] + b_3 e^{5t}[1, 5, 25] \\ &= [b_1 e^{-t} + b_2 e^{2t} + b_3 e^{5t},\ -b_1 e^{-t} + 2b_2 e^{2t} + 5b_3 e^{5t},\ b_1 e^{-t} + 4b_2 e^{2t} + 25b_3 e^{5t}]. \end{aligned}$$

Since the first entry of this result equals $f_1(t) = y$, the general continuously differentiable solution to the original third-order differential equation is

$$y = b_1 e^{-t} + b_2 e^{2t} + b_3 e^{5t}.$$

The technique used in Example 4 can be generalized to many homogeneous higher-order differential equations $y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$. In Exercise 5(a), you are asked to show that this equation can be represented as a linear system $\mathbf{F}'(t) = \mathbf{AF}(t)$, where $\mathbf{F}(t) = [f_1(t),\ f_2(t), \ldots,\ f_n(t)]$, with $f_1(t) = y$, $f_2(t) = y', \ldots, f_n(t) = y^{(n-1)}$ and where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} \end{bmatrix}.$$

The corresponding linear system can then be solved using Theorem 8.9, as in Example 4.

Several startling patterns were revealed in Example 4. First, notice the similarity between the original differential equation $y''' - 6y'' + 3y' + 10y = 0$ and $p_{\mathbf{A}}(x) = x^3 - 6x^2 + 3x + 10$. This observation leads to the following theorem, which you are asked to prove in Exercise 5(b).

**Theorem 8.10** *If $y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$ is represented as a linear system $\mathbf{F}'(t) = \mathbf{AF}(t)$, where $\mathbf{F}(t)$ and $\mathbf{A}$ are as just described, then*

$$p_{\mathbf{A}}(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

Hence, from now on, we can avoid the long calculations necessary to determine $p_{\mathbf{A}}(x)$. When solving differential equations, $p_{\mathbf{A}}(x)$ is always derived from this shortcut. The equation $p_{\mathbf{A}}(x) = 0$ is called the **characteristic equation** of the original differential equation. The roots of this equation, the eigenvalues of $\mathbf{A}$, are frequently called the **characteristic values** of the differential equation.

Also, notice in Example 4 that the eigenspace $E_\lambda$ for each eigenvalue $\lambda$ is one-dimensional and is spanned by the vector $[1, \lambda, \lambda^2]$. More generally, you are asked to prove the following in Exercise 6:

**Theorem 8.11** *If $y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0$ is represented as a linear system $\mathbf{F}'(t) = \mathbf{AF}(t)$, where $\mathbf{F}(t)$ and $\mathbf{A}$ are as just described, and if $\lambda$ is any eigenvalue for $\mathbf{A}$, then the eigenspace $E_\lambda$ is one-dimensional and is spanned by the vector $[1, \lambda, \lambda^2, \ldots, \lambda^{n-1}]$.*

Combining the preceding theorems, we can state the solution set for many higher-order homogeneous differential equations directly (and avoid linear algebra techniques altogether), as follows:

**Theorem 8.12** *Consider the differential equation*

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_2y'' + a_1y' + a_0y = 0.$$

*Suppose that $\lambda_1, \ldots, \lambda_n$ are $n$ distinct solutions to the characteristic equation*

$$x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0 = 0.$$

*Then all continuously differentiable solutions of the differential equation have the form*

$$y = b_1e^{\lambda_1 t} + b_2e^{\lambda_2 t} + \cdots + b_ne^{\lambda_n t}.$$

**Example 5**

To solve the homogeneous differential equation

$$y'''' + 2y''' - 28y'' - 50y' + 75y = 0,$$

we first find its characteristic values by solving the characteristic equation

$$x^4 + 2x^3 - 28x^2 - 50x + 75 = 0.$$

By factoring, or using an appropriate numerical technique, we obtain four distinct characteristic values. These are $\lambda_1 = -5$, $\lambda_2 = -3$, $\lambda_3 = 1$, and $\lambda_4 = 5$. Thus, by Theorem 8.12, the continuously differentiable solutions for the original differential equation are precisely those functions of the form

$$y = b_1e^{-5t} + b_2e^{-3t} + b_3e^t + b_4e^{5t}.$$

Notice that the technique in Example 5 cannot be used if the differential equation has fewer than $n$ distinct characteristic values. If you can find only $k$ distinct characteristic values for an $n$th-order equation, with $k < n$, then the process yields only a $k$-dimensional subspace of the full $n$-dimensional solution space. As with first-order systems, finding the complete solution set in such a case requires the use of complex eigenvalues, complex eigenvectors, and generalized eigenvectors.

## New Vocabulary

characteristic equation (of a higher-order differential equation)
characteristic values (of a higher-order differential equation)
continuously differentiable function
first-order linear homogeneous system of differential equations
higher-order homogeneous differential equation

## Highlights

- If $\mathbf{A}$ is a diagonalizable $n \times n$ matrix, the continuously differentiable solutions for the first-order system $\mathbf{F}'(t) = \mathbf{AF}(t)$ are $\mathbf{F}(t) = b_1e^{\lambda_1 t}\mathbf{v}_1 + b_2e^{\lambda_2 t}\mathbf{v}_2 + \cdots + b_ne^{\lambda_n t}\mathbf{v}_n$, where $b_1, \ldots, b_n \in \mathbb{R}$, and $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ is an ordered basis for $\mathbb{R}^n$ of eigenvectors for $\mathbf{A}$ corresponding to the (not necessarily distinct) eigenvalues $\lambda_1, \ldots, \lambda_n$.
- If $\mathbf{A}$ is a nondiagonalizable $n \times n$ matrix, solutions for the first-order system $\mathbf{F}'(t) = \mathbf{AF}(t)$ can be found using complex eigenvalues and/or generalized eigenvectors. (Details for solving such systems can be found in the accompanying web section "Solving First-Order Systems of Linear Homogeneous Differential Equations.")
- If the characteristic equation $x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0 = 0$ has $n$ distinct solutions $\lambda_1, \ldots, \lambda_n$, then all continuously differentiable solutions of the differential equation $y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_2y'' + a_1y' + a_0y = 0$ have the form $y = b_1e^{\lambda_1 t} + b_2e^{\lambda_2 t} + \cdots + b_ne^{\lambda_n t}$.

## Exercises for Section 8.9

1. In each part of this exercise, the given matrix represents $\mathbf{A}$ in a first-order system of the form $\mathbf{F}'(t) = \mathbf{AF}(t)$. Use Theorem 8.9 to find the general form of a solution to each system.

★ (a) $\begin{bmatrix} 13 & -28 \\ 6 & -13 \end{bmatrix}$

(b) $\begin{bmatrix} 18 & -15 \\ 20 & -17 \end{bmatrix}$

★ (c) $\begin{bmatrix} 1 & 4 & 4 \\ -1 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

★ (d) $\begin{bmatrix} -5 & -6 & 15 \\ -6 & -5 & 15 \\ -6 & -6 & 16 \end{bmatrix}$

(e) $\begin{bmatrix} -1 & 0 & -2 & 2 \\ -3 & 5 & 1 & -9 \\ 0 & 4 & 5 & -12 \\ -1 & 4 & 3 & -10 \end{bmatrix}$

2. Find the solution set for each given homogeneous differential equation.

★ (a) $y'' + y' - 6y = 0$

(b) $y''' - 5y'' - y' + 5y = 0$

★ (c) $y'''' - 6y'' + 8y = 0$

3. Let $\mathbf{A}$ be an $n \times n$ matrix with linearly independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ corresponding, respectively, to the eigenvalues $\lambda_1, \ldots, \lambda_k$. Prove that

$$\mathbf{F}(t) = b_1 e^{\lambda_1 t}\mathbf{v}_1 + b_2 e^{\lambda_2 t}\mathbf{v}_2 + \cdots + b_k e^{\lambda_k t}\mathbf{v}_k$$

is a solution for the first-order system $\mathbf{F}'(t) = \mathbf{AF}(t)$, for every choice of $b_1, \ldots, b_k \in \mathbb{R}$.

4. This exercise involves initial conditions for first-order systems.

(a) Let $\mathbf{A}$ be a diagonalizable $n \times n$ matrix, and let $\mathbf{v}$ be a fixed vector in $\mathbb{R}^n$. Show there is a unique function $\mathbf{F}(t)$ that satisfies the first-order system $\mathbf{F}'(t) = \mathbf{AF}(t)$ such that $\mathbf{F}(0) = \mathbf{v}$. (The vector $\mathbf{v}$ is called an **initial condition** for the system.)

★ (b) Find the unique solution to $\mathbf{F}'(t) = \mathbf{AF}(t)$ with initial condition $\mathbf{F}(0) = \mathbf{v}$, where

$$\mathbf{A} = \begin{bmatrix} -11 & -6 & 16 \\ -4 & -1 & 4 \\ -12 & -6 & 17 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = [1, -4, 0].$$

5. This exercise is related to Theorem 8.10. (Compare this exercise to Exercise 8 in Section 8.6, if you covered that section.)

(a) Verify that the homogeneous differential equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

can be represented as $\mathbf{F}'(t) = \mathbf{AF}(t)$, where $\mathbf{F}(t) = [f_1(t), f_2(t), \ldots, f_n(t)]$, with $f_1(t) = y$, $f_2(t) = y'$, $\ldots$, $f_n(t) = y^{(n-1)}$, and where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} \end{bmatrix}.$$

► (b) Prove Theorem 8.10. (Hint: Use induction on $n$ and a cofactor expansion on the first column of $(x\mathbf{I}_n - \mathbf{A})$, for the matrix $\mathbf{A}$ in part (a).)

6. This exercise establishes Theorem 8.11. Let $\mathbf{A}$ be the $n \times n$ matrix from Exercise 5, for some $a_0, a_1, \ldots, a_{n-1} \in \mathbb{R}$. (Compare this exercise to Exercise 9 in Section 8.6, if you covered that section.)

(a) Calculate $\mathbf{A}\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$, for a general $n$-vector $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$.

**(b)** Let $\lambda$ be an eigenvalue for $\mathbf{A}$. Show that $[1, \lambda, \lambda^2, \ldots, \lambda^{n-1}]$ is an eigenvector corresponding to $\lambda$. (Hint: Use Theorem 8.10.)

**(c)** Show that if $\mathbf{v}$ is a vector with first coordinate $c$ such that $\mathbf{Av} = \lambda\mathbf{v}$, for some $\lambda \in \mathbb{R}$, then $\mathbf{v} = c[1, \lambda, \lambda^2, \ldots, \lambda^{n-1}]$.

**(d)** Conclude that the eigenspace $E_\lambda$ for an eigenvalue $\lambda$ of $\mathbf{A}$ is always one-dimensional.

★ **7.** True or False:

**(a)** $\mathbf{F}(t) = \mathbf{0}$ is always a solution of $\mathbf{F}'(t) = \mathbf{AF}(t)$.

**(b)** The set of all continuously differentiable solutions of $\mathbf{F}'(t) = \mathbf{AF}(t)$ is a vector space.

**(c)** $\mathbf{F}'(t) = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \mathbf{F}(t)$ has solution set $\left\{ \begin{bmatrix} b_1 e^t + b_2 e^{3t} \\ b_2 e^{3t} \end{bmatrix} \middle| \; b_1, b_2 \in \mathbb{R} \right\}$.

**(d)** $\mathbf{F}'(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{F}(t)$ has no nontrivial solutions because $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is not diagonalizable.

## 8.10 Least-Squares Solutions for Inconsistent Systems

*Prerequisite: Section 6.2, Orthogonal Complements*

When attempting to solve a system of linear equations $\mathbf{Ax} = \mathbf{b}$, there is always the possibility that the system is inconsistent. However, in practical situations, even if no solutions to $\mathbf{Ax} = \mathbf{b}$ exist, it is usually helpful to find an approximate solution; that is, a vector $\mathbf{v}$ such that $\mathbf{Av}$ is as close as possible to $\mathbf{b}$.

### Finding Approximate Solutions

If $\mathbf{A}$ is an $m \times n$ matrix, consider the linear transformation $L\colon \mathbb{R}^n \to \mathbb{R}^m$ given by $L(\mathbf{x}) = \mathbf{Ax}$. If $\mathbf{b} \in \mathbb{R}^m$, then any solution to the linear system $\mathbf{Ax} = \mathbf{b}$ is a pre-image for $\mathbf{b}$ under $L$. However, if $\mathbf{b} \notin \text{range}(L)$, the system is inconsistent, but we can calculate an approximate solution to the system $\mathbf{Ax} = \mathbf{b}$ by finding a pre-image under $L$ of a vector in the subspace $\mathcal{W} = \text{range}(L)$ that is as close as possible to $\mathbf{b}$. Theorem 6.18 implies that, among the vectors in $\mathcal{W}$, $\mathbf{proj}_{\mathcal{W}}\mathbf{b}$ has minimal distance to $\mathbf{b}$. The following theorem shows that $\mathbf{proj}_{\mathcal{W}}\mathbf{b}$ is the *unique* closest vector in $\mathcal{W}$ to $\mathbf{b}$ and that the set of pre-images $L^{-1}(\{\mathbf{proj}_{\mathcal{W}}\mathbf{b}\})$ can be found by solving the linear system $(\mathbf{A}^T\mathbf{A})\mathbf{x} = \mathbf{A}^T\mathbf{b}$.

**Theorem 8.13** *Let* $\mathbf{A}$ *be an* $m \times n$ *matrix, let* $\mathbf{b} \in \mathbb{R}^m$*, and let* $\mathcal{W}$ *be the subspace* $\{\mathbf{Ax} \mid \mathbf{x} \in \mathbb{R}^n\}$*. Then the following three conditions on a vector* $\mathbf{v} \in \mathbb{R}^n$ *are equivalent:*

**(1)** $\mathbf{Av} = \mathbf{proj}_{\mathcal{W}}\mathbf{b}$

**(2)** $\|\mathbf{Av} - \mathbf{b}\| \le \|\mathbf{Az} - \mathbf{b}\|$ *for all* $\mathbf{z} \in \mathbb{R}^n$

**(3)** $(\mathbf{A}^T\mathbf{A})\mathbf{v} = \mathbf{A}^T\mathbf{b}$.

*Such a vector* $\mathbf{v}$ *is called a* ***least-squares solution*** *to the linear system* $\mathbf{Ax} = \mathbf{b}$.

The inequality $\|\mathbf{Av} - \mathbf{b}\| \le \|\mathbf{Az} - \mathbf{b}\|$ in Theorem 8.13 implies that there is no better approximation than $\mathbf{v}$ for a solution to $\mathbf{Ax} = \mathbf{b}$ because the distance from $\mathbf{Av}$ to $\mathbf{b}$ is never larger than the distance from $\mathbf{Az}$ to $\mathbf{b}$ for any other vector $\mathbf{z}$. Of course, if $\mathbf{Ax} = \mathbf{b}$ is consistent, then $\mathbf{v}$ is an actual solution to $\mathbf{Ax} = \mathbf{b}$ (see Exercise 4).

The inequality $\|\mathbf{Av} - \mathbf{b}\| \le \|\mathbf{Az} - \mathbf{b}\|$ also shows why $\mathbf{v}$ is called a least-squares solution. Since calculating a norm involves finding a sum of squares, this inequality implies that the solution $\mathbf{v}$ produces the least possible value for the sum of the squares of the differences in each coordinate between $\mathbf{Az}$ and $\mathbf{b}$ over all possible vectors $\mathbf{z}$.

*Proof.* Let $\mathbf{A}$ and $\mathbf{b}$ be as given in the statement of the theorem, and let $L\colon \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation given by $L(\mathbf{x}) = \mathbf{Ax}$. Then $\mathcal{W} = \{\mathbf{Ax} \mid \mathbf{x} \in \mathbb{R}^n\} = \text{range}(L)$.

Our first goal is to prove (1) if and only if (2). Now, let $\mathbf{Av} = \mathbf{proj}_{\mathcal{W}}\mathbf{b}$. Since $\mathbf{Az} \in \mathcal{W}$, Theorem 6.18 shows that $\|\mathbf{Av} - \mathbf{b}\| \le \|\mathbf{Az} - \mathbf{b}\|$ for all $\mathbf{z} \in \mathbb{R}^n$.

Conversely, suppose $\|\mathbf{Av} - \mathbf{b}\| \le \|\mathbf{Az} - \mathbf{b}\|$ for all $\mathbf{z} \in \mathbb{R}^n$. Let $\mathbf{p} = \mathbf{proj}_{\mathcal{W}}\mathbf{b}$. We need to show that $\mathbf{Av} = \mathbf{p}$. Now $\mathbf{p} \in \mathcal{W}$, so $\mathbf{p}$ is a vector of the form $\mathbf{Az}$ for some $\mathbf{z} \in \mathbb{R}^n$. Hence, $\|\mathbf{Av} - \mathbf{b}\| \le \|\mathbf{p} - \mathbf{b}\|$ by assumption. But $\|\mathbf{p} - \mathbf{b}\| \le \|\mathbf{Av} - \mathbf{b}\|$ by Theorem 6.18. Therefore, $\|\mathbf{Av} - \mathbf{b}\| = \|\mathbf{p} - \mathbf{b}\|$.

Now $\mathbf{Av}, \mathbf{p} \in \mathcal{W}$, so $\mathbf{Av} - \mathbf{p} \in \mathcal{W}$. Also, $\mathbf{p} - \mathbf{b} = -(\mathbf{b} - \mathbf{p}) = -\mathbf{proj}_{\mathcal{W}^\perp}\mathbf{b} \in \mathcal{W}^\perp$, by Corollary 6.16. Thus, $(\mathbf{Av} - \mathbf{p}) \cdot (\mathbf{p} - \mathbf{b}) = 0$. Therefore,

$$\begin{aligned}\|\mathbf{Av}-\mathbf{b}\|^2 &= \|(\mathbf{Av}-\mathbf{p})+(\mathbf{p}-\mathbf{b})\|^2\\ &= ((\mathbf{Av}-\mathbf{p})+(\mathbf{p}-\mathbf{b}))\cdot((\mathbf{Av}-\mathbf{p})+(\mathbf{p}-\mathbf{b}))\\ &= \|\mathbf{Av}-\mathbf{p}\|^2+2(\mathbf{Av}-\mathbf{p})\cdot(\mathbf{p}-\mathbf{b})+\|\mathbf{p}-\mathbf{b}\|^2\\ &= \|\mathbf{Av}-\mathbf{p}\|^2+\|\mathbf{p}-\mathbf{b}\|^2.\end{aligned}$$

But $\|\mathbf{Av}-\mathbf{b}\| = \|\mathbf{p}-\mathbf{b}\|$, implying $\|\mathbf{Av}-\mathbf{p}\|^2 = 0$. Hence, $\mathbf{Av}-\mathbf{p}=\mathbf{0}$, or $\mathbf{Av}=\mathbf{p}$. This completes our first goal.

To finish the proof, we will prove (1) if and only if (3). First, suppose $\mathbf{A}^T\mathbf{Av}=\mathbf{A}^T\mathbf{b}$. We will prove that $\mathbf{Av}=\mathbf{proj}_{\mathcal{W}}\mathbf{b}$. Let $\mathbf{u}=\mathbf{b}-\mathbf{Av}$, and hence $\mathbf{b}=\mathbf{Av}+\mathbf{u}$. If we can show that $\mathbf{Av}\in\mathcal{W}$ and $\mathbf{u}\in\mathcal{W}^\perp$, then we will have $\mathbf{Av}=\mathbf{proj}_{\mathcal{W}}\mathbf{b}$ by the uniqueness assertion in the Projection Theorem (Theorem 6.15). But $\mathbf{Av}\in\mathcal{W}$, since $\mathcal{W}$ consists precisely of vectors of this form. Also, $\mathbf{u}=\mathbf{b}-\mathbf{Av}$, and so $\mathbf{A}^T\mathbf{u}=\mathbf{A}^T\mathbf{b}-\mathbf{A}^T\mathbf{Av}=\mathbf{0}$, since $\mathbf{A}^T\mathbf{Av}=\mathbf{A}^T\mathbf{b}$. Now $\mathbf{A}^T\mathbf{u}=\mathbf{0}$ implies that $\mathbf{u}$ is orthogonal to every row of $\mathbf{A}^T$, and hence $\mathbf{u}$ is orthogonal to every column of $\mathbf{A}$. But recall from Section 5.3 that the columns of $\mathbf{A}$ span $\mathcal{W}=\text{range}(L)$. Hence, $\mathbf{u}\in\mathcal{W}^\perp$ by Theorem 6.10, completing this half of the proof.

Conversely, suppose $\mathbf{Av}=\mathbf{proj}_{\mathcal{W}}\mathbf{b}$. Then $\mathbf{b}=\mathbf{Av}+\mathbf{u}$, where $\mathbf{u}\in\mathcal{W}^\perp$. Hence, $\mathbf{A}^T\mathbf{u}=\mathbf{0}$, since $\mathbf{u}$ must be orthogonal to the rows of $\mathbf{A}^T$, which form a spanning set for $\mathcal{W}$. Therefore,

$$\mathbf{b}=\mathbf{Av}+\mathbf{u} \implies \mathbf{A}^T\mathbf{b}=\mathbf{A}^T\mathbf{Av}+\mathbf{A}^T\mathbf{u} \implies \mathbf{A}^T\mathbf{b}=\mathbf{A}^T\mathbf{Av}.$$

□

**Example 1**

Consider the inconsistent linear system

$$\begin{cases}7x+7y+5z=15\\ 4x \qquad +\ z=\ 1\\ 2x+\ y+\ z=\ 4\\ 5x+8y+5z=16\end{cases}.$$

Letting $\mathbf{A}=\begin{bmatrix}7&7&5\\4&0&1\\2&1&1\\5&8&5\end{bmatrix}$ and $\mathbf{b}=\begin{bmatrix}15\\1\\4\\16\end{bmatrix}$, we will find a least-squares solution to $\mathbf{Ax}=\mathbf{b}$. By part (3) of Theorem 8.13, we need to solve the linear system $\mathbf{A}^T\mathbf{Ax}=\mathbf{A}^T\mathbf{b}$. Now,

$$\mathbf{A}^T\mathbf{A}=\begin{bmatrix}94&91&66\\91&114&76\\66&76&52\end{bmatrix} \quad\text{and}\quad \mathbf{A}^T\mathbf{b}=\begin{bmatrix}197\\237\\160\end{bmatrix}.$$

Row reducing $\left[\begin{array}{ccc|c}94&91&66&197\\91&114&76&237\\66&76&52&160\end{array}\right]$ to obtain $\left[\begin{array}{ccc|c}1&0&0&-7\\0&1&0&-12\\0&0&1&29.5\end{array}\right]$ shows that $\mathbf{v}=[-7,-12,29.5]$ is the desired solution. Notice that

$$\mathbf{Av}=\begin{bmatrix}7&7&5\\4&0&1\\2&1&1\\5&8&5\end{bmatrix}\begin{bmatrix}-7\\-12\\29.5\end{bmatrix}=\begin{bmatrix}14.5\\1.5\\3.5\\16.5\end{bmatrix},$$

and so $\mathbf{Av}$ comes close to producing the vector $\mathbf{b}$.

In fact, for any $\mathbf{z}\in\mathbb{R}^3$, $\|\mathbf{Av}-\mathbf{b}\|\le\|\mathbf{Az}-\mathbf{b}\|$. For example, if $\mathbf{z}=[-11,-19,45]$, the unique solution to the first three equations in the system, then $\|\mathbf{Az}-\mathbf{b}\| = \|[15,1,4,18]-[15,1,4,16]\| = \|[0,0,0,2]\| = 2$. But, $\|\mathbf{Av}-\mathbf{b}\| = \|[14.5,1.5,3.5,16.5]-[15,1,4,16]\| = \|[-0.5,0.5,-0.5,0.5]\| = 1$, which is less than $\|\mathbf{Az}-\mathbf{b}\|$.

## Non-Unique Least-Squares Solutions

Theorem 8.13 shows that if $\mathbf{v}$ is a least-squares solution for a linear system $\mathbf{Ax}=\mathbf{b}$, then $\mathbf{Av}=\mathbf{proj}_{\mathcal{W}}\mathbf{b}$, where $\mathcal{W}=\{\mathbf{Ax}\mid\mathbf{x}\in\mathbb{R}^n\}$. Now, even though $\mathbf{proj}_{\mathcal{W}}\mathbf{b}$ is uniquely determined, there may be more than one vector $\mathbf{v}$ with $\mathbf{Av}=\mathbf{proj}_{\mathcal{W}}\mathbf{b}$. In such a case, there are infinitely many least-squares solutions for $\mathbf{Ax}=\mathbf{b}$, all of which produce the same value for $\mathbf{Ax}$.

**Example 2**

Consider the system $\mathbf{Ax} = \mathbf{b}$, where

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 1 & 3 \\ 2 & -7 & 9 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 9 \\ 8 \\ -1 \end{bmatrix}.$$

We find a least-squares solution to $\mathbf{Ax} = \mathbf{b}$ by solving the linear system $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$. Now,

$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 24 & -4 & 28 \\ -4 & 59 & -63 \\ 28 & -63 & 91 \end{bmatrix} \text{ and } \mathbf{A}^T\mathbf{b} = \begin{bmatrix} 48 \\ 42 \\ 6 \end{bmatrix}.$$

Row reducing $\left[\begin{array}{ccc|c} 24 & -4 & 28 & 48 \\ -4 & 59 & -63 & 42 \\ 28 & -63 & 91 & 6 \end{array}\right]$ to obtain $\left[\begin{array}{ccc|c} 1 & 0 & 1 & \frac{15}{7} \\ 0 & 1 & -1 & \frac{6}{7} \\ 0 & 0 & 0 & 0 \end{array}\right]$ shows that this system has infinitely many solutions. The solution set is $S = \{[\frac{15}{7} - c, \frac{6}{7} + c, c] \mid c \in \mathbb{R}\}$. Two particular solutions are $\mathbf{v}_1 = [\frac{15}{7}, \frac{6}{7}, 0]$, and $\mathbf{v}_2 = [3, 0, -\frac{6}{7}]$. You can verify that $\mathbf{Av}_1 = \mathbf{Av}_2 = \left[\frac{48}{7}, \frac{66}{7}, -\frac{12}{7}\right]$. In general, multiplying $\mathbf{A}$ by any vector in $S$ produces the result $[\frac{48}{7}, \frac{66}{7}, -\frac{12}{7}]$. Every vector in $S$ is a least-squares solution for $\mathbf{Ax} = \mathbf{b}$. They all produce the same result for $\mathbf{Ax}$, which is as close as possible to $\mathbf{b}$.

## Approximate Eigenvalues and Eigenvectors

When solving for eigenvalues and eigenvectors for a square matrix $\mathbf{C}$, a problem can arise if the exact value of an eigenvalue $\lambda$ is not known, but only a close approximation $\lambda'$ instead. Then, since $\lambda'$ is not the precise eigenvalue, the matrix $\lambda'\mathbf{I} - \mathbf{C}$ is nonsingular. This makes it impossible to solve $(\lambda'\mathbf{I} - \mathbf{C})\mathbf{x} = \mathbf{0}$ directly for an eigenvector because only the trivial solution exists. One of several possible approaches to this problem[11] is to use the technique of least-squares to find an approximate eigenvector associated with the approximate eigenvalue $\lambda'$. To do this, first add an extra equation to the system $(\lambda'\mathbf{I} - \mathbf{C})\mathbf{x} = \mathbf{0}$ to force the solution to be nontrivial. One possibility is to require that the sum of the coordinates of the solution equals 1. Even though this new nonhomogeneous system formed is inconsistent, a least-squares solution for this expanded system frequently serves as the desired approximate eigenvector. We illustrate this technique in the following example:

**Example 3**

Consider the matrix

$$\mathbf{C} = \begin{bmatrix} 2 & -3 & -1 \\ 7 & -6 & -1 \\ -16 & 14 & 3 \end{bmatrix},$$

which has eigenvalues $\sqrt{5}$, $-\sqrt{5}$, and $-1$.

Suppose the best estimate we have for the eigenvalue $\lambda = \sqrt{5} \approx 2.23606$ is $\lambda' = \frac{9}{4} = 2.25$. Then

$$\lambda'\mathbf{I}_3 - \mathbf{C} = \begin{bmatrix} \frac{1}{4} & 3 & 1 \\ -7 & \frac{33}{4} & 1 \\ 16 & -14 & -\frac{3}{4} \end{bmatrix},$$

which is nonsingular. (Its determinant is $\frac{13}{64}$.) Hence, the system $(\lambda'\mathbf{I}_3 - \mathbf{C})\mathbf{x} = \mathbf{0}$ has only the trivial solution. We now force a nontrivial solution $\mathbf{x}$ by adding the condition that the sum of the coordinates of $\mathbf{x}$ equals 1. This produces the system

[11] Numerical techniques exist for finding approximate eigenvectors that produce more accurate results than the method of least-squares. The major problem with the least-squares technique is that the accuracy of the approximate eigenvector is limited by the accuracy of the approximate eigenvalue used. Other numerical methods, such as an adaptation of the inverse power method, are iterative and adjust the approximation for the eigenvalue while solving for the eigenvector. For more information on the inverse power method and other numerical techniques for solving for eigenvalues and eigenvectors, consult a text on numerical methods in your library. One classic text is *Numerical Analysis*, 10th ed., by Burden, Faires, and Burden (published by Cengage, 2015).

$$\begin{cases} \frac{1}{4}x_1 + \ \ 3x_2 + \ \ x_3 = 0 \\ -7x_1 + \frac{33}{4}x_2 + \ \ x_3 = 0 \\ 16x_1 - 14x_2 - \frac{3}{4}x_3 = 0 \\ \ \ x_1 + \ \ \ x_2 + \ \ x_3 = 1 \end{cases}.$$

However, this new system is inconsistent since the first three equations together have only the trivial solution, which does not satisfy the last equation. We will find a least-squares solution to this system. Let

$$\mathbf{A} = \begin{bmatrix} \frac{1}{4} & 3 & 1 \\ -7 & \frac{33}{4} & 1 \\ 16 & -14 & -\frac{3}{4} \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Then

$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} \frac{4897}{16} & -280 & -\frac{71}{4} \\ -280 & \frac{4385}{16} & \frac{91}{4} \\ -\frac{71}{4} & \frac{91}{4} & \frac{57}{16} \end{bmatrix} \quad \text{and} \quad \mathbf{A}^T\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Row reducing

$$\left[\begin{array}{ccc|c} \frac{4897}{16} & -280 & -\frac{71}{4} & 1 \\ -280 & \frac{4385}{16} & \frac{91}{4} & 1 \\ -\frac{71}{4} & \frac{91}{4} & \frac{57}{16} & 1 \end{array}\right] \quad \text{produces} \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & -0.50 \\ 0 & 1 & 0 & -0.69 \\ 0 & 0 & 1 & 2.19 \end{array}\right],$$

where we have rounded the results to two places after the decimal point. Hence, $\mathbf{v} = [-0.50, -0.69, 2.19]$ is an approximate eigenvector for $\mathbf{C}$ corresponding to the approximate eigenvalue $\lambda' = \frac{9}{4}$. In fact, $(\lambda'\mathbf{I}_3 - \mathbf{C})\mathbf{v} = [-0.005, -0.0025, 0.0175]$, which is close to the zero vector. This implies that $\mathbf{Cv}$ is very close to $\lambda'\mathbf{v}$. In fact, the maximum difference among the three coordinates ($\approx 0.0175$) is about the same magnitude as the error in the estimation of the eigenvalue ($\approx 0.01394$). Also, a lengthy computation would show that the unit vector $\mathbf{v}/\left\|\mathbf{v}\right\| \approx [-0.21, -0.29, 0.93]$ agrees with an actual unit eigenvector for $\mathbf{C}$ corresponding to $\lambda = \sqrt{5}$ in every coordinate, after rounding to the first two places after the decimal point.

There may be a problem with the technique described in Example 3 if the actual eigenspace for $\lambda$ is orthogonal to the vector $\mathbf{t} = [1, 1, \ldots, 1]$ since our added requirement implies that the dot product of the approximate eigenvector with $\mathbf{t}$ equals 1. If this problem arises, simply change the requirement to specify that the dot product with any nonzero vector of your choice (other than $\mathbf{t}$) equals 1 and try again.[12]

## Least-Squares Polynomials

In Theorem 8.4 of Section 8.3, we presented a technique for finding a polynomial function $\mathbf{p}$ in $\mathcal{P}_k$ that comes closest to passing through a given set of data points $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$. This produces a linear system whose intended solution is a polynomial that passes through all $n$ data points. However, if the desired degree $k$ of the polynomial is less than $n + 1$, then the linear system is inconsistent (in most cases). Thus, we find that a least-squares solution to the system produces a **least-squares polynomial** that approximates the given data.

Theorem 8.4 is actually a corollary of Theorem 8.13 in this section. We ask you to prove Theorem 8.4 in Exercise 6 using Theorem 8.13. See Section 8.3 if you have further interest in least-squares polynomials.

## New Vocabulary

least-squares polynomial (for a given set of data points)

least-squares solution to the linear system $\mathbf{Ax} = \mathbf{b}$

[12] For a more detailed technical analysis of the process of finding approximate eigenvectors using the method of least-squares, see "Using Least-Squares to Find an Approximate Eigenvector," Electronic Journal of Linear Algebra, Volume 16, pp. 99–110, 2007, by D. Hecker and D. Lurie at https://doi.org/10.13001/1081-3810.1186.

## Highlights

- If the linear system $\mathbf{Ax} = \mathbf{b}$ is inconsistent (that is, if $\mathbf{b}$ is not in the range of $L(\mathbf{x}) = \mathbf{Ax}$), then a least-squares solution to $\mathbf{Ax} = \mathbf{b}$ is a vector $\mathbf{v}$ for which $\mathbf{Av}$ is as close as possible to $\mathbf{b}$.
- If $\mathbf{A}$ is an $m \times n$ matrix, and $L(\mathbf{x}) = \mathbf{Ax}$, then for any vector $\mathbf{b} \in \mathbb{R}^n$, the vector $\mathbf{proj}_{\mathcal{W}}\mathbf{b}$ is the *unique* closest vector to $\mathbf{b}$ in $\mathcal{W} = \text{range}(L)$.
- The least-squares solutions to $\mathbf{Ax} = \mathbf{b}$ are the solutions of the linear system $(\mathbf{A}^T\mathbf{A})\mathbf{x} = \mathbf{A}^T\mathbf{b}$. (These are precisely the vectors that map to $\mathbf{proj}_{\mathcal{W}}\mathbf{b}$ under $L(\mathbf{x}) = \mathbf{Ax}$, where $\mathcal{W} = \text{range}(L)$.)
- If $\mathbf{C}$ is a matrix having an eigenvalue $\lambda$, but only an approximate value $\lambda'$ for $\lambda$ is known, then approximate eigenvectors for $\lambda$ can often be obtained by finding least-squares solutions to $(\lambda'\mathbf{I} - \mathbf{C})\mathbf{x} = \mathbf{0}$ under an additional constraint such as $x_1 + x_2 + \cdots + x_n = 1$ (to ensure a nontrivial solution).

## Exercises for Section 8.10

We strongly recommend that you use a computer or calculator to help you perform the required computations in these exercises.

1. In each part, find the set of all least-squares solutions for the linear system $\mathbf{Ax} = \mathbf{b}$ for the given matrix $\mathbf{A}$ and vector $\mathbf{b}$. If there is more than one least-squares solution, find at least two particular least-squares solutions. Finally, illustrate the inequality $\|\mathbf{Av} - \mathbf{b}\| \leq \|\mathbf{Az} - \mathbf{b}\|$ by computing $\|\mathbf{Av} - \mathbf{b}\|$ for a particular least-squares solution $\mathbf{v}$ and $\|\mathbf{Az} - \mathbf{b}\|$ for the given vector $\mathbf{z}$.

★ (a) $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 4 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix}, \mathbf{z} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(b) $\mathbf{A} = \begin{bmatrix} 5 & 2 \\ 3 & 1 \\ 4 & 3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 12 \\ 15 \\ 14 \end{bmatrix}, \mathbf{z} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

★ (c) $\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 2 & 5 \\ 1 & 0 & -7 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ 6 \end{bmatrix}, \mathbf{z} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

(d) $\mathbf{A} = \begin{bmatrix} 3 & 1 & 0 & -1 \\ 5 & 3 & 2 & 2 \\ 2 & 2 & 2 & 3 \\ 7 & 5 & 4 & 5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ 14 \\ 10 \\ 25 \end{bmatrix}, \mathbf{z} = \begin{bmatrix} -1 \\ 6 \\ 0 \\ 0 \end{bmatrix}$

2. In practical applications, we are frequently interested in only those solutions having nonnegative entries in every coordinate. In each part, find the set of all such least-squares solutions to the linear system $\mathbf{Ax} = \mathbf{b}$ for the given matrix $\mathbf{A}$ and vector $\mathbf{b}$.

★ (a) $\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 3 & -2 & 4 \\ 7 & 0 & 4 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

(b) $\mathbf{A} = \begin{bmatrix} 2 & 3 & 3 \\ 1 & -1 & 1 \\ 1 & 9 & 3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 5 \\ 2 \\ 7 \end{bmatrix}$

3. In each part, find an approximate eigenvector $\mathbf{v}$ for the given matrix $\mathbf{C}$ corresponding to the given approximate eigenvalue $\lambda'$ using the technique in Example 3. Round the entries of $\mathbf{v}$ to two places after the decimal point. Then compute $(\lambda'\mathbf{I} - \mathbf{C})\mathbf{v}$ to estimate the error in your answer.

(a) $\mathbf{C} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}, \lambda' = \frac{15}{4}$

★ (b) $\mathbf{C} = \begin{bmatrix} 3 & -3 & -2 \\ -5 & 5 & 4 \\ 11 & -12 & -9 \end{bmatrix}, \lambda' = \frac{3}{2}$

(c) $\mathbf{C} = \begin{bmatrix} 1 & 18 & -7 \\ -1 & 12 & -5 \\ -3 & 32 & -13 \end{bmatrix}, \lambda' = \frac{9}{4}$

4. Prove that if a linear system $\mathbf{Ax} = \mathbf{b}$ is consistent, then the set of least-squares solutions for the system equals the set of actual solutions.
5. Let $\mathbf{A}$ be an $m \times n$ matrix, and let $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^n$. Prove that if $\mathbf{A}^T\mathbf{A}\mathbf{v}_1 = \mathbf{A}^T\mathbf{A}\mathbf{v}_2$, then $\mathbf{A}\mathbf{v}_1 = \mathbf{A}\mathbf{v}_2$.

► 6. Use Theorem 8.13 to prove Theorem 8.4 in Section 8.3.

★ 7. True or False:

(a) A least-squares solution to an inconsistent system is a vector $\mathbf{v}$ that satisfies as many equations in the system as possibly can be satisfied.

(b) For any matrix $\mathbf{A}$, the matrix $\mathbf{A}^T\mathbf{A}$ is square and symmetric.

**(c)** Every system $\mathbf{Ax} = \mathbf{b}$ must have at least one least-squares solution.
**(d)** If $\mathbf{v}_1$ and $\mathbf{v}_2$ are both least-squares solutions to $\mathbf{Ax} = \mathbf{b}$, then $\mathbf{Av}_1 = \mathbf{Av}_2$.
**(e)** In this section, the least-squares process is applied to solve for eigenvectors in cases in which only an estimate of the eigenvalue is known.

## 8.11 Quadratic Forms

*Prerequisite: Section 6.3, Orthogonal Diagonalization*

In Section 8.7, we used a change of coordinates to simplify a general second-degree equation (conic section) in two variables $x$ and $y$. In this section, we generalize this process to any finite number of variables, using orthogonal diagonalization.

### Quadratic Forms

**Definition** A **quadratic form** on $\mathbb{R}^n$ is a function $Q\colon \mathbb{R}^n \to \mathbb{R}$ of the form

$$Q([x_1, \ldots, x_n]) = \sum_{1 \le i \le j \le n} c_{ij} x_i x_j,$$

for some real numbers $c_{ij}$, $1 \le i \le j \le n$.

Thus, a quadratic form on $\mathbb{R}^n$ is a polynomial in $n$ variables in which each term has degree 2.

**Example 1**

The function $Q_1([x_1, x_2, x_3]) = 7x_1^2 + 5x_1x_2 - 6x_2^2 + 9x_2x_3 + 14x_3^2$ is a quadratic form on $\mathbb{R}^3$. $Q_1$ is a polynomial in three variables in which each term has degree 2. Note that the coefficient $c_{13}$ of the $x_1x_3$ term is zero.

The function $Q_2([x, y]) = 8x^2 - 3y^2 + 12xy$ is a quadratic form on $\mathbb{R}^2$ with coefficients $c_{11} = 8$, $c_{22} = -3$, and $c_{12} = 12$. On $\mathbb{R}^2$, a quadratic form consists of the $x^2$, $y^2$, and $xy$ terms from the general form for the equation of a conic section.

In general, a quadratic form $Q$ on $\mathbb{R}^n$ can be expressed as $Q(\mathbf{x}) = \mathbf{x}^T\mathbf{Cx}$, where $\mathbf{x}$ is a column matrix and $\mathbf{C}$ is the upper triangular matrix whose entries on and above the main diagonal are given by the coefficients $c_{ij}$ in the definition of a quadratic form above. For example, the quadratic forms $Q_1$ and $Q_2$ in Example 1 can be expressed as

$$Q_1\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = [x_1, x_2, x_3]\begin{bmatrix} 7 & 5 & 0 \\ 0 & -6 & 9 \\ 0 & 0 & 14 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{and}$$

$$Q_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = [x, y]\begin{bmatrix} 8 & 12 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

However, this representation for a quadratic form is not the most useful one for our purposes. Instead, we will replace the upper triangular matrix $\mathbf{C}$ with a symmetric matrix $\mathbf{A}$.

**Theorem 8.14** *Let $Q\colon \mathbb{R}^n \to \mathbb{R}$ be a quadratic form. Then there is a unique symmetric $n \times n$ matrix $\mathbf{A}$ such that $Q(\mathbf{x}) = \mathbf{x}^T\mathbf{Ax}$.*

*Proof.* (Abridged) The uniqueness of the matrix $\mathbf{A}$ in the theorem is unimportant in what follows. Its proof is left for you to provide in Exercise 3.

To prove the existence of $\mathbf{A}$, let $Q([x_1, \ldots, x_n]) = \sum_{1 \le i \le j \le n} c_{ij} x_i x_j$. If $\mathbf{C} = [c_{ij}]$ is the upper triangular matrix of coefficients for $Q$, then define $\mathbf{A} = \frac{1}{2}\left(\mathbf{C} + \mathbf{C}^T\right)$. Notice that $\mathbf{A}$ is symmetric (see Theorem 1.14 in Section 1.4). A straightforward calculation of $\mathbf{x}^T\mathbf{Ax}$ shows that the coefficient of its $x_ix_j$ term is $c_{ij}$. (Verify.) Hence, $\mathbf{x}^T\mathbf{Ax} = Q(\mathbf{x})$. □

---

**Example 2**

Consider the quadratic form $Q_3\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = 17x_1^2 + 8x_1x_2 - 9x_2^2$. Then the corresponding symmetric matrix $\mathbf{A}$ for $Q_3$ is $\begin{bmatrix}c_{11} & \frac{1}{2}c_{12}\\ \frac{1}{2}c_{12} & c_{22}\end{bmatrix} = \begin{bmatrix}17 & 4\\ 4 & -9\end{bmatrix}$. You can verify that

$$Q_3\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = [x_1, x_2]\begin{bmatrix}17 & 4\\ 4 & -9\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}.$$

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## Orthogonal Change of Basis

The next theorem indicates how the symmetric matrix for a quadratic form is altered when we perform an orthogonal change of coordinates.

**Theorem 8.15** *Let $Q\colon \mathbb{R}^n \to \mathbb{R}$ be a quadratic form given by $Q(x) = \mathbf{x}^T\mathbf{A}\mathbf{x}$, for some symmetric matrix $\mathbf{A}$. Let $B$ be an orthonormal basis for $\mathbb{R}^n$. Let $\mathbf{P}$ be the transition matrix from $B$-coordinates to standard coordinates, and let $\mathbf{K} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$. Then $\mathbf{K}$ is symmetric and $Q(\mathbf{x}) = [\mathbf{x}]_B^T\mathbf{K}[\mathbf{x}]_B$.*

*Proof.* Since $B$ is an orthonormal basis, $\mathbf{P}$ is an orthogonal matrix by Theorem 6.8. Hence, $\mathbf{P}^{-1} = \mathbf{P}^T$. Now, $[\mathbf{x}]_B = \mathbf{P}^{-1}\mathbf{x} = \mathbf{P}^T\mathbf{x}$, and thus, $[\mathbf{x}]_B^T = (\mathbf{P}^T\mathbf{x})^T = \mathbf{x}^T\mathbf{P}$. Therefore,

$$Q(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x} \;=\; \mathbf{x}^T\mathbf{P}\mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{P}^{-1}\mathbf{x} \;=\; [\mathbf{x}]_B^T\mathbf{P}^{-1}\mathbf{A}\mathbf{P}\,[\mathbf{x}]_B\,.$$

Letting $\mathbf{K} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$, we have $Q(\mathbf{x}) = [\mathbf{x}]_B^T\mathbf{K}[\mathbf{x}]_B$. Finally, notice that $\mathbf{K}$ is symmetric, since

$$\mathbf{K}^T = \left(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}\right)^T = \left(\mathbf{P}^T\mathbf{A}\mathbf{P}\right)^T = \mathbf{P}^T\mathbf{A}^T\left(\mathbf{P}^T\right)^T = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{K}.$$

□

---

**Example 3**

Consider the quadratic form $Q([x, y, z]) = 2xy + 4xz + 2yz - y^2 + 3z^2$. Then

$$Q\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = [x, y, z]\,\mathbf{A}\begin{bmatrix}x\\y\\z\end{bmatrix}, \quad \text{where} \quad \mathbf{A} = \begin{bmatrix}0 & 1 & 2\\ 1 & -1 & 1\\ 2 & 1 & 3\end{bmatrix}.$$

Consider the orthonormal basis $B = \left(\frac{1}{3}[2, 1, 2],\ \frac{1}{3}[2, -2, -1],\ \frac{1}{3}[1, 2, -2]\right)$ for $\mathbb{R}^3$. We will find the symmetric matrix for $Q$ with respect to this new basis $B$.

The transition matrix from $B$-coordinates to standard coordinates is the orthogonal matrix

$$\mathbf{P} = \frac{1}{3}\begin{bmatrix}2 & 2 & 1\\ 1 & -2 & 2\\ 2 & -1 & -2\end{bmatrix} \quad \text{and so} \quad \mathbf{P}^{-1} = \mathbf{P}^T = \frac{1}{3}\begin{bmatrix}2 & 1 & 2\\ 2 & -2 & -1\\ 1 & 2 & -2\end{bmatrix}.$$

Then,

$$\mathbf{K} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \frac{1}{9}\begin{bmatrix}35 & -7 & -11\\ -7 & -13 & 4\\ -11 & 4 & -4\end{bmatrix}.$$

Let $[u, v, w]$ be the representation of the vector $[x, y, z]$ in $B$-coordinates; that is, $[x, y, z]_B = [u, v, w]$. Then, by Theorem 8.15, $Q$ can be expressed as

$$Q\left(\begin{bmatrix} u \\ v \\ w \end{bmatrix}\right) = [u, v, w]\left(\frac{1}{9}\begin{bmatrix} 35 & -7 & -11 \\ -7 & -13 & 4 \\ -11 & 4 & -4 \end{bmatrix}\right)\begin{bmatrix} u \\ v \\ w \end{bmatrix}$$
$$= \frac{35}{9}u^2 - \frac{13}{9}v^2 - \frac{4}{9}w^2 - \frac{14}{9}uv - \frac{22}{9}uw + \frac{8}{9}vw.$$

Let us check this formula for $Q$ in a particular case. If $[x, y, z] = [9, 2, -1]$, then the original formula for $Q$ yields

$$Q([9, 2, -1]) = (2)(9)(2) + (4)(9)(-1) + (2)(2)(-1) - (2)^2 + (3)(-1)^2 = -5.$$

On the other hand,

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ -1 \end{bmatrix}_B = \mathbf{P}^{-1}\begin{bmatrix} 9 \\ 2 \\ -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & 1 & 2 \\ 2 & -2 & -1 \\ 1 & 2 & -2 \end{bmatrix}\begin{bmatrix} 9 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 5 \end{bmatrix}.$$

Calculating $Q$ using the formula for $B$-coordinates, we get

$$Q([u, v, w]) = \frac{35}{9}(6)^2 - \frac{13}{9}(5)^2 - \frac{4}{9}(5)^2 - \frac{14}{9}(6)(5) - \frac{22}{9}(6)(5) + \frac{8}{9}(5)(5) = -5,$$

which agrees with our previous calculation for $Q$.

## The Principal Axes Theorem

We are now ready to prove the main result of this section—given any quadratic form $Q$ on $\mathbb{R}^n$, an orthonormal basis $B$ for $\mathbb{R}^n$ can be chosen so that the expression for $Q$ in $B$-coordinates contains no "mixed-product" terms (that is, $Q$ contains only "square" terms).

**Theorem 8.16** **(Principal Axes Theorem)** *Let $Q\colon \mathbb{R}^n \to \mathbb{R}$ be a quadratic form. Then there is an orthonormal basis $B$ for $\mathbb{R}^n$ such that $Q(\mathbf{x}) = [\mathbf{x}]_B^T\mathbf{D}[\mathbf{x}]_B$ for some diagonal matrix $\mathbf{D}$. That is, if $[\mathbf{x}]_B = \mathbf{y} = [y_1, y_2, \ldots, y_n]$, then*

$$Q(\mathbf{x}) = d_{11}y_1^2 + d_{22}y_2^2 + \cdots + d_{nn}y_n^2.$$

*Proof.* Let $Q$ be a quadratic form on $\mathbb{R}^n$. Then by Theorem 8.14, there is a symmetric $n \times n$ matrix $\mathbf{A}$ such that $Q(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}$. Now, by Corollary 6.23, $\mathbf{A}$ can be orthogonally diagonalized; that is, there is an orthogonal matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$ is diagonal. Let $B$ be the orthonormal basis for $\mathbb{R}^n$ given by the columns of $\mathbf{P}$. Then Theorem 8.15 implies that $Q(\mathbf{x}) = [\mathbf{x}]_B^T\mathbf{D}[\mathbf{x}]_B$. □

The process of finding a diagonal matrix for a given quadratic form $Q$ is referred to as **diagonalizing** $Q$. We now outline the method for diagonalizing a quadratic form, as presented in the proof of Theorem 8.16.

**Method for Diagonalizing a Quadratic Form (Quadratic Form Method)**
Given a quadratic form $Q\colon \mathbb{R}^n \to \mathbb{R}$,

**Step 1**: Find a symmetric $n \times n$ matrix $\mathbf{A}$ such that $Q(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}$.

**Step 2**: Apply Steps 3 through 8 of the Orthogonal Diagonalization Method in Section 6.3, using the matrix $\mathbf{A}$. This process yields an orthonormal basis $B$, an orthogonal matrix $\mathbf{P}$ whose columns are the vectors in $B$, and a diagonal matrix $\mathbf{D}$ with $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$.

**Step 3**: Then $Q(\mathbf{x}) = [\mathbf{x}]_B^T\,\mathbf{D}\,[\mathbf{x}]_B$, with $[\mathbf{x}]_B = \mathbf{P}^{-1}\mathbf{x} = \mathbf{P}^T\mathbf{x}$. If $[\mathbf{x}]_B = [y_1, y_2, \ldots, y_n]$, then $Q(\mathbf{x}) = d_{11}y_1^2 + d_{22}y_2^2 + \cdots + d_{nn}y_n^2$.

**Example 4**

Let $Q([x, y, z]) = \frac{1}{121}(183x^2 + 266y^2 + 35z^2 + 12xy + 408xz + 180yz)$. We will diagonalize $Q$.

**Step 1**: Note that $Q(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is the symmetric matrix

$$\frac{1}{121}\begin{bmatrix} 183 & 6 & 204 \\ 6 & 266 & 90 \\ 204 & 90 & 35 \end{bmatrix}.$$

**Step 2**: We apply Steps 3 through 8 of the Orthogonal Diagonalization Method to $\mathbf{A}$. We list the results here but leave the details of the calculations for you to check.

**(3)** A quick computation gives

$$p_{\mathbf{A}}(x) = x^3 - 4x^2 + x + 6 = (x-3)(x-2)(x+1).$$

Therefore, the eigenvalues of $\mathbf{A}$ are $\lambda_1 = 3$, $\lambda_2 = 2$, and $\lambda_3 = -1$.

**(4)** Next, we find a basis for each eigenspace for $\mathbf{A}$. To find a basis for $E_{\lambda_1}$, we solve the system $(3\mathbf{I}_3 - \mathbf{A})\mathbf{x} = \mathbf{0}$, which yields the basis $\{[7, 6, 6]\}$. Similarly, we solve appropriate systems to find:

$$\text{Basis for } E_{\lambda_2} = \{[6, -9, 2]\}$$
$$\text{Basis for } E_{\lambda_3} = \{[6, 2, -9]\}.$$

**(5)** Since each eigenspace from (4) is one-dimensional, we need only normalize each basis vector to find orthonormal bases for $E_{\lambda_1}$, $E_{\lambda_2}$, and $E_{\lambda_3}$:

$$\text{Orthonormal basis for } E_{\lambda_1} = \left\{\tfrac{1}{11}[7, 6, 6]\right\}$$
$$\text{Orthonormal basis for } E_{\lambda_2} = \left\{\tfrac{1}{11}[6, -9, 2]\right\}$$
$$\text{Orthonormal basis for } E_{\lambda_3} = \left\{\tfrac{1}{11}[6, 2, -9]\right\}.$$

**(6)** Let $B$ be the ordered orthonormal basis $\left(\frac{1}{11}[7, 6, 6], \frac{1}{11}[6, -9, 2], \frac{1}{11}[6, 2, -9]\right)$.

**(7)** The desired diagonal matrix for $Q$ with respect to the basis $B$ is

$$\mathbf{D} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

which has eigenvalues $\lambda_1 = 3$, $\lambda_2 = 2$, and $\lambda_3 = -1$ along the main diagonal.

**(8)** The transition matrix $\mathbf{P}$ from $B$-coordinates to standard coordinates is the matrix whose columns are the vectors in $B$—namely,

$$\mathbf{P} = \frac{1}{11}\begin{bmatrix} 7 & 6 & 6 \\ 6 & -9 & 2 \\ 6 & 2 & -9 \end{bmatrix}.$$

Of course, $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$. In this case, $\mathbf{P}$ is not only orthogonal but is symmetric as well, so $\mathbf{P}^{-1} = \mathbf{P}^T = \mathbf{P}$. (Be careful! $\mathbf{P}$ will not always be symmetric.)

This concludes Step 2.

**Step 3**: Let $[x, y, z]_B = [u, v, w]$. Then using $\mathbf{D}$, we have $Q = 3u^2 + 2v^2 - w^2$. Notice that $Q$ has only "square" terms, since $\mathbf{D}$ is diagonal.

For a particular example, let $[x, y, z] = [2, 6, -1]$. Then

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \mathbf{P}^{-1}\begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 7 & 6 & 6 \\ 6 & -9 & 2 \\ 6 & 2 & -9 \end{bmatrix}\begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 3 \end{bmatrix}.$$

Hence, $Q([2, 6, -1]) = 3(4)^2 + 2(-4)^2 - (3)^2 = 71$. As an independent check, notice that plugging $[2, 6, -1]$ into the original equation for $Q$ produces the same result.

## New Vocabulary

diagonalizing a quadratic form
positive definite quadratic form
positive semidefinite quadratic form
Principal Axes Theorem
quadratic form
Quadratic Form Method

## Highlights

- A quadratic form on $\mathbb{R}^n$ is a function $Q\colon \mathbb{R}^n \to \mathbb{R}$ of the form $Q([x_1, \ldots, x_n]) = \sum_{1 \le i \le j \le n} c_{ij} x_i x_j$, for some real numbers $c_{ij}$, $1 \le i \le j \le n$.
- For any quadratic form $Q\colon \mathbb{R}^n \to \mathbb{R}$, there is a unique symmetric $n \times n$ matrix $\mathbf{A}$ such that $Q(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}$.
- Let $Q\colon \mathbb{R}^n \to \mathbb{R}$ be a quadratic form given by $Q(x) = \mathbf{x}^T\mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is symmetric. If $B$ is an orthonormal basis for $\mathbb{R}^n$, and $\mathbf{P}$ is the transition matrix from $B$-coordinates to standard coordinates, then $Q(\mathbf{x}) = [\mathbf{x}]_B^T \mathbf{K} [\mathbf{x}]_B$, where $\mathbf{K} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is a symmetric matrix.
- By the Principal Axes Theorem, every quadratic form has a representation with no mixed-product terms. In particular, if $Q\colon \mathbb{R}^n \to \mathbb{R}$ is a quadratic form, there is an orthonormal basis $B$ for $\mathbb{R}^n$ for $Q$ such that $Q(\mathbf{x}) = [\mathbf{x}]_B^T \mathbf{D} [\mathbf{x}]_B$, for some diagonal matrix $\mathbf{D}$. Consequently, if $[\mathbf{x}]_B = [y_1, y_2, \ldots, y_n]$, then $Q(\mathbf{x}) = d_{11}y_1^2 + d_{22}y_2^2 + \cdots + d_{nn}y_n^2$.

## Exercises for Section 8.11

1. In each part of this exercise, a quadratic form $Q\colon \mathbb{R}^n \to \mathbb{R}$ is given. Find an upper triangular matrix $\mathbf{C}$ and a symmetric matrix $\mathbf{A}$ such that, for every $\mathbf{x} \in \mathbb{R}^n$, $Q(\mathbf{x}) = \mathbf{x}^T\mathbf{C}\mathbf{x} = \mathbf{x}^T\mathbf{A}\mathbf{x}$.
   ★ (a) $Q([x, y]) = 8x^2 - 9y^2 + 12xy$
   (b) $Q([x, y]) = 7x^2 + 11y^2 - 17xy$
   ★ (c) $Q([x_1, x_2, x_3]) = 5x_1^2 - 2x_2^2 + 4x_1x_2 - 3x_1x_3 + 5x_2x_3$
2. In each part of this exercise, use the Quadratic Form Method to diagonalize the given quadratic form $Q\colon \mathbb{R}^n \to \mathbb{R}$. Your answers should include the matrices $\mathbf{A}$, $\mathbf{P}$, and $\mathbf{D}$ defined in that method, as well as the orthonormal basis $B$. Finally, calculate $Q(\mathbf{x})$ for the given vector $\mathbf{x}$ in the following two different ways: first, using the given formula for $Q$, and second, calculating $Q = [\mathbf{x}]_B^T \mathbf{D} [\mathbf{x}]_B$ where $[\mathbf{x}]_B = \mathbf{P}^{-1}\mathbf{x}$ and $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$.
   ★ (a) $Q([x, y]) = 43x^2 + 57y^2 - 48xy$; $\mathbf{x} = [1, -8]$
   (b) $Q([x_1, x_2, x_3]) = -5x_1^2 + 37x_2^2 + 49x_3^2 + 32x_1x_2 + 80x_1x_3 + 32x_2x_3$; $\mathbf{x} = [7, -2, 1]$
   ★ (c) $Q([x_1, x_2, x_3]) = 18x_1^2 - 68x_2^2 + x_3^2 + 96x_1x_2 - 60x_1x_3 + 36x_2x_3$; $\mathbf{x} = [4, -3, 6]$
   (d) $Q([x_1, x_2, x_3, x_4]) = x_1^2 + 5x_2^2 + 864x_3^2 + 864x_4^2 - 24x_1x_3 + 24x_1x_4 + 120x_2x_3 + 120x_2x_4 + 1152x_3x_4$; $\mathbf{x} = [5, 9, -3, -2]$
3. Let $Q\colon \mathbb{R}^n \to \mathbb{R}$ be a quadratic form, and let $\mathbf{A}$ and $\mathbf{B}$ be symmetric matrices such that $Q(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x} = \mathbf{x}^T\mathbf{B}\mathbf{x}$. Prove that $\mathbf{A} = \mathbf{B}$ (the uniqueness assertion from Theorem 8.14). (Hint: Use $\mathbf{x} = \mathbf{e}_i$ to show that $a_{ii} = b_{ii}$. Then use $\mathbf{x} = \mathbf{e}_i + \mathbf{e}_j$ to prove that $a_{ij} = b_{ij}$ when $i \neq j$.)
★ 4. Let $Q\colon \mathbb{R}^n \to \mathbb{R}$ be a quadratic form. Is the upper triangular representation for $Q$ necessarily unique? That is, if $\mathbf{C}_1$ and $\mathbf{C}_2$ are upper triangular $n \times n$ matrices with $Q(\mathbf{x}) = \mathbf{x}^T\mathbf{C}_1\mathbf{x} = \mathbf{x}^T\mathbf{C}_2\mathbf{x}$, for all $\mathbf{x} \in \mathbb{R}^n$, must $\mathbf{C}_1 = \mathbf{C}_2$? Prove your answer.
5. A quadratic form $Q(\mathbf{x})$ on $\mathbb{R}^n$ is **positive definite** if and only if both of the following conditions hold:
   (i) $Q(\mathbf{x}) \ge 0$, for all $\mathbf{x} \in \mathbb{R}^n$.
   (ii) $Q(\mathbf{x}) = 0$ if and only if $\mathbf{x} = \mathbf{0}$.
   A quadratic form having only property (i) is said to be **positive semidefinite**.
   Let $Q$ be a quadratic form on $\mathbb{R}^n$, and let $\mathbf{A}$ be the symmetric matrix such that $Q(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}$.
   (a) Prove that $Q$ is positive definite if and only if every eigenvalue of $\mathbf{A}$ is positive.
   (b) Prove that $Q$ is positive semidefinite if and only if every eigenvalue of $\mathbf{A}$ is nonnegative.
★ 6. True or False:
   (a) If $Q(\mathbf{x}) = \mathbf{x}^T\mathbf{C}\mathbf{x}$ is a quadratic form, and $\mathbf{A} = \frac{1}{2}(\mathbf{C} + \mathbf{C}^T)$, then $Q(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}$.
   (b) $Q(x, y) = xy$ is not a quadratic form because it has no $x^2$ or $y^2$ terms.
   (c) If $\mathbf{x}^T\mathbf{A}\mathbf{x} = \mathbf{x}^T\mathbf{B}\mathbf{x}$ for every $\mathbf{x} \in \mathbb{R}^n$, then $\mathbf{A} = \mathbf{B}$.
   (d) Every quadratic form can be diagonalized.
   (e) If $\mathbf{A}$ is a symmetric matrix and $Q(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}$ is a quadratic form that diagonalizes to $Q(\mathbf{x}) = [\mathbf{x}]_B^T \mathbf{D} [\mathbf{x}]_B$, then the main diagonal entries of $\mathbf{D}$ are the eigenvalues of $\mathbf{A}$.

Chapter 9

# Numerical Techniques

**A Calculating Mindset**

In this chapter, we present several additional computational techniques that are widely used in numerical linear algebra. When performing calculations, exact solutions are not always possible because we often round our results at each step, enabling roundoff errors to accumulate readily. In such cases, we may only be able to obtain the desired numerical answer within a certain margin of error. In what follows, we will examine some iterative processes that can minimize such roundoff errors—in particular, for solving systems of linear equations or for finding eigenvalues. We will also examine three methods (**LDU** Decomposition, **QR** Factorization, and Singular Value Decomposition) for factoring a matrix into a product of simpler matrices—techniques that are particularly useful for solving certain linear systems.

Throughout this text, we have urged the use of a calculator or computer with appropriate software to perform tedious calculations once you have mastered a computational technique. The algorithms discussed in this chapter are especially facilitated by employing a calculator or computer to save time and decrease drudgery.

## 9.1 Numerical Techniques for Solving Systems

*Prerequisite: Section 2.3, Equivalent Systems, Rank, and Row Space*

In this section, we discuss some considerations for solving linear systems by calculator or computer and investigate some alternate techniques for solving systems, including partial pivoting, the Jacobi Method, and the Gauss-Seidel Method.

### Computational Accuracy

One basic problem in using a computational device in linear algebra is that real numbers cannot always be represented exactly in its memory. Because the physical storage space of any device is limited, a predetermined amount of space is assigned in the memory for the storage of any real number. Thus, only the most significant digits of any real number can be stored.[1] Nonterminating decimals, such as $\frac{1}{3} = 0.333333\ldots$ or $e = 2.718281828459045\ldots$, can never be represented fully. Using the first few decimal places of such numbers may be enough for most practical purposes, but it is not completely accurate.

As calculations are performed, all computational results are truncated and rounded to fit within the limited storage space allotted. Numerical errors caused by this process are called **roundoff errors**. Unfortunately, if many operations are performed, roundoff errors can compound, thus producing a significant error in the final result. This is one reason that Gaussian Elimination is computationally more accurate than the Gauss-Jordan Method. Since fewer arithmetic operations generally need to be performed, Gaussian Elimination allows less chance for roundoff errors to compound.

### Ill-Conditioned Systems

Sometimes the number of significant digits used in computations has a great effect on the answers. For example, consider the similar systems

$$\text{(A)} \begin{cases} 2x_1 + x_2 = 2 \\ 2.005x_1 + x_2 = 7 \end{cases} \quad \text{and} \quad \text{(B)} \begin{cases} 2x_1 + x_2 = 2 \\ 2.01x_1 + x_2 = 7 \end{cases}.$$

The linear equations of these systems are graphed in Fig. 9.1.

[1] The first $n$ significant digits of a decimal number are its leftmost $n$ digits, beginning with the first nonzero digit. For example, consider the real numbers $r_1 = 47.26835$, $r_2 = 9.00473$, and $r_3 = 0.000456$. Approximating these by stopping after the first three significant digits and rounding to the nearest digit, we get $r_1 \approx 47.3$, $r_2 \approx 9.00$, and $r_3 \approx 0.000456$ (since the first nonzero digit in $r_3$ is 4).

Elementary Linear Algebra. https://doi.org/10.1016/B978-0-12-822978-1.00019-5

Even though the coefficients of systems (A) and (B) are almost identical, the solutions to the systems are very different.

$$\text{Solution to (A)} = (1000, -1998) \quad \text{and} \quad \text{solution to (B)} = (500, -998).$$

Systems like these, in which a very small change in a coefficient leads to a very large change in the solution set, are called **ill-conditioned systems**. In this case, there is a geometric way to see that these systems are ill-conditioned: the pair of lines in each system are almost parallel. Therefore, a small change in one line can move the point of intersection very far along the other line, as in Fig. 9.1.

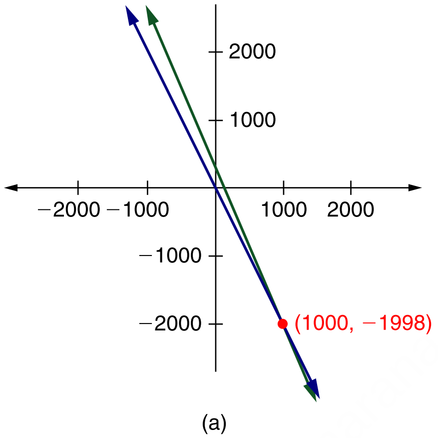



**FIGURE 9.1** (a) Lines of system (A); (b) lines of system (B)

Suppose the coefficients in system (A) had been obtained after a series of long calculations. A slight difference in the roundoff error of those calculations could have led to a very different final solution set. Thus, we need to be very careful when working with ill-conditioned systems. Special methods have been developed for recognizing ill-conditioned systems, and a technique known as **iterative refinement** is used when the coefficients are known only to a certain degree of accuracy. These methods are beyond the scope of this book, but further details can be found in *Numerical Analysis*, 10th ed., by Burden, Faires, and Burden (published by Cengage, 2015).

## Partial Pivoting

A common problem in numerical linear algebra occurs when dividing by real numbers that are very close to zero—for example, during the row reduction process when the entry to be converted to a pivot is extremely small. This small number might be inaccurate itself because of a previous roundoff error. Performing a Type (I) row operation with this number might result in additional roundoff error.

Even when dealing with accurate small numbers, we can still have problems. When we divide every entry of a row by a very small number, the remaining row entries could become much larger (in absolute value) than the other matrix entries. Then, when these larger row entries are added to the (smaller) entries of another row in a Type (II) operation, the most significant digits of the larger row entries may not be affected at all. That is, the data stored in smaller row entries may not be playing their proper role in determining the final solution set. As more computations are performed, these roundoff errors can accumulate, making the final result inaccurate.

### Example 1

Consider the linear system

$$\begin{cases} 0.0006x_1 - \phantom{30}x_2 + \phantom{5}x_3 = 10 \\ \phantom{0.}0.03x_1 + 30x_2 - 5x_3 = 15 \\ \phantom{0.}0.04x_1 + 40x_2 - 7x_3 = 19 \end{cases}.$$

The unique solution is $(x_1, x_2, x_3) = (5000, -4, 3)$. But if we attempt to solve the system by row reduction and round all computations to four significant figures, we get an inaccurate result. For example, using Gaussian Elimination on

$$\left[\begin{array}{ccc|c} 0.0006 & -1 & 1 & 10 \\ 0.03 & 30 & -5 & 15 \\ 0.04 & 40 & -7 & 19 \end{array}\right],$$

we obtain:

| **Row Operations** | **Resulting Matrices** |
|---|---|
| (I): $\langle 1\rangle \leftarrow (1/0.0006)\,\langle 1\rangle$ | $\left[\begin{array}{ccc\|c} 1 & -1667 & 1667 & 16670 \\ 0.03 & 30 & -5 & 15 \\ 0.04 & 40 & -7 & 19 \end{array}\right]$ |
| (II): $\langle 2\rangle \leftarrow -0.03\,\langle 1\rangle + \langle 2\rangle$ <br> (II): $\langle 3\rangle \leftarrow -0.04\,\langle 1\rangle + \langle 3\rangle$ | $\left[\begin{array}{ccc\|c} 1 & -1667 & 1667 & 16670 \\ 0 & 80.01 & -55.01 & -485.1 \\ 0 & 106.7 & -73.68 & -647.8 \end{array}\right]$ |
| (I): $\langle 2\rangle \leftarrow (1/80.01)\,\langle 2\rangle$ | $\left[\begin{array}{ccc\|c} 1 & -1667 & 1667 & 16670 \\ 0 & 1 & -0.6876 & -6.064 \\ 0 & 106.7 & -73.68 & -647.8 \end{array}\right]$ |
| (II): $\langle 3\rangle \leftarrow -106.7\,\langle 2\rangle + \langle 3\rangle$ | $\left[\begin{array}{ccc\|c} 1 & -1667 & 1667 & 16670 \\ 0 & 1 & -0.6876 & -6.064 \\ 0 & 0 & -0.3131 & -0.7712 \end{array}\right]$ |
| (I): $\langle 3\rangle \leftarrow (-1/0.3131) + \langle 3\rangle$ | $\left[\begin{array}{ccc\|c} 1 & -1667 & 1667 & 16670 \\ 0 & 1 & -0.6876 & -6.064 \\ 0 & 0 & 1 & 2.463 \end{array}\right].$ |

Back substitution produces the solution $(x_1, x_2, x_3) = (5279, -4.370, 2.463)$. This inaccurate answer is largely the result of dividing row 1 through by 0.0006, a number much smaller than the other entries of the matrix, in the first step of the row reduction.

A technique known as **partial pivoting** can be employed to avoid roundoff errors like those encountered in Example 1. Whenever we move to a new pivot column, we first determine whether the entry that would normally become the next pivot has any entries *below* it with a *greater* absolute value. If so, choose the entry among these having the *maximum absolute value*. (If two or more entries have the maximum absolute value, choose any one of those.) Then we switch rows to place the chosen entry into the desired pivot position before continuing the row reduction process.

### Example 2

We use partial pivoting on the system in Example 1. The initial augmented matrix is

$$\left[\begin{array}{ccc|c} 0.0006 & -1 & 1 & 10 \\ 0.03 & 30 & -5 & 15 \\ 0.04 & 40 & -7 & 19 \end{array}\right].$$

The entry in the first column with the largest absolute value is in the third row, so we interchange the first and third rows to obtain

$$\text{(III): } \langle 1\rangle \leftrightarrow \langle 3\rangle \qquad \left[\begin{array}{ccc|c} 0.04 & 40 & -7 & 19 \\ 0.03 & 30 & -5 & 15 \\ 0.0006 & -1 & 1 & 10 \end{array}\right].$$

Continuing the row reduction, we obtain

| **Row Operations** | **Resulting Matrices** |
|---|---|
| (I): $\langle 1\rangle \leftarrow (1/0.04)\,\langle 1\rangle$ | $\left[\begin{array}{ccc\|c} 1 & 1000 & -175.0 & 475.0 \\ 0.03 & 30 & -5 & 15 \\ 0.0006 & -1 & 1 & 10 \end{array}\right]$ |
| (II): $\langle 2\rangle \leftarrow -0.03\,\langle 1\rangle + \langle 2\rangle$ <br> (II): $\langle 3\rangle \leftarrow -0.0006\,\langle 1\rangle + \langle 3\rangle$ | $\left[\begin{array}{ccc\|c} 1 & 1000 & -175.0 & 475.0 \\ 0 & 0.000 & 0.2500 & 0.7500 \\ 0 & -1.600 & 1.105 & 9.715 \end{array}\right]$ |
| (III): $\langle 2\rangle \leftrightarrow \langle 3\rangle$ | $\left[\begin{array}{ccc\|c} 1 & 1000 & -175.0 & 475.0 \\ 0 & -1.600 & 1.105 & 9.715 \\ 0 & 0 & 0.2500 & 0.7500 \end{array}\right]$ |
| (I): $\langle 2\rangle \leftarrow (-1/1.600)\,\langle 2\rangle$ | $\left[\begin{array}{ccc\|c} 1 & 1000 & -175.0 & 475.0 \\ 0 & 1 & -0.6906 & -6.072 \\ 0 & 0 & 0.2500 & 0.7500 \end{array}\right]$ |
| (I): $\langle 3\rangle \leftarrow (1/0.2500)\,\langle 3\rangle$ | $\left[\begin{array}{ccc\|c} 1 & 1000 & -175.0 & 475.0 \\ 0 & 1 & -0.6906 & -6.072 \\ 0 & 0 & 1 & 3.000 \end{array}\right].$ |

Back substitution produces the solution $(x_1, x_2, x_3) = (5000, -4.000, 3.000)$. Therefore, by partial pivoting, we have obtained the correct solution, a big improvement over the answer obtained in Example 1 without partial pivoting.

For many systems, the technique of partial pivoting is powerful enough to provide reasonably accurate answers. However, in more difficult cases, partial pivoting is not enough. An even more useful technique is **total pivoting** (also called **full pivoting** or **complete pivoting**), in which columns as well as rows are interchanged. The strategy in total pivoting is to select the entry with the largest absolute value from all the remaining rows and columns to become the next pivot.

## Iterative Techniques: Jacobi and Gauss-Seidel Methods

When we have a rough approximation of the unique solution to a certain $n \times n$ linear system, an **iterative method** may be the fastest way to obtain the actual solution. We use the initial approximation to generate a second (preferably better) approximation. We then use the second approximation to generate a third, and so on. The process stops if the approximations "stabilize"—that is, if the difference between successive approximations becomes negligible. In this section, we illustrate the following two iterative methods: the **Jacobi Method** and the **Gauss-Seidel Method**.

For these iterative methods, it is convenient to express linear systems in a slightly different form. Suppose we are given the following system of $n$ equations in $n$ unknowns:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\ \quad\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n = b_n \end{cases}.$$

If the coefficient matrix has rank $n$, every row and column of the reduced row echelon form of the coefficient matrix contains a pivot. In this case, it is always possible to rearrange the equations so that the coefficient of $x_i$ is nonzero in the $i$th equation, for $1 \le i \le n$. Let us assume that the equations have already been rearranged in this way. Solving for $x_i$ in the $i$th equation in terms of the remaining unknowns, we obtain

$$\begin{cases} x_1 = \qquad\qquad c_{12}x_2 + c_{13}x_3 + \cdots + c_{1n}x_n + d_1 \\ x_2 = c_{21}x_1 \qquad\quad + c_{23}x_3 + \cdots + c_{2n}x_n + d_2 \\ x_3 = c_{31}x_1 + c_{32}x_2 \qquad\quad + \cdots + c_{3n}x_n + d_3 \\ \;\vdots \quad \vdots \qquad \vdots \qquad \vdots \;\; \vdots \qquad\quad \vdots \qquad \vdots \\ x_n = c_{n1}x_1 + c_{n2}x_2 + c_{n3}x_3 + \cdots \qquad\quad + d_n \end{cases},$$

where each $c_{ij}$ and $d_i$ represents a new coefficient obtained after we reorder the equations and solve for each $x_i$.

For example, suppose we are given the system

$$\begin{cases} 3x_1 - 2x_2 + \phantom{0}x_3 = \phantom{-}11 \\ 2x_1 + 7x_2 - 3x_3 = -14 \\ 9x_1 - \phantom{0}x_2 - 4x_3 = \phantom{-}17 \end{cases}.$$

Solving for $x_1$ in the first equation, $x_2$ in the second equation, and $x_3$ in the third equation, we obtain

$$\begin{cases} x_1 = \qquad\quad \frac{2}{3}x_2 - \frac{1}{3}x_3 + \frac{11}{3} \\ x_2 = -\frac{2}{7}x_1 \qquad + \frac{3}{7}x_3 - 2 \\ x_3 = \frac{9}{4}x_1 - \frac{1}{4}x_2 \qquad - \frac{17}{4} \end{cases}.$$

We now introduce the Jacobi Method.

**Method for Solving a Linear System Iteratively (Jacobi Method)**

Assume a linear system of $n$ equations in $n$ variables $x_1, x_2, \ldots, x_n$ is given whose coefficient matrix has rank $n$.

**Step 1:** First, express the given system in the following form:

$$\begin{cases} x_1 = \quad\quad c_{12}x_2 + c_{13}x_3 + \cdots + c_{1n}x_n + d_1 \\ x_2 = c_{21}x_1 \quad\quad + c_{23}x_3 + \cdots + c_{2n}x_n + d_2 \\ x_3 = c_{31}x_1 + c_{32}x_2 \quad\quad + \cdots + c_{3n}x_n + d_3 \\ \ \vdots \quad\ \vdots \quad\quad \vdots \quad\quad \vdots \quad \vdots \quad\quad\quad \vdots \quad\quad \vdots \\ x_n = c_{n1}x_1 + c_{n2}x_2 + c_{n3}x_3 + \cdots \quad\quad + d_n \end{cases}$$

**Step 2:** Substitute an initial approximation for $x_1, x_2, \ldots, x_n$ into the right-hand side to obtain new values for $x_1, x_2, \ldots, x_n$ on the left-hand side.

**Step 3:** Substitute the most recently obtained values for $x_1, x_2, \ldots, x_n$ into the right-hand side to obtain a newer set of values on the left-hand side. Repeat this step unless the values on the left-hand side have "stabilized."

**Step 4:** The final set of values for $x_1, x_2, \ldots, x_n$ represent a good approximation for a solution to the linear system.

## Example 3

We solve

$$\begin{cases} 8x_1 + \ \ x_2 - \ \ 2x_3 = -11 \\ 2x_1 + 9x_2 + \quad x_3 = \ \ 22 \\ -x_1 - 2x_2 + 11x_3 = -15 \end{cases}$$

with the Jacobi Method. The true solution is $(x_1, x_2, x_3) = (-2, 3, -1)$. Let us use $x_1 = -1.5$, $x_2 = 2.5$, and $x_3 = -0.5$ as an initial approximation (or guess) of the solution.

First, we rewrite the system in the form

$$\begin{cases} x_1 = -\frac{1}{8}x_2 + \ \frac{1}{4}x_3 - \frac{11}{8} \\ x_2 = -\frac{2}{9}x_1 - \ \frac{1}{9}x_3 + \frac{22}{9} \\ x_3 = \ \frac{1}{11}x_1 + \frac{2}{11}x_2 - \frac{15}{11} \end{cases}.$$

In the following calculations, we round all results to three decimal places. Plugging the initial guess into the right-hand side of each equation, we get

$$\begin{cases} x_1 = \quad -\frac{1}{8}(2.5) + \frac{1}{4}(-0.5) - \frac{11}{8} \\ x_2 = -\frac{2}{9}(-1.5) - \frac{1}{9}(-0.5) + \frac{22}{9} \\ x_3 = \ \frac{1}{11}(-1.5) + \ \frac{2}{11}(2.5) - \frac{15}{11} \end{cases},$$

yielding the new values $x_1 = -1.813$, $x_2 = 2.833$, $x_3 = -1.045$. We then plug these values into the right-hand side of each equation to obtain

$$\begin{cases} x_1 = \quad -\frac{1}{8}(2.833) + \frac{1}{4}(-1.045) - \frac{11}{8} \\ x_2 = -\frac{2}{9}(-1.813) - \frac{1}{9}(-1.045) + \frac{22}{9} \\ x_3 = \ \frac{1}{11}(-1.813) + \ \frac{2}{11}(2.833) - \frac{15}{11} \end{cases},$$

yielding the values $x_1 = -1.990$, $x_2 = 2.963$, $x_3 = -1.013$. Repeating this process, we get the values in the following chart:

| | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|
| Initial values | −1.500 | 2.500 | −0.500 |
| After 1 step | −1.813 | 2.833 | −1.045 |
| After 2 steps | −1.990 | 2.963 | −1.013 |
| After 3 steps | −1.999 | 2.999 | −1.006 |
| After 4 steps | −2.001 | 3.000 | −1.000 |
| After 5 steps | −2.000 | 3.000 | −1.000 |
| After 6 steps | −2.000 | 3.000 | −1.000 |

After six steps, the values for $x_1$, $x_2$, and $x_3$ have stabilized at the true solution.

In Example 3, we could have used any starting values for $x_1$, $x_2$, and $x_3$ as the initial approximation. In the absence of any information about the solution, we can begin with $x_1 = x_2 = x_3 = 0$. If we use the Jacobi Method on the system in Example 3 with $x_1 = x_2 = x_3 = 0$ as the initial values, we obtain the following chart (again, rounding each result to three decimal places):

| | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|
| Initial values | 0.000 | 0.000 | 0.000 |
| After 1 step | −1.375 | 2.444 | −1.364 |
| After 2 steps | −2.022 | 2.902 | −1.044 |
| After 3 steps | −1.999 | 3.010 | −1.020 |
| After 4 steps | −2.006 | 3.002 | −0.998 |
| After 5 steps | −2.000 | 3.001 | −1.000 |
| After 6 steps | −2.000 | 3.000 | −1.000 |
| After 7 steps | −2.000 | 3.000 | −1.000 |

In this case, the Jacobi Method still produces the correct solution, although an extra step is required.

The **Gauss-Seidel Method** is similar to the Jacobi Method except that as each new value $x_i$ is obtained, it is used immediately in place of the previous value for $x_i$ when plugging values into the right-hand side of the equations.

**Alternate Method for Solving a Linear System Iteratively (Gauss-Seidel Method)**

Assume a linear system having $n$ equations and $n$ variables $x_1, x_2, \ldots, x_n$ is given, whose coefficient matrix has rank $n$.

**Step 1:** First, express the given system in the form indicated in Step 1 of the Jacobi Method.

**Step 2:** Substitute an initial approximation for $x_1, x_2, \ldots, x_n$ into the first equation on the right-hand side to obtain a new value for $x_1$ on the left-hand side. Continue substituting for $x_1, x_2, \ldots, x_n$ into each of the remaining equations, in order, but using the most recently obtained value for each $x_i$ each time.

**Step 3:** Revisit each of the equations in order. At the $i$th equation, substitute the most recently obtained values for $x_1, x_2, \ldots, x_n$ into that equation on the right-hand side to obtain a newer value for $x_i$ on the left-hand side. Repeat this step unless the values on the left-hand side have "stabilized."

**Step 4:** The final set of values for $x_1, x_2, \ldots, x_n$ represent a good approximation for a solution to the linear system.

### Example 4

Consider the system

$$\begin{cases} 8x_1 + \phantom{9}x_2 - \phantom{1}2x_3 = -11 \\ 2x_1 + 9x_2 + \phantom{11}x_3 = \phantom{-}22 \\ -x_1 - 2x_2 + 11x_3 = -15 \end{cases}$$

of Example 3. We solve this system with the Gauss-Seidel Method, using the initial approximation $x_1 = x_2 = x_3 = 0$. Again, we begin by rewriting the system in the form

$$\begin{cases} x_1 = -\frac{1}{8}x_2 + \frac{1}{4}x_3 - \frac{11}{8} \\ x_2 = -\frac{2}{9}x_1 - \frac{1}{9}x_3 + \frac{22}{9} \\ x_3 = \frac{1}{11}x_1 + \frac{2}{11}x_2 - \frac{15}{11} \end{cases}.$$

Plugging the initial approximation into the right-hand side of the first equation, we get

$$x_1 = -\frac{1}{8}(0) + \frac{1}{4}(0) - \frac{11}{8} = -1.375.$$

We now plug this new value for $x_1$ and the current value for $x_3$ into the right-hand side of the second equation to get

$$x_2 = -\frac{2}{9}(-1.375) - \frac{1}{9}(0) + \frac{22}{9} = 2.750.$$

We then plug the new values for $x_1$ and $x_2$ into the right-hand side of the third equation to get

$$x_3 = \frac{1}{11}(-1.375) + \frac{2}{11}(2.750) - \frac{15}{11} = -0.989.$$

The process is then repeated as many times as necessary with the newest values of $x_1$, $x_2$, and $x_3$ used in each case. The results are given in the following chart (rounding all results to three decimal places):

| | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|
| Initial values | 0.000 | 0.000 | 0.000 |
| After 1 step | −1.375 | 2.750 | −0.989 |
| After 2 steps | −1.966 | 2.991 | −0.999 |
| After 3 steps | −1.999 | 3.000 | −1.000 |
| After 4 steps | −2.000 | 3.000 | −1.000 |
| After 5 steps | −2.000 | 3.000 | −1.000 |

After five steps, we see that the values for $x_1$, $x_2$, and $x_3$ have stabilized to the correct solution.

For certain classes of linear systems, the Jacobi and Gauss-Seidel Methods will always stabilize to the correct solution for any given initial approximation (see Exercise 7). In most ordinary applications, the Gauss-Seidel Method takes fewer steps than the Jacobi Method, but for some systems, the Jacobi Method is superior to the Gauss-Seidel Method. However, for other systems, neither method produces the correct answer (see Exercise 8).[2]

## Comparing Iterative and Row Reduction Methods

When are iterative methods useful? A major advantage of iterative methods is that roundoff errors are not given a chance to "accumulate," as they are in Gaussian Elimination and the Gauss-Jordan Method, because each iteration essentially creates a new approximation to the solution. The only roundoff error that we need to consider with an iterative method is the error involved in the most recent step.

Also, in many applications, the coefficient matrix for a given system contains a large number of zeroes. Such matrices are said to be **sparse**. When a linear system has a sparse matrix, each equation in the system may involve very few variables. If so, each step of the iterative process is relatively easy. However, neither the Gauss-Jordan Method nor Gaussian Elimination would be very attractive in such a case because the cumulative effect of many row operations would tend to replace the zero coefficients with nonzero numbers. But even if the coefficient matrix is not sparse, iterative methods often give more accurate answers when large matrices are involved because fewer arithmetic operations are performed overall.

On the other hand, when iterative methods take an extremely large number of steps to stabilize or do not stabilize at all, it is much better to use the Gauss-Jordan Method or Gaussian Elimination.

## New Vocabulary

Gauss-Seidel Method
ill-conditioned systems
iterative methods
Jacobi Method
partial pivoting
roundoff errors
sparse (coefficient) matrix
total (full, complete) pivoting

## Highlights

- The partial pivoting technique is used to avoid roundoff errors that could be caused when dividing a row by an entry that is relatively small in comparison to its remaining row entries.
- In partial pivoting, for each new pivot column in turn, check whether there is an entry having a greater absolute value in that column *below* the current pivot row. If so, choose the entry among these having the *maximum absolute value*. (If two or more entries have the maximum absolute value, choose any one of those.) Then we switch rows to place the chosen entry into the desired pivot position before continuing the row reduction process.
- In both the Jacobi Method and the Gauss-Seidel Method, to find a solution to a linear system with variables $x_1, x_2, \ldots, x_n$, the equations are first rearranged so that in the $i$th equation the coefficient of $x_i$ is nonzero, and rewritten so that $x_i$ is expressed in terms of the other variables. Beginning with an initial guess at the solution, new values for $x_1, x_2, \ldots, x_n$ are

[2] In cases where the Jacobi and Gauss-Seidel Methods do not stabilize, related iterative techniques (known as **relaxation methods**) may still work. For further details, see *Numerical Analysis*, 10th ed., by Burden, Faires, and Burden (published by Cengage, 2015).

repeatedly obtained by substituting the current values of $x_1, x_2, \ldots, x_n$ into the equations. The Methods are successful if the values for $x_1, x_2, \ldots, x_n$ eventually stabilize, thereby producing the correct solution.

- In each iteration of the Jacobi Method, the current values for $x_1, x_2, \ldots, x_n$ are substituted into every equation in the system *simultaneously* to obtain the next set of values for $x_1, x_2, \ldots, x_n$. But, in the Gauss-Seidel Method, the substitution is carried out *one equation at a time*, repeatedly cycling through the equations in order, with the most recent $x_i$ value obtained from the $i$th equation immediately replacing the former $x_i$ value before proceeding to the next substitution.
- The Gauss-Seidel Method generally takes fewer steps to stabilize, but there are linear systems for which the Jacobi Method is superior.
- A major advantage of iterative methods is that roundoff errors are not compounded. Iterative methods are often effective on sparse matrices.

## Exercises for Section 9.1

**Note**: You should use a calculator or appropriate computer software to solve these problems.

1. In each part of this exercise, find the exact solution sets for the two given systems. Are the systems ill-conditioned? Why or why not?

★ (a) $\begin{cases} 5x - 2y = 10 \\ 5x - 1.995y = 17.5 \end{cases}, \quad \begin{cases} 5x - 2y = 10 \\ 5x - 1.99y = 17.5 \end{cases}$

(b) $\begin{cases} 6x - z = 400 \\ 3y - z = 400 \\ 25x + 12y - 8z = 3600 \end{cases}, \quad \begin{cases} 6x - 1.01z = 400 \\ 3y - z = 400 \\ 25x + 12y - 8z = 3600 \end{cases}$

2. First, use Gaussian Elimination *without* partial pivoting to solve each of the following systems. Then, solve each system using Gaussian Elimination *with* partial pivoting. Which solution is more accurate? In each case, round all numbers in the problem to three significant digits before beginning, and round the results after each row operation is performed to three significant digits.

★ (a) $\begin{cases} 0.00072x - 4.312y = -0.9846 \\ 2.31x - 9876.0y = -130.8 \end{cases}$

(b) $\begin{cases} 0.0004x_1 - 0.6234x_2 - 2.123x_3 = 5.581 \\ 0.0832x_1 - 26.17x_2 - 1.759x_3 = -3.305 \\ 0.09512x_1 + 0.1458x_2 + 55.13x_3 = 11.168 \end{cases}$

★ (c) $\begin{cases} 0.00032x_1 + 0.2314x_2 + 0.127x_3 = -0.03456 \\ -241x_1 - 217x_2 - 8x_3 = -576 \\ 49x_1 + 45x_2 + 2.4x_3 = 283.2 \end{cases}$

3. Repeat Exercise 2, but round all computations to four significant digits.
4. Solve each of the following systems using the Jacobi Method. Round all results to three decimal places, and stop when successive values of the variables agree to three decimal places. Let the initial values of all variables be zero. List the values of the variables after each step of the iteration.

★ (a) $\begin{cases} 5x_1 + x_2 = 26 \\ 3x_1 + 7x_2 = -42 \end{cases}$

(b) $\begin{cases} 9x_1 - x_2 - x_3 = -7 \\ 2x_1 - 8x_2 - x_3 = 35 \\ x_1 + 2x_2 + 11x_3 = 22 \end{cases}$

★ (c) $\begin{cases} 7x_1 + x_2 - 2x_3 = -62 \\ -x_1 + 6x_2 + x_3 = 27 \\ 2x_1 - x_2 - 6x_3 = 26 \end{cases}$

(d) $\begin{cases} 10x_1 + x_2 - 2x_3 + x_4 = 9 \\ -x_1 - 9x_2 + x_3 - 2x_4 = 15 \\ -2x_1 + x_2 + 7x_3 + x_4 = 21 \\ x_1 - x_2 - x_3 + 13x_4 = -27 \end{cases}$

5. Repeat Exercise 4 using the Gauss-Seidel Method instead of the Jacobi Method.

★ 6. A square matrix is **strictly diagonally dominant** if the absolute value of each diagonal entry is larger than the sum of the absolute values of the remaining entries in its row. That is, if **A** is an $n \times n$ matrix, then **A** is strictly diagonally dominant if, for $1 \le i \le n$, $|a_{ii}| > \sum_{\substack{1 \le j \le n \\ j \ne i}} |a_{ij}|$. Which of the following matrices are strictly diagonally dominant?

**(a)** $\begin{bmatrix} -3 & 1 \\ -2 & 4 \end{bmatrix}$

**(b)** $\begin{bmatrix} 2 & 2 \\ 4 & 3 \end{bmatrix}$

**(c)** $\begin{bmatrix} -6 & 2 & 1 \\ 2 & 5 & -2 \\ -1 & 4 & 7 \end{bmatrix}$

**(d)** $\begin{bmatrix} 15 & 9 & -3 \\ 3 & 6 & 4 \\ 7 & -2 & 11 \end{bmatrix}$

**(e)** $\begin{bmatrix} 6 & 2 & 3 \\ 4 & 5 & 1 \\ 7 & 1 & 9 \end{bmatrix}$

**7.** The Jacobi and Gauss-Seidel Methods stabilize to the correct solution (for any choice of initial values) if the equations can be rearranged to make the coefficient matrix for the system strictly diagonally dominant (see Exercise 6). For the following systems, rearrange the equations accordingly, and then perform the Gauss-Seidel Method. Use initial values of zero for all variables. Round all results to three decimal places. List the values of the variables after each step of the iteration, and give the final solution set in each case.

★ **(a)** $\begin{cases} 2x_1 + 13x_2 + x_3 = 0 \\ x_1 - 2x_2 + 15x_3 = 26 \\ 8x_1 - x_2 + 3x_3 = 25 \end{cases}$

**(b)** $\begin{cases} -3x_1 - x_2 - 7x_3 = -39 \\ 10x_1 + x_2 + x_3 = 37 \\ x_1 + 9x_2 + 2x_3 = -58 \end{cases}$

★ **(c)** $\begin{cases} x_1 + x_2 + 13x_3 + 2x_4 = 120 \\ 9x_1 + 2x_2 - x_3 + x_4 = 49 \\ -2x_1 + 3x_2 - x_3 - 14x_4 = 110 \\ -x_1 - 17x_2 - 3x_3 + 2x_4 = 86 \end{cases}$

★ **8.** Show that neither the Jacobi Method nor the Gauss-Seidel Method seems to stabilize when applied to the following system by observing what happens during the first six steps of the Jacobi Method and the first four steps of the Gauss-Seidel Method. Let the initial value of all variables be zero, and round all results to three decimal places. Then find the solution using Gaussian Elimination.

$$\begin{cases} x_1 - 5x_2 - x_3 = 16 \\ 6x_1 - x_2 - 2x_3 = 13 \\ 7x_1 + x_2 + x_3 = 12 \end{cases}$$

**9.** This exercise compares the stability of the Gauss-Seidel and Jacobi Methods for a particular system.

**(a)** For the following system, show that with initial values of zero for each variable, the Gauss-Seidel Method stabilizes to the correct solution. Round all results to three decimal places, and give the values of the variables after each step of the iteration.

$$\begin{cases} 2x_1 + x_2 + x_3 = 7 \\ x_1 + 2x_2 + x_3 = 8 \\ x_1 + x_2 + 2x_3 = 9 \end{cases}$$

**(b)** Work out the first eight steps of the Jacobi Method for the system in part (a) (again using initial values of zero for each variable), and observe that this method does not stabilize. (On alternate passes, the results oscillate between values near $x_1 = 3$, $x_2 = 4$, $x_3 = 5$, and $x_1 = -1$, $x_2 = 0$, $x_3 = 1$.)

★ **10.** True or False:

**(a)** Roundoff error occurs when fewer digits are used to represent a number than are actually required.

**(b)** An ill-conditioned system of linear equations is a system in which some of the coefficients are unknown.

**(c)** In partial pivoting, we use row swaps to ensure that each entry to be converted into a pivot is as small as possible in absolute value.

**(d)** Iterative methods generally tend to introduce less roundoff error than Gauss-Jordan row reduction.

**(e)** In the Jacobi Method, the new value of $x_i$ is immediately used to compute $x_{i+1}$ (for $i < n$) on the same iteration.

**(f)** The first approximate solution obtained using initial values of 0 for all variables in the system $\begin{cases} x - 2y = 6 \\ 2x + 3y = 15 \end{cases}$ using the Gauss-Seidel Method is $x = 6$, $y = 5$.

## 9.2 LDU Decomposition

*Prerequisite: Section 2.4, Inverses of Matrices*

In this section, we show that many nonsingular matrices can be written as the product of a lower triangular matrix **L**, a diagonal matrix **D**, and an upper triangular matrix **U**. As you will see, this **LDU** Decomposition is useful in solving certain types of linear systems. Although **LDU** Decomposition is used here only to solve systems having square coefficient matrices, this technique can be generalized to solve systems with nonsquare coefficient matrices as well.

### Calculating the LDU Decomposition

For a given matrix **A**, we can find matrices **L**, **D**, and **U** such that $\mathbf{A} = \mathbf{LDU}$ by using row reduction. It is not necessary to bring **A** completely to reduced row echelon form. Instead, we put **A** into row echelon form.

In our discussion, we need to give a name to a row operation of Type (II) in which the pivot row is used to zero out an entry *below* it. Let us call this a **lower Type (II)** row operation. Notice that a matrix can be put in row echelon form using only Type (I) and lower Type (II) operations if you do not need to interchange any rows.

Throughout this section, we assume that row reduction into row echelon form is performed exactly as described in Section 2.1 for Gaussian Elimination. Beware! If you try to be "creative" in your choice of row operations and stray from this standard method of row reduction, you may obtain incorrect answers.

We can now state the main theorem of this section, as follows:

**Theorem 9.1** (**LDU Decomposition Theorem**) *Let* **A** *be a nonsingular* $n \times n$ *matrix. If* **A** *can be row reduced to row echelon form using only Type (I) and lower Type (II) operations, then* $\mathbf{A} = \mathbf{LDU}$ *where* **L** *is an* $n \times n$ *lower triangular matrix,* **D** *is an* $n \times n$ *diagonal matrix, and* **U** *is an* $n \times n$ *upper triangular matrix and where all main diagonal entries of* **L** *and* **U** *equal* 1.

*Furthermore, this decomposition of* **A** *is unique; that is, if* $\mathbf{A} = \mathbf{L}'\mathbf{D}'\mathbf{U}'$, *where* $\mathbf{L}'$ *is* $n \times n$ *lower triangular,* $\mathbf{D}'$ *is* $n \times n$ *diagonal, and* $\mathbf{U}'$ *is* $n \times n$ *upper triangular with all main diagonal entries of* $\mathbf{L}'$ *and* $\mathbf{U}'$ *equal to* 1*, then* $\mathbf{L}' = \mathbf{L}$, $\mathbf{D}' = \mathbf{D}$, *and* $\mathbf{U}' = \mathbf{U}$.

We now outline the proof of this theorem, which illustrates how to calculate the **LDU** Decomposition for a matrix **A** when it exists. We omit the proof of uniqueness, since that property is not needed for the applications.

*Proof.* (Outline) Suppose that **A** is a nonsingular $n \times n$ matrix and we can reduce **A** to row echelon form using only Type (I) and lower Type (II) row operations. Let **U** be the row echelon form matrix obtained from this process. Then **U** is an upper triangular matrix (why?). Since **A** is nonsingular, all of the main diagonal entries of **U** must equal 1 (why?). Now, $\mathbf{U} = R_t(R_{t-1}(\cdots(R_2(R_1(\mathbf{A})))\cdots))$ where $R_1, \ldots, R_t$ are the Type (I) and lower Type (II) row operations used to obtain **U** from **A**. Hence,

$$\begin{aligned}\mathbf{A} &= R_1^{-1}(R_2^{-1}(\cdots(R_{t-1}^{-1}(R_t^{-1}(\mathbf{U})))\cdots))\\ &= R_1^{-1}(R_2^{-1}(\cdots(R_{t-1}^{-1}(R_t^{-1}(\mathbf{I}_n\mathbf{U})))\cdots))\\ &= R_1^{-1}(R_2^{-1}(\cdots(R_{t-1}^{-1}(R_t^{-1}(\mathbf{I}_n)))\cdots))\mathbf{U},\end{aligned}$$

by Theorem 2.1. Let $\mathbf{K} = R_1^{-1}(R_2^{-1}(\cdots(R_{t-1}^{-1}(R_t^{-1}(\mathbf{I}_n)))\cdots))$. Then $\mathbf{A} = \mathbf{KU}$.

Consulting Table 2.1 in Section 2.3, we see that each of $R_1^{-1}, R_2^{-1}, \ldots, R_t^{-1}$ is also either Type (I) or lower Type (II). Now, since $\mathbf{I}_n$ is lower triangular and applying Type (I) and lower Type (II) row operations to a lower triangular matrix always produces a lower triangular matrix (why?), it follows that **K** is a lower triangular matrix. Thus, **K** has the general form

$$\begin{bmatrix} k_{11} & 0 & 0 & \cdots & 0 \\ k_{21} & k_{22} & 0 & \cdots & 0 \\ k_{31} & k_{32} & k_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_{n1} & k_{n2} & k_{n3} & \cdots & k_{nn} \end{bmatrix}.$$

In fact, if we are careful to follow the standard method of row reduction, we get the following values for the entries of **K**:

$$\begin{cases} k_{ii} = \frac{1}{c} & \text{if we performed (I): } \langle i \rangle \leftarrow c\,\langle i \rangle \text{ to create a pivot in column } i \\ k_{ij} = -c & \text{if we performed (II): } \langle j \rangle \leftarrow c\,\langle i \rangle + \langle j \rangle \text{ to zero out the } (i,\ j) \text{ entry (where } i > j) \end{cases}.$$

Thus, the main diagonal entries of **K** are the reciprocals of the constants used in the Type (I) operations, and the entries of **K** below the main diagonal are the additive inverses of the constants used in the lower Type (II) operations (verify!). In particular, all of the main diagonal entries of **K** are nonzero.

Finally, **K** can be expressed as **LD**, where

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \frac{k_{21}}{k_{11}} & 1 & 0 & \cdots & 0 \\ \frac{k_{31}}{k_{11}} & \frac{k_{32}}{k_{22}} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{k_{n1}}{k_{11}} & \frac{k_{n2}}{k_{22}} & \frac{k_{n3}}{k_{33}} & \cdots & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{D} = \begin{bmatrix} k_{11} & 0 & 0 & \cdots & 0 \\ 0 & k_{22} & 0 & \cdots & 0 \\ 0 & 0 & k_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & k_{nn} \end{bmatrix}.$$

Therefore, we have $\mathbf{A} = \mathbf{KU} = \mathbf{LDU}$, with **L** lower triangular, **D** diagonal, **U** upper triangular, and all main diagonal entries of **L** and **U** equal to 1. □

In the next example, we decompose a nonsingular matrix **A** into **LDU** form. As in the proof of Theorem 9.1, we first decompose **A** into **KU** form, with $\mathbf{K} = \mathbf{LD}$. We then find the matrices **L** and **D** using **K**.

---

**Example 1**

We express

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 2 & 5 \\ 4 & 1 & 9 \end{bmatrix}$$

in **LDU** form. To do this, we first convert **A** into row echelon form **U**. Notice that only Type (I) and lower Type (II) row operations are used.

| **Row Operations** | **Resulting Matrices** |
|---|---|
| (I): $\langle 1 \rangle \leftarrow \frac{1}{2}\langle 1 \rangle$ | $\begin{bmatrix} 1 & \frac{1}{2} & 2 \\ 3 & 2 & 5 \\ 4 & 1 & 9 \end{bmatrix}$ |
| (II): $\langle 2 \rangle \leftarrow -3\langle 1 \rangle + \langle 2 \rangle$ | $\begin{bmatrix} 1 & \frac{1}{2} & 2 \\ 0 & \frac{1}{2} & -1 \\ 4 & 1 & 9 \end{bmatrix}$ |
| (II): $\langle 3 \rangle \leftarrow -4\langle 1 \rangle + \langle 3 \rangle$ | $\begin{bmatrix} 1 & \frac{1}{2} & 2 \\ 0 & \frac{1}{2} & -1 \\ 0 & -1 & 1 \end{bmatrix}$ |
| (I): $\langle 2 \rangle \leftarrow 2\langle 2 \rangle$ | $\begin{bmatrix} 1 & \frac{1}{2} & 2 \\ 0 & 1 & -2 \\ 0 & -1 & 1 \end{bmatrix}$ |
| (II): $\langle 3 \rangle \leftarrow 1\langle 2 \rangle + \langle 3 \rangle$ | $\begin{bmatrix} 1 & \frac{1}{2} & 2 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{bmatrix}$ |
| (I): $\langle 3 \rangle \leftarrow -1\langle 3 \rangle$ | $\begin{bmatrix} 1 & \frac{1}{2} & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{U}.$ |

Using the formulas in the proof of Theorem 9.1 for $k_{ii}$ and $k_{ij}$, we have

$$\mathbf{K} = \begin{bmatrix} 2 & 0 & 0 \\ 3 & \frac{1}{2} & 0 \\ 4 & -1 & -1 \end{bmatrix}.$$

For example, $k_{22} = \frac{1}{2}$ because it is the reciprocal of the constant $c = 2$ used in the row operation (I): $\langle 2\rangle \leftarrow 2\langle 2\rangle$ in order to convert the $(2, 2)$ entry to 1. Similarly, $k_{31} = 4$ because it is the additive inverse of the constant $c = -4$ used in the row operation (II): $\langle 3\rangle \leftarrow -4\langle 1\rangle + \langle 3\rangle$ to zero out the $(3, 1)$ entry of **A**.

Finally, **K** can be broken into a product **LD** as follows: take the main diagonal entries of **D** to be those of **K** and create **L** by dividing each column of **K** by the main diagonal entry in that column. Performing these steps yields

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{D} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

You should verify that $\mathbf{A} = \mathbf{LDU}$.

We can summarize this general procedure as follows:

**Method for Decomposing (If Possible) a Nonsingular $n \times n$ Matrix A into a Product of the Form LDU, where L is Lower Triangular, D is Diagonal, U is Upper Triangular, and All Main Diagonal Entries of L and U Equal 1 (LDU Decomposition)**

**Step 1:** Use (only) Type (I) and lower Type (II) operations to find an upper triangular matrix **U** having all main diagonal entries equal to 1 such that **U** is row equivalent to **A**. (If at least one row operation of Type (III) is needed in this process, stop.)
**Step 2:** Create an $n \times n$ lower triangular matrix **K** so that $k_{ii} = \frac{1}{c}$ if we performed (I): $\langle i\rangle \leftarrow c\,\langle i\rangle$ to create a pivot in column $i$ in Step 1, and $k_{ij} = -c$ if we performed (II): $\langle j\rangle \leftarrow c\,\langle i\rangle + \langle j\rangle$ to zero out the $(i,\ j)$ entry (where $i > j$) in Step 1.
**Step 3:** Create an $n \times n$ diagonal matrix **D** whose main diagonal entries agree with those of **K**.
**Step 4:** Create an $n \times n$ lower triangular matrix **L** so that for all $i \geq j$, $l_{ij} = k_{ij}/k_{jj}$. (Note that all main diagonal entries of **L** are therefore equal to 1.)
**Step 5:** Then the product **LDU** is equal to **A**.

## Solving a System Using LDU Decomposition

When solving a system of linear equations with coefficient matrix **A**, it is often useful to leave the **LDU** Decomposition of **A** in **KU** form. We can then find the solution of the system using substitution techniques, as in the next example.

### Example 2

We solve

$$\begin{cases} -4x_1 + 5x_2 - 2x_3 = \phantom{-}5 \\ -3x_1 + 2x_2 - \phantom{2}x_3 = \phantom{-}4 \\ \phantom{-4}x_1 + \phantom{5}x_2 \phantom{- 2x_3} = -1 \end{cases}$$

by decomposing the coefficient matrix $\mathbf{A} = \begin{bmatrix} -4 & 5 & -2 \\ -3 & 2 & -1 \\ 1 & 1 & 0 \end{bmatrix}$ into **KU** form. First, putting **A** into row echelon form **U**, we have

| **Row Operations** | **Resultant Matrices** |
|---|---|
| (I): $\langle 1\rangle \leftarrow -\frac{1}{4}\langle 1\rangle$ | $\begin{bmatrix} 1 & -\frac{5}{4} & \frac{1}{2} \\ -3 & 2 & -1 \\ 1 & 1 & 0 \end{bmatrix}$ |
| (II): $\langle 2\rangle \leftarrow 3\langle 1\rangle + \langle 2\rangle$<br>(II): $\langle 3\rangle \leftarrow -1\langle 1\rangle + \langle 3\rangle$ | $\begin{bmatrix} 1 & -\frac{5}{4} & \frac{1}{2} \\ 0 & -\frac{7}{4} & \frac{1}{2} \\ 0 & \frac{9}{4} & -\frac{1}{2} \end{bmatrix}$ |
| (I): $\langle 2\rangle \leftarrow -\frac{4}{7}\langle 2\rangle$ | $\begin{bmatrix} 1 & -\frac{5}{4} & \frac{1}{2} \\ 0 & 1 & -\frac{2}{7} \\ 0 & \frac{9}{4} & -\frac{1}{2} \end{bmatrix}$ |
| (II): $\langle 3\rangle \leftarrow -\frac{9}{4}\langle 2\rangle + \langle 3\rangle$ | $\begin{bmatrix} 1 & -\frac{5}{4} & \frac{1}{2} \\ 0 & 1 & -\frac{2}{7} \\ 0 & 0 & \frac{1}{7} \end{bmatrix}$ |
| (I): $\langle 3\rangle \leftarrow 7\langle 3\rangle$ | $\begin{bmatrix} 1 & -\frac{5}{4} & \frac{1}{2} \\ 0 & 1 & -\frac{2}{7} \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{U}.$ |

Then

$$\mathbf{K} = \begin{bmatrix} -4 & 0 & 0 \\ -3 & -\frac{7}{4} & 0 \\ 1 & \frac{9}{4} & \frac{1}{7} \end{bmatrix}$$

because the main diagonal entries of $\mathbf{K}$ are the reciprocals of the constants used in the Type (I) operations and the entries of $\mathbf{K}$ below the main diagonal are the additive inverses of the constants used in the lower Type (II) operations.

Now the original system can be written as

$$\mathbf{A}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ -1 \end{bmatrix}, \qquad \text{or,} \qquad \mathbf{KU}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ -1 \end{bmatrix}.$$

If we let

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \mathbf{U}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \qquad \text{then we have} \qquad \mathbf{K}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ -1 \end{bmatrix}.$$

Both of the last two systems can be solved using substitution. We solve the second system for the $y$-values, and once they are known, we solve the first system for the $x$-values.

The second system,

$$\mathbf{K}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ -1 \end{bmatrix},$$

is equivalent to

$$\begin{cases} -4y_1 & = 5 \\ -3y_1 - \frac{7}{4}y_2 & = 4 \\ y_1 + \frac{9}{4}y_2 + \frac{1}{7}y_3 & = -1 \end{cases}.$$

The first equation gives $y_1 = -\frac{5}{4}$. Substituting this solution into the second equation and solving for $y_2$, we get $-3\left(-\frac{5}{4}\right) - \frac{7}{4}y_2 = 4$, or $y_2 = -\frac{1}{7}$. Finally, substituting for $y_1$ and $y_2$ in the third equation, we get $-\frac{5}{4} + \frac{9}{4}(-\frac{1}{7}) + \frac{1}{7}y_3 = -1$, or $y_3 = 4$. But then the first system,

$$\mathbf{U}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix},$$

is equivalent to

$$\begin{cases} x_1 - \frac{5}{4}x_2 + \frac{1}{2}x_3 = -\frac{5}{4} \\ \quad\quad x_2 - \frac{2}{7}x_3 = -\frac{1}{7} \\ \quad\quad\quad\quad x_3 = \quad 4 \end{cases}.$$

This time, we solve the equations in *reverse* order. The last equation gives $x_3 = 4$. Then $x_2 - \frac{2}{7}(4) = -\frac{1}{7}$, or $x_2 = 1$. Finally, $x_1 - \frac{5}{4}(1) + \frac{1}{2}(4) = -\frac{5}{4}$, or $x_1 = -2$. Therefore, $(x_1, x_2, x_3) = (-2, 1, 4)$.

Solving a system of linear equations using ($\mathbf{KU} =$) **LDU** Decomposition has an advantage over Gaussian Elimination when there are many systems to be solved with the same coefficient matrix **A**. In that case, **K** and **U** need to be calculated just once, and the solutions to each system can be obtained relatively efficiently using substitution. We saw a similar philosophy in Section 2.4 when we discussed the practicality of solving several systems that had the same coefficient matrix by using the inverse of that matrix.

In our discussion of **LDU** Decomposition, we have not encountered Type (III) row operations. If we need to use Type (III) row operations to reduce a nonsingular matrix **A** to row echelon form, it turns out that $\mathbf{A} = \mathbf{PLDU}$, for some matrix **P** formed by rearranging the rows of the $n \times n$ identity matrix, and with **L**, **D**, and **U** as before. (Rearranging the rows of **P** essentially corresponds to putting the equations of the system in the correct order first so that no Type (III) row operations are needed thereafter.) However, the **PLDU** Decomposition thus obtained is not necessarily unique.

## New Vocabulary

**LDU** Decomposition (for a matrix)
lower Type (II) row operation
**PLDU** Decomposition (for a matrix)

## Highlights

- If a nonsingular matrix **A** can be placed in row echelon form using only Type (I) and lower Type (II) row operations, then $\mathbf{A} = \mathbf{LDU}$, where **L** is lower triangular with all main diagonal entries equal to 1, **D** is diagonal, and **U** is upper triangular with all main diagonal entries equal to 1. Such an **LDU** Decomposition of **A** is unique.
- The **LDU** Decomposition for a matrix **A** can be obtained by first decomposing **A** into **KU** form, with **K** lower triangular and **U** upper triangular, so that **U** is row equivalent to **A** and has all main diagonal entries equal to 1. The entries of the matrix **K** are determined from the row operations applied to **A** as follows: $k_{ii} = \frac{1}{c}$ if we performed (I): $\langle i \rangle \leftarrow c\,\langle i \rangle$ to create a pivot in column $i$, and $k_{ij} = -c$ if we performed (II): $\langle j \rangle \leftarrow c\,\langle i \rangle + \langle j \rangle$ to zero out the $(i,\ j)$ entry (where $i > j$). Then, **K** can be decomposed as **LD**, where:

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \frac{k_{21}}{k_{11}} & 1 & 0 & \cdots & 0 \\ \frac{k_{31}}{k_{11}} & \frac{k_{32}}{k_{22}} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{k_{n1}}{k_{11}} & \frac{k_{n2}}{k_{22}} & \frac{k_{n3}}{k_{33}} & \cdots & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{D} = \begin{bmatrix} k_{11} & 0 & 0 & \cdots & 0 \\ 0 & k_{22} & 0 & \cdots & 0 \\ 0 & 0 & k_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & k_{nn} \end{bmatrix}.$$

- It is often convenient to solve a linear system $\mathbf{AX} = \mathbf{B}$ as follows: First, decompose **A** into **KU** form to obtain $\mathbf{K}(\mathbf{UX}) = \mathbf{B}$, and let $\mathbf{Y} = \mathbf{UX}$. Next, solve $\mathbf{KY} = \mathbf{B}$ for **Y** using substitution. Finally, solve $\mathbf{UX} = \mathbf{Y}$ for **X** using back substitution.
- If Type (III) row operations are needed to place an $n \times n$ matrix **A** in row echelon form, then $\mathbf{A} = \mathbf{PLDU}$, with **L**, **D**, **U** as before, and with **P** obtained from an appropriate rearrangement of the rows of $\mathbf{I}_n$.

## Exercises for Section 9.2

1. Find the **LDU** Decomposition for each of the following matrices:

★ (a) $\begin{bmatrix} 2 & -4 \\ -6 & 17 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & 1 \\ \frac{3}{2} & -\frac{3}{2} \end{bmatrix}$

★ (c) $\begin{bmatrix} -1 & 4 & -2 \\ 2 & -6 & -4 \\ 2 & 0 & -25 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 6 & -4 \\ 5 & 11 & 10 \\ 1 & 9 & -29 \end{bmatrix}$

★ (e) $\begin{bmatrix} -3 & 1 & 1 & -1 \\ 4 & -2 & -3 & 5 \\ 6 & -1 & 1 & -2 \\ -2 & 2 & 4 & -7 \end{bmatrix}$

(f) $\begin{bmatrix} -3 & -12 & 6 & 9 \\ -6 & -26 & 12 & 20 \\ 9 & 42 & -17 & -28 \\ 3 & 8 & -8 & -18 \end{bmatrix}$

2. This exercise shows that not every nonsingular matrix has an **LDU** Decomposition.

(a) Show that the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has no **LDU** Decomposition by showing that there are no values $w$, $x$, $y$, and $z$ such that

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ w & 1 \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}}_{\mathbf{D}} \underbrace{\begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix}}_{\mathbf{U}}.$$

(b) The result of part (a) does not contradict Theorem 9.1. Why not?

3. For each system, find the **KU** Decomposition (where $\mathbf{K} = \mathbf{LD}$) for the coefficient matrix, and use it to solve the system by substitution, as in Example 2.

★ (a) $\begin{cases} -x_1 + 5x_2 = -9 \\ 2x_1 - 13x_2 = 21 \end{cases}$

(b) $\begin{cases} 2x_1 - 4x_2 + 10x_3 = 34 \\ 2x_1 - 5x_2 + 7x_3 = 29 \\ x_1 - 5x_2 - x_3 = 8 \end{cases}$

★ (c) $\begin{cases} -x_1 + 3x_2 - 2x_3 = -13 \\ 4x_1 - 9x_2 - 7x_3 = 28 \\ -2x_1 + 11x_2 - 31x_3 = -68 \end{cases}$

(d) $\begin{cases} 3x_1 - 15x_2 + 6x_3 + 6x_4 = 60 \\ x_1 - 7x_2 + 8x_3 + 2x_4 = 30 \\ -5x_1 + 24x_2 - 3x_3 - 18x_4 = -115 \\ x_1 - 2x_2 - 7x_3 - x_4 = -4 \end{cases}$

★ 4. True or False:

(a) Every nonsingular matrix has a unique **LDU** Decomposition.

(b) The entries of the matrix **K** (as defined in this section) can be obtained just by examining the row operations that were used to reduce **A** to row echelon form.

(c) The operation $R$ given by $\langle 2 \rangle \leftarrow -2\langle 3 \rangle + \langle 2 \rangle$ is a lower Type (II) row operation.

(d) If $\mathbf{A} = \mathbf{KU}$ (as described in this section), then $\mathbf{AX} = \mathbf{B}$ is solved by first solving for $\mathbf{Y}$ in $\mathbf{UY} = \mathbf{B}$ and then solving for $\mathbf{X}$ in $\mathbf{KX} = \mathbf{Y}$.

## 9.3 The Power Method for Finding Eigenvalues

### *Prerequisite: Section 3.4, Eigenvalues and Diagonalization*

The only method given in Section 3.4 for finding the eigenvalues of an $n \times n$ matrix **A** is to calculate the characteristic polynomial of **A** and find its roots. However, if $n$ is large, $p_{\mathbf{A}}(x)$ is often difficult to calculate. Also, numerical techniques may be required to find its roots. Finally, if an eigenvalue $\lambda$ is not known to a high enough degree of accuracy, we may have difficulty finding a corresponding eigenvector $\mathbf{v}$, because the matrix $\lambda\mathbf{I}-\mathbf{A}$ in the equation $(\lambda\mathbf{I}-\mathbf{A})\mathbf{v}=\mathbf{0}$ may not be singular for the given value of $\lambda$.

Therefore, in this section we present a numerical technique known as the Power Method for finding the largest eigenvalue (in absolute value) of a matrix and a corresponding eigenvector. Such an eigenvalue is called a **dominant eigenvalue**.

All calculations for the examples and exercises in this section were performed on a calculator that stores numbers with 12-digit accuracy, but only the first 4 significant digits are printed here. Your own computations may differ slightly if you are using a different number of significant digits. If you do not have a calculator with the ability to perform matrix calculations, use an appropriate linear algebra software package. You might also consider writing your own Power Method program, since the algorithm involved is not difficult.

## The Power Method

Suppose $\mathbf{A}$ is a diagonalizable $n \times n$ matrix having (not necessarily distinct) eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, with $\lambda_1$ being the dominant eigenvalue. The **Power Method** can be used to find $\lambda_1$ and an associated eigenvector. In fact, it often works in cases where $\mathbf{A}$ is not diagonalizable, but it is not guaranteed to work in such a case.

The idea behind the Power Method is as follows: choose any unit $n$-vector $\mathbf{v}$ and calculate $(\mathbf{A}^k\mathbf{v})/\left\|\mathbf{A}^k\mathbf{v}\right\|$ for some large positive integer $k$. The result should be a good approximation for a unit eigenvector corresponding to $\lambda_1$.

Why? First, we prove in Chapter 5 that every $\mathbf{v} \in \mathbb{R}^n$ can be expressed in the form $\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$, where $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a complete set of fundamental eigenvectors for $\mathbf{A}$ corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_n$, as found in the Diagonalization Method of Section 3.4. Then,

$$\begin{aligned}\mathbf{A}^k\mathbf{v} &= a_1\mathbf{A}^k\mathbf{v}_1 + a_2\mathbf{A}^k\mathbf{v}_2 + \cdots + a_n\mathbf{A}^k\mathbf{v}_n \\ &= a_1\lambda_1^k\mathbf{v}_1 + a_2\lambda_2^k\mathbf{v}_2 + \cdots + a_n\lambda_n^k\mathbf{v}_n.\end{aligned}$$

Because $|\lambda_1| > |\lambda_i|$ for $2 \le i \le n$, we see that for large $k$, $\left|\lambda_1^k\right|$ is significantly larger than $\left|\lambda_i^k\right|$, since the ratio $|\lambda_i|^k / |\lambda_1|^k$ approaches 0 as $k \to \infty$. Thus, the term $a_1\lambda_1^k\mathbf{v}_1$ dominates the expression for $\mathbf{A}^k\mathbf{v}$ for large enough values of $k$.[3] If we normalize $\mathbf{A}^k\mathbf{v}$, we have $\mathbf{u} = \left(\mathbf{A}^k\mathbf{v}\right)/\left\|\mathbf{A}^k\mathbf{v}\right\| \approx \left(a_1\lambda_1^k\mathbf{v}_1\right)/\left\|a_1\lambda_1^k\mathbf{v}_1\right\|$, which is a scalar multiple of $\mathbf{v}_1$, and thus, $\mathbf{u}$ is a unit eigenvector corresponding to $\lambda_1$.

Finally, $\mathbf{A}\mathbf{u} \approx \lambda_1\mathbf{u}$, and so $\|\mathbf{A}\mathbf{u}\|$ approximates $|\lambda_1|$. The sign of $\lambda_1$ is determined by checking whether $\mathbf{A}\mathbf{u}$ is in the same direction as $\mathbf{u}$ or in the opposite direction. We now outline the Power Method in detail.

**Method for Finding the Dominant Eigenvalue of a Square Matrix (Power Method)**
Let $\mathbf{A}$ be an $n \times n$ matrix.

**Step 1:** Choose an arbitrary unit $n$-vector $\mathbf{u}_0$.
**Step 2:** Create a sequence of unit $n$-vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots$ by repeating Steps 2(a) through 2(d) until one of the terminal conditions in Steps 2(c) or 2(d) is reached or until it becomes clear that the method is not converging to an answer.

**(a)** Given $\mathbf{u}_{k-1}$, calculate $\mathbf{w}_k = \mathbf{A}\mathbf{u}_{k-1}$.
**(b)** Calculate $\mathbf{u}_k = \mathbf{w}_k/\|\mathbf{w}_k\|$.
**(c)** If $\mathbf{u}_{k-1}$ equals $\mathbf{u}_k$ to the desired degree of accuracy, let $\lambda = \|\mathbf{w}_k\|$ and go to Step 3.
**(d)** If $\mathbf{u}_{k-1}$ equals $-\mathbf{u}_k$ to the desired degree of accuracy, let $\lambda = -\|\mathbf{w}_k\|$ and go to Step 3.

**Step 3:** The last $\mathbf{u}_k$ vector calculated in Step 2 is an approximate eigenvector of $\mathbf{A}$ corresponding to the (approximate) eigenvalue $\lambda$.

Notice that in the Power Method, we normalize each new vector *after* multiplying by $\mathbf{A}$, while in our prior discussion we normalized the final vector $\mathbf{A}^k\mathbf{v}$. However, the fact that matrix and scalar multiplication commute and that both approaches result in a unit vector should convince you that the two techniques are equivalent.

It is possible (but unlikely) to get $\mathbf{w}_k = \mathbf{0}$ in Step 2(a) of the Power Method, which makes Step 2(b) impossible to perform. In this case, $\mathbf{u}_{k-1}$ is an eigenvector for $\mathbf{A}$ corresponding to $\lambda = 0$. You can then return to Step 1, choosing a different $\mathbf{u}_0$, in hope of finding another eigenvalue for $\mathbf{A}$.

[3] Theoretically, a problem may arise if $a_1 = 0$. However, in most practical situations, this will not happen. If the method does not work and you suspect it is because $a_1 = 0$, try using a different vector for $\mathbf{v}$ that is not a linear combination of those you have already tried.

**Example 1**

Let

$$\mathbf{A} = \begin{bmatrix} -16 & 6 & 30 \\ 4 & 1 & -8 \\ -9 & 3 & 17 \end{bmatrix}.$$

We use the Power Method to find the dominant eigenvalue for $\mathbf{A}$ and a corresponding eigenvector correct to four decimal places.

**Step 1:** We choose $\mathbf{u}_0 = [1, 0, 0]$.

**Step 2:** A first pass through this step gives the following:

(a) $\mathbf{w}_1 = \mathbf{A}\mathbf{u}_0 \approx [-16, 4, -9]$.

(b) $\|\mathbf{w}_1\| = \sqrt{(-16)^2 + 4^2 + (-9)^2} \approx 18.79$.

So $\mathbf{u}_1 = \mathbf{w}_1 / \|\mathbf{w}_1\| \approx [-0.8516, 0.2129, -0.4790]$.

Because $\mathbf{u}_0$ and $\pm\mathbf{u}_1$ do not agree to four decimal places, we return to Step 2(a). Subsequent iterations of Step 2 lead to the results in the following table:

| $k$ | $\mathbf{w}_k = \mathbf{A}\mathbf{u}_{k-1}$ | $\|\mathbf{w}_k\|$ | $\mathbf{u}_k = \frac{\mathbf{w}_k}{\|\mathbf{w}_k\|}$ |
|---|---|---|---|
| 1 | $[-16, 4, -9]$ | 18.79 | $[-0.8516, 0.2129, -0.4790]$ |
| 2 | $[0.5322, 0.6387, 0.1597]$ | 0.8466 | $[0.6287, 0.7544, 0.1886]$ |
| 3 | $[0.1257, 1.760, -0.1886]$ | 1.775 | $[0.0708, 0.9918, -0.1063]$ |
| 4 | $[1.629, 2.125, 0.5313]$ | 2.730 | $[0.5968, 0.7784, 0.1946]$ |
| 5 | $[0.9601, 1.609, 0.2725]$ | 1.893 | $[0.5071, 0.8498, 0.1439]$ |
| 6 | $[1.302, 1.727, 0.4317]$ | 2.205 | $[0.5904, 0.7830, 0.1958]$ |
| 7 | $[1.125, 1.578, 0.3635]$ | 1.972 | $[0.5704, 0.8004, 0.1843]$ |
| 8 | $[1.207, 1.607, 0.4018]$ | 2.050 | $[0.5889, 0.7841, 0.1960]$ |
| 9 | $[1.164, 1.571, 0.3851]$ | 1.993 | $[0.5840, 0.7884, 0.1932]$ |
| 10 | $[1.184, 1.578, 0.3946]$ | 2.012 | $[0.5885, 0.7844, 0.1961]$ |
| 11 | $[1.173, 1.570, 0.3905]$ | 1.998 | $[0.5873, 0.7855, 0.1954]$ |
| 12 | $[1.179, 1.571, 0.3928]$ | 2.003 | $[0.5884, 0.7844, 0.1961]$ |
| 13 | $[1.176, 1.569, 0.3918]$ | 2.000 | $[0.5881, 0.7847, 0.1959]$ |
| 14 | $[1.177, 1.570, 0.3924]$ | 2.001 | $[0.5884, 0.7845, 0.1961]$ |
| 15 | $[1.176, 1.569, 0.3921]$ | 2.000 | $[0.5883, 0.7845, 0.1961]$ |
| 16 | $[1.177, 1.569, 0.3923]$ | 2.000 | $[0.5884, 0.7845, 0.1961]$ |
| 17 | $[1.177, 1.569, 0.3922]$ | 2.000 | $[0.5883, 0.7845, 0.1961]$ |
| 18 | $[1.177, 1.569, 0.3922]$ | 2.000 | $[0.5883, 0.7845, 0.1961]$ |

After 18 iterations, we find that $\mathbf{u}_{17}$ and $\mathbf{u}_{18}$ agree to four decimal places. Therefore, Step 2 terminates with $\lambda = 2.000$.

**Step 3:** Thus, $\lambda = 2.000$ is the dominant eigenvalue for $\mathbf{A}$ with corresponding unit eigenvector $\mathbf{u}_{18} = [0.5883, 0.7845, 0.1961]$.

We can check that the Power Method gives the correct result in this particular case. A quick calculation shows that for the given matrix $\mathbf{A}$, $p_{\mathbf{A}}(x) = x^3 - 2x^2 - x + 2 = (x-2)(x-1)(x+1)$. Thus, $\lambda_1 = 2$ is the dominant eigenvalue for $\mathbf{A}$.

Solving the system $(2\mathbf{I}_3 - \mathbf{A})\mathbf{v} = \mathbf{0}$ produces an eigenvector $\mathbf{v} = [3, 4, 1]$ corresponding to $\lambda_1 = 2$. Normalizing $\mathbf{v}$ yields a unit eigenvector $\mathbf{v}/\|\mathbf{v}\| \approx [0.5883, 0.7845, 0.1961]$.

## Problems With the Power Method

Unfortunately, the Power Method does not always work. Note that it depends on the fact that multiplying by $\mathbf{A}$ magnifies the size of an eigenvector for the dominant eigenvalue more than for any other eigenvector for $\mathbf{A}$. For example, if $\mathbf{A}$ is a diagonalizable matrix, the Power Method fails if both $\pm\lambda$ are eigenvalues of $\mathbf{A}$ with the largest absolute value. In particular, suppose $\mathbf{A}$ is a $3 \times 3$ matrix with eigenvalues $\lambda_1 = 2$, $\lambda_2 = -2$, and $\lambda_3 = 1$ and corresponding eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Multiplying $\mathbf{A}$ by any vector $\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3$ produces $\mathbf{A}\mathbf{v} = 2a_1\mathbf{v}_1 - 2a_2\mathbf{v}_2 + a_3\mathbf{v}_3$. The contribution of neither eigenvector $\mathbf{v}_1$ nor $\mathbf{v}_2$ dominates over the other, since both terms are doubled simultaneously.

The next example illustrates that the Power Method is not guaranteed to work for a nondiagonalizable matrix.

**Example 2**

Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 7 & -15 & -24 \\ -12 & 25 & 42 \\ 6 & -15 & -23 \end{bmatrix}.$$

This matrix has only one eigenvalue, $\lambda = 1$, with the single fundamental eigenvector $\mathbf{v}_1 = [3, -2, 2]$. The Power Method cannot be used to find this eigenvalue, since some vectors in $\mathbb{R}^3$ that cannot be expressed as a linear combination of fundamental eigenvectors have their magnitudes increased when multiplied by $\mathbf{A}$ while the magnitude of $\mathbf{v}_1$ is fixed by $\mathbf{A}$. If we attempt the Power Method anyway, starting with $\mathbf{u}_0 = [1, 0, 0]$, the following results are produced:

| $k$ | $\mathbf{w}_k = \mathbf{A}\mathbf{u}_{k-1}$ | $\|\mathbf{w}_k\|$ | $\mathbf{u}_k = \frac{\mathbf{w}_k}{\|\mathbf{w}_k\|}$ |
|---|---|---|---|
| 1 | $[7, -12, 6]$ | 15.13 | $[0.4626, -0.7930, 0.3965]$ |
| 2 | $[5.617, -8.723, 5.551]$ | 11.77 | $[0.4774, -0.7413, 0.4718]$ |
| 3 | $[3.139, -4.448, 3.134]$ | 6.282 | $[0.4998, -0.7081, 0.4989]$ |
| ⋮ | ⋮ | ⋮ | ⋮ |
| 25 | $[0.3434, 0.3341, 0.3434]$ | 0.5894 | $[0.5825, 0.5668, 0.5825]$ |
| 26 | $[-18.41, 31.65, -18.41]$ | 40.98 | $[-0.4492, 0.7723, -0.4492]$ |
| 27 | $[-3.949, 5.833, -3.949]$ | 8.075 | $[-0.4890, 0.7223, -0.4890]$ |
| ⋮ | ⋮ | ⋮ | ⋮ |
| 50 | $[2.589, -5.325, 2.589]$ | 6.462 | $[0.4006, -0.8240, 0.4006]$ |
| 51 | $[5.551, -8.583, 5.551]$ | 11.63 | $[0.4772, -0.7379, 0.4772]$ |
| 52 | $[2.957, -4.132, 2.957]$ | 5.879 | $[0.5029, -0.7029, 0.5029]$ |
| ⋮ | ⋮ | ⋮ | ⋮ |

As you can see, there is no evidence of any convergence at all in either the $\|\mathbf{w}_k\|$ or $\mathbf{u}_k$ columns. If the Power Method were successful, these would be converging to, respectively, the absolute value of the dominant eigenvalue and a corresponding unit eigenvector.

One disadvantage of the Power Method is that it can only be used to find the dominant eigenvalue for a matrix. There are additional numerical techniques for calculating other eigenvalues. One such technique is the **Inverse Power Method,** which finds the *smallest* eigenvalue of a matrix essentially by using the Power Method on the inverse of the matrix. If you are interested in learning more about this technique and other more sophisticated methods for finding eigenvalues, check such classic references as *Numerical Analysis*, 10th ed., by Burden, Faires, and Burden (published by Cengage, 2015).

## New Vocabulary

dominant eigenvalue
Inverse Power Method
Power Method (for finding a dominant eigenvalue)

## Highlights

- The Power Method is used to find a dominant eigenvalue (one having the largest absolute value), if one exists, and a corresponding eigenvector.
- To apply the Power Method to a square matrix $\mathbf{A}$, begin with an initial guess $\mathbf{u}_0$ for the eigenvector of the dominant eigenvalue. Then, for $i \geq 1$, calculate $\mathbf{u}_i = \mathbf{A}\mathbf{u}_{i-1}/||\mathbf{A}\mathbf{u}_{i-1}||$, until consecutive vectors $\mathbf{u}_i$ are either identical or opposite. If $\mathbf{u}_k$ denotes the last vector calculated in this process, then $\mathbf{u}_k$ is an approximate eigenvector of $\mathbf{A}$, and $||\mathbf{A}\mathbf{u}_k||$ is the absolute value of the dominant eigenvalue for $\mathbf{A}$.
- The Power Method is very useful, but is not always guaranteed to converge if the given matrix is nondiagonalizable.
- The Inverse Power Method (if convergent) calculates the eigenvalue having smallest absolute value.

## Exercises for Section 9.3

**1.** Use the Power Method on each of the given matrices, starting with the given vector,[4] to find the dominant eigenvalue and a corresponding unit eigenvector for each matrix. Perform as many iterations as needed until two successive vectors agree in every entry in the first $m$ digits after the decimal point for the given value of $m$. Carry out all calculations using as many significant digits as are feasible with your calculator or computer software.

★ **(a)** $\begin{bmatrix} 2 & 36 \\ 36 & 23 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, m = 2$

**(b)** $\begin{bmatrix} 3 & 5 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, m = 2$

★ **(c)** $\begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 1 \\ 3 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, m = 2$

**(d)** $\begin{bmatrix} 3 & 1 & 1 & 2 \\ 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 2 & 3 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, m = 2$

★ **(e)** $\begin{bmatrix} -10 & 2 & -1 & 11 \\ 4 & 2 & -3 & 6 \\ -44 & 7 & 3 & 28 \\ -17 & 4 & 1 & 12 \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \\ 2 \\ 3 \end{bmatrix}, m = 3$

**(f)** $\begin{bmatrix} 5 & 3 & -4 & 6 \\ -2 & -1 & 6 & -10 \\ -6 & -6 & 8 & -7 \\ -2 & -2 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -5 \\ -6 \\ -1 \end{bmatrix}, m = 4$

**2.** In each part of this exercise, show that the Power Method does *not* work on the given matrix using $[1, 0, 0]$ as an initial vector. Explain why the method fails in each case.

**(a)** $\begin{bmatrix} -21 & 10 & -74 \\ 25 & -9 & 80 \\ 10 & -4 & 33 \end{bmatrix}$

**(b)** $\begin{bmatrix} 13 & -10 & 8 \\ -8 & 11 & -4 \\ -40 & 40 & -23 \end{bmatrix}$

**3.** This exercise examines a property of diagonalizable matrices involving the ratio of a dominant eigenvalue to another eigenvalue.

**(a)** Suppose that $\mathbf{A}$ is a diagonalizable $2 \times 2$ matrix with eigenvalues $\lambda_1$ and $\lambda_2$ such that $|\lambda_1| > |\lambda_2| \neq 0$. Let $\mathbf{v}_1$, $\mathbf{v}_2$ be unit eigenvectors in $\mathbb{R}^2$ corresponding to $\lambda_1$ and $\lambda_2$, respectively. Assume each vector $\mathbf{x} \in \mathbb{R}^2$ can be expressed uniquely in the form $\mathbf{x} = a\mathbf{v}_1 + b\mathbf{v}_2$. (This follows from results in Section 4.4.) Finally, suppose $\mathbf{u}_0$ is the initial vector used in the Power Method for finding the dominant eigenvalue of $\mathbf{A}$. Expressing $\mathbf{u}_i$ in that method as $a_i\mathbf{v}_1 + b_i\mathbf{v}_2$, prove that for all $i \geq 0$,

$$\frac{|a_i|}{|b_i|} = \left|\frac{\lambda_1}{\lambda_2}\right|^i \cdot \frac{|a_0|}{|b_0|},$$

assuming that $b_i \neq 0$. (Note that the ratio $\frac{|a_i|}{|b_i|} \to \infty$, so, for large values of $i$, the term $b_i\mathbf{v}_2$ in $\mathbf{u}_i$ will be insignificant in contrast to the term $a_i\mathbf{v}_1$. Thus, $\mathbf{u}_i$ will approximately equal a multiple of $\mathbf{v}_1$.)

★ **(b)** Suppose $\mathbf{A}$ is a diagonalizable $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ such that $|\lambda_1| > |\lambda_j|$, for $2 \leq j \leq n$. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be fundamental eigenvectors for $\mathbf{A}$ corresponding to $\lambda_1, \ldots, \lambda_n$, respectively. Assume that every vector $\mathbf{x} \in \mathbb{R}^n$ can be expressed uniquely in the form $\mathbf{x} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b\mathbf{v}_n$. (This follows from results in Section 4.4.) Finally, suppose the initial vector in the Power Method is $\mathbf{u}_0 = a_{01}\mathbf{v}_1 + \cdots + a_{0n}\mathbf{v}_n$ and the $i$th iteration yields $\mathbf{u}_i = a_{i1}\mathbf{v}_1 + \cdots + a_{in}\mathbf{v}_n$. Prove that, for $2 \leq j \leq n$, $\lambda_j \neq 0$, and $a_{0j} \neq 0$, we have

$$\frac{|a_{i1}|}{|a_{ij}|} = \left|\frac{\lambda_1}{\lambda_j}\right|^i \frac{|a_{01}|}{|a_{0j}|}.$$

★ **4.** True or False:

**(a)** If the Power Method succeeds in finding a dominant eigenvalue $\lambda$ for a matrix $\mathbf{A}$, then we must have $\lambda = ||\mathbf{A}\mathbf{u}_{k-1}||$, where $\mathbf{u}_k$ is the final vector found in the process.

**(b)** The Power Method does not find the dominant eigenvalue of a matrix $\mathbf{A}$ if the initial vector used is an eigenvector for a different eigenvalue for $\mathbf{A}$.

[4] In parts (e) and (f), the initial vector $\mathbf{u}_0$ is not a unit vector. This does not affect the outcome of the Power Method since all subsequent vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots$ will be unit vectors.

(c) Starting with the vector [1, 0, 0, 0], the Power Method produces the eigenvalue 4 for the $4 \times 4$ matrix **A** having all entries equal to 1.
(d) If 2 and $-3$ are eigenvalues for a $2 \times 2$ matrix **A**, then the Power Method produces an eigenvector for **A** corresponding to the eigenvalue 2 because $2 > -3$.

## 9.4 QR Factorization

*Prerequisite: Section 6.1, Orthogonal Bases and the Gram-Schmidt Process*

In this section, we show that any matrix **A** with linearly independent columns can be factored into a product of two matrices, one having orthonormal columns, and the other being nonsingular and upper triangular. Such a product is often called a **QR** Factorization for **A**.

### QR Factorization Theorem

The proof of the following theorem illustrates the method for **QR** Factorization.

**Theorem 9.2** *Let* **A** *be an $n \times k$ matrix, with $n \geq k$, whose $k$ columns are linearly independent. Then* $\mathbf{A} = \mathbf{QR}$*, where* **Q** *is an $n \times k$ matrix whose columns form an orthonormal basis for the subspace of $\mathbb{R}^n$ spanned by the columns of* **A***, and* **R** *is a nonsingular upper triangular $k \times k$ matrix.*

The matrix **R** in Theorem 9.2 as constructed in the following proof has its main diagonal entries all positive. If this additional restriction is placed on **R**, then the **QR** Factorization of **A** is unique. You are asked to prove this in Exercise 3.

*Proof.* Let **A** be an $n \times k$ matrix with linearly independent columns $\mathbf{w}_1, \ldots, \mathbf{w}_k$, respectively. Apply the Gram-Schmidt Process to $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ to obtain an orthogonal set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$. That is,

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{w}_1, \\
\mathbf{v}_2 &= \mathbf{w}_2 - \left(\frac{\mathbf{w}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1, \\
\mathbf{v}_3 &= \mathbf{w}_3 - \left(\frac{\mathbf{w}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{w}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2, \\
&\text{etc.}
\end{aligned}$$

Notice that if $\mathcal{W}$ is the subspace of $\mathbb{R}^n$ spanned by $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an orthogonal basis for $\mathcal{W}$ by Theorem 6.4.

Now, let **Q** be the $n \times k$ matrix with columns $\mathbf{u}_1, \ldots, \mathbf{u}_k$, where

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{||\mathbf{v}_1||}, \ldots, \mathbf{u}_k = \frac{\mathbf{v}_k}{||\mathbf{v}_k||}.$$

Then $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthonormal basis for $\mathcal{W}$, and the columns of **Q** form an orthonormal set.

We can finish the proof if we can find a nonsingular upper triangular matrix **R** such that $\mathbf{A} = \mathbf{QR}$. To find the entries of **R**, let us express each $\mathbf{w}_i$ ($i$th column of **A**) as a linear combination of the $\mathbf{u}_i$'s (columns of **Q**). Now, from the Gram-Schmidt Process, we know

$$\begin{aligned}
\mathbf{w}_1 &= \mathbf{v}_1 = ||\mathbf{v}_1||\mathbf{u}_1, \quad \text{and} \\
\mathbf{w}_2 &= \left(\frac{\mathbf{w}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 + \mathbf{v}_2 \\
&= \left(\frac{\mathbf{w}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)||\mathbf{v}_1||\mathbf{u}_1 + ||\mathbf{v}_2||\mathbf{u}_2 \\
&= \frac{\mathbf{w}_2 \cdot \mathbf{v}_1}{\|\mathbf{v}_1\|}\mathbf{u}_1 + ||\mathbf{v}_2||\mathbf{u}_2 \\
&= (\mathbf{w}_2 \cdot \mathbf{u}_1)\,\mathbf{u}_1 + ||\mathbf{v}_2||\mathbf{u}_2.
\end{aligned}$$

By an argument similar to that for $\mathbf{w}_2$, it is easy to show that

$$\begin{aligned}\mathbf{w}_3 &= \left(\frac{\mathbf{w}_3\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\right)\mathbf{v}_1 + \left(\frac{\mathbf{w}_3\cdot\mathbf{v}_2}{\mathbf{v}_2\cdot\mathbf{v}_2}\right)\mathbf{v}_2 + \mathbf{v}_3\\ &= (\mathbf{w}_3\cdot\mathbf{u}_1)\,\mathbf{u}_1 + (\mathbf{w}_3\cdot\mathbf{u}_2)\,\mathbf{u}_2 + ||\mathbf{v}_3||\mathbf{u}_3.\end{aligned}$$

In general,

$$\begin{aligned} i\text{th column of } \mathbf{A} &= \mathbf{w}_i\\ &= (\mathbf{w}_i\cdot\mathbf{u}_1)\,\mathbf{u}_1 + (\mathbf{w}_i\cdot\mathbf{u}_2)\,\mathbf{u}_2 + \cdots + (\mathbf{w}_i\cdot\mathbf{u}_{i-1})\,\mathbf{u}_{i-1} + ||\mathbf{v}_i||\mathbf{u}_i.\\ &= [\mathbf{u}_1,\ldots,\mathbf{u}_k]\begin{bmatrix}\mathbf{w}_i\cdot\mathbf{u}_1\\ \mathbf{w}_i\cdot\mathbf{u}_2\\ \vdots\\ ||\mathbf{v}_i||\\ 0\\ \vdots\\ 0\end{bmatrix} = \mathbf{Q}\begin{bmatrix}\mathbf{w}_i\cdot\mathbf{u}_1\\ \mathbf{w}_i\cdot\mathbf{u}_2\\ \vdots\\ ||\mathbf{v}_i||\\ 0\\ \vdots\\ 0\end{bmatrix}. \longleftarrow i^{th} \text{ row}\end{aligned}$$

Thus, $\mathbf{A} = \mathbf{QR}$, where

$$\mathbf{R} = \begin{bmatrix} ||\mathbf{v}_1|| & \mathbf{w}_2\cdot\mathbf{u}_1 & \mathbf{w}_3\cdot\mathbf{u}_1 & \cdots & \mathbf{w}_k\cdot\mathbf{u}_1\\ 0 & ||\mathbf{v}_2|| & \mathbf{w}_3\cdot\mathbf{u}_2 & \cdots & \mathbf{w}_k\cdot\mathbf{u}_2\\ 0 & 0 & ||\mathbf{v}_3|| & \cdots & \mathbf{w}_k\cdot\mathbf{u}_3\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & ||\mathbf{v}_k||\end{bmatrix}.$$

Finally, note that since $\{\mathbf{v}_1,\ldots,\mathbf{v}_k\}$ is a basis for $\mathcal{W}$, all $||\mathbf{v}_i|| \neq 0$. Thus, $\mathbf{R}$ is nonsingular, since it is upper triangular with all main diagonal entries nonzero. □

Notice in the special case when $\mathbf{A}$ is square, the matrix $\mathbf{Q}$ is square also, and then by part (2) of Theorem 6.7, $\mathbf{Q}$ is an orthogonal matrix. However, in all cases, $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}_k$ because the columns of $\mathbf{Q}$ are orthonormal. After multiplying both sides of $\mathbf{A} = \mathbf{QR}$ by $\mathbf{Q}^T$ on the left, we obtain $\mathbf{R} = \mathbf{Q}^T\mathbf{A}$.

The technique for **QR** Factorization is summarized as follows:

**Method for Decomposing a Matrix With Linearly Independent Columns into the Product of Two Matrices, With the Columns of the First Forming an Orthonormal Basis for the Span of the Original Columns, and the Second a Nonsingular Upper Triangular Matrix (QR Factorization)**

Let $\mathbf{A}$ be an $n \times k$ matrix, with $n \geq k$, having columns $\mathbf{w}_1,\ldots,\mathbf{w}_k$ which are linearly independent. To find the **QR** Factorization of $\mathbf{A}$:

**Step 1:** Use the Gram-Schmidt Process on $\{\mathbf{w}_1,\ldots,\mathbf{w}_k\}$ to obtain an orthogonal set of vectors $\{\mathbf{v}_1,\ldots,\mathbf{v}_k\}$.
**Step 2:** Normalize $\{\mathbf{v}_1,\ldots,\mathbf{v}_k\}$ to create an orthonormal set of vectors $\{\mathbf{u}_1,\ldots,\mathbf{u}_k\}$.
**Step 3:** Create the $n \times k$ matrix $\mathbf{Q}$ whose columns are $\mathbf{u}_1,\ldots,\mathbf{u}_k$, respectively.
**Step 4:** Create the $k \times k$ matrix $\mathbf{R} = \mathbf{Q}^T\mathbf{A}$. Then $\mathbf{A} = \mathbf{QR}$.

In using the Gram-Schmidt Process in Section 6.1, we often replaced certain vectors with scalar multiples in order to avoid fractions. We can perform a similar procedure here. Replacing the $\mathbf{v}_i$ vectors obtained in the Gram-Schmidt Process with suitable *positive* scalar multiples will not affect the final orthonormal vectors $\mathbf{u}_i$ that are obtained, and thus the matrix $\mathbf{Q}$ will not change. However, if some vector $\mathbf{v}_i$ is replaced with a *negative* scalar multiple $c_i\mathbf{v}_i$, then all entries in the corresponding $i$th column of $\mathbf{Q}$ and $i$th row of $\mathbf{R}$ will have the opposite sign from what they would have had if the positive scalar $|c_i|$ had been used instead. Therefore, if any of the $\mathbf{v}_i$'s are replaced with negative scalar multiples during the Gram-Schmidt Process, $\mathbf{R}$ will have one or more negative entries on its main diagonal.

The **QR** Factorization Method is illustrated in the following example, where only positive scalar multiples are used in the Gram-Schmidt Process:

## Example 1

We find the **QR** Factorization for the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We label the columns of $\mathbf{A}$ as $\mathbf{w}_1 = [1,0,1,0]$, $\mathbf{w}_2 = [0,1,1,0]$, and $\mathbf{w}_3 = [0,1,0,1]$, and let $\mathcal{W}$ be the subspace of $\mathbb{R}^4$ generated by these vectors. We will use the Gram-Schmidt Process to find an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for $\mathcal{W}$, and then an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ for $\mathcal{W}$.

Beginning the Gram-Schmidt Process, we obtain

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{w}_1 = [1,0,1,0], \quad \text{and} \\ \mathbf{v}_2 &= \mathbf{w}_2 - \left(\frac{\mathbf{w}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 \\ &= [0,1,1,0] - \frac{[0,1,1,0]\cdot[1,0,1,0]}{[1,0,1,0]\cdot[1,0,1,0]}[1,0,1,0] \\ &= [0,1,1,0] - \frac{1}{2}[1,0,1,0] \\ &= \left[-\frac{1}{2}, 1, \frac{1}{2}, 0\right]. \end{aligned}$$

Multiplying this vector by a factor of $c_2 = 2$ to avoid fractions, we let $\mathbf{v}_2 = [-1,2,1,0]$. Finally,

$$\begin{aligned} \mathbf{v}_3 &= \mathbf{w}_3 - \left(\frac{\mathbf{w}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 + \left(\frac{\mathbf{w}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 \\ &= [0,1,0,1] - \frac{[0,1,0,1]\cdot[1,0,1,0]}{[1,0,1,0]\cdot[1,0,1,0]}[1,0,1,0] - \frac{[0,1,0,1]\cdot[-1,2,1,0]}{[-1,2,1,0]\cdot[-1,2,1,0]}[-1,2,1,0] \\ &= [0,1,0,1] - 0[1,0,1,0] - \frac{2}{6}[-1,2,1,0] \\ &= \left[\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}, 1\right]. \end{aligned}$$

Multiplying this vector by a factor of $c_3 = 3$ to avoid fractions, we obtain $\mathbf{v}_3 = [1,1,-1,3]$. Normalizing $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, we get

$$\mathbf{u}_1 = \left[\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0\right], \quad \mathbf{u}_2 = \left[-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, 0\right], \quad \mathbf{u}_3 = \left[\frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, -\frac{1}{2\sqrt{3}}, \frac{3}{2\sqrt{3}}\right].$$

From the preceding Method, we know that these vectors are the columns of $\mathbf{Q}$. Also, we know that

$$\begin{aligned} \mathbf{R} = \mathbf{Q}^T\mathbf{A} &= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 \\ \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & \frac{3}{2\sqrt{3}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{\sqrt{6}}{2} & \frac{2}{\sqrt{6}} \\ 0 & 0 & \frac{2\sqrt{3}}{3} \end{bmatrix}. \end{aligned}$$

You should check that **QR** really does equal **A**.

## QR Factorization and Least Squares

Suppose $\mathbf{AX} = \mathbf{B}$ is an inconsistent linear system; that is, a system with no solutions. Exercise 9 of Section 8.3 and all of Section 8.10 show how the technique of least squares can be used to find values that come "close" to satisfying all the equations in this system. Specifically, the solutions of the related system $\mathbf{A}^T\mathbf{AX} = \mathbf{A}^T\mathbf{B}$ are called **least-squares solutions** for the original system $\mathbf{AX} = \mathbf{B}$.

The **QR** Factorization Method affords a way of finding certain least-squares solutions, as shown in the following theorem:

**Theorem 9.3** *Suppose* $\mathbf{A}$ *is an* $n \times k$ *matrix, with* $n \geq k$*, whose* $k$ *columns are linearly independent. Then the least-squares solution of the linear system* $\mathbf{AX} = \mathbf{B}$ *is given by* $\mathbf{X} = \mathbf{R}^{-1}\mathbf{Q}^T\mathbf{B}$*, where* $\mathbf{Q}$ *and* $\mathbf{R}$ *are the matrices obtained from the* **QR** *Factorization of* $\mathbf{A}$.

*Proof.* From the preceding remarks, the least-squares solutions of $\mathbf{AX} = \mathbf{B}$ are the solutions of $\mathbf{A}^T\mathbf{AX} = \mathbf{A}^T\mathbf{B}$. Let $\mathbf{A} = \mathbf{QR}$, where $\mathbf{Q}$ and $\mathbf{R}$ are the matrices obtained from the **QR** Factorization of $\mathbf{A}$. Then, $(\mathbf{QR})^T(\mathbf{QR})\mathbf{X} = (\mathbf{QR})^T\mathbf{B}$, which gives $\mathbf{R}^T\mathbf{Q}^T\mathbf{QRX} = \mathbf{R}^T\mathbf{Q}^T\mathbf{B}$. But the columns of $\mathbf{Q}$ are orthonormal, so $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}_k$. Thus, $\mathbf{R}^T\mathbf{RX} = \mathbf{R}^T\mathbf{Q}^T\mathbf{B}$. Since $\mathbf{R}^{-1}$ exists (by the previous theorem), and since $(\mathbf{R}^{-1})^T = (\mathbf{R}^T)^{-1}$, the matrix $\mathbf{R}^T$ is also nonsingular, and we have $(\mathbf{R}^T)^{-1}\mathbf{R}^T\mathbf{RX} = (\mathbf{R}^T)^{-1}\mathbf{R}^T\mathbf{Q}^T\mathbf{B}$, which reduces to $\mathbf{RX} = \mathbf{Q}^T\mathbf{B}$, and hence $\mathbf{X} = \mathbf{R}^{-1}\mathbf{Q}^T\mathbf{B}$, as desired. □

In practice, it is often easier, and involves less roundoff error, to find the least-squares solutions of $\mathbf{AX} = \mathbf{B}$ by solving $\mathbf{RX} = \mathbf{Q}^T\mathbf{B}$ using back substitution. This process is illustrated in the next example.

### Example 2

Consider the linear system

$$\begin{cases} x \qquad\quad = 3 \\ \quad y + z = 9 \\ x + y \quad = 7.5 \\ \qquad\quad z = 5 \end{cases},$$

which is clearly inconsistent, since the first and last equations imply $x = 3$, $z = 5$, and the two middle equations then give two different values for $y$ ($y = 4$ or $y = 4.5$). We will find a least-squares solution for this system which will come "close" to satisfying all of the equations. We express the system in the form $\mathbf{AX} = \mathbf{B}$, with

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 3 \\ 9 \\ 7.5 \\ 5 \end{bmatrix}.$$

Note that $\mathbf{A}$ is the matrix from Example 1.

Recall from Example 1 that the **QR** Factorization of $\mathbf{A}$ is given by $\mathbf{A} = \mathbf{QR}$, where

$$\mathbf{Q} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{2\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{2\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{2\sqrt{3}} \\ 0 & 0 & \frac{3}{2\sqrt{3}} \end{bmatrix} \approx \begin{bmatrix} 0.707107 & -0.408208 & 0.288675 \\ 0 & 0.816497 & 0.288675 \\ 0.707107 & 0.408248 & -0.288675 \\ 0 & 0 & 0.866025 \end{bmatrix}, \text{ and } \mathbf{R} = \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{\sqrt{6}}{2} & \frac{2}{\sqrt{6}} \\ 0 & 0 & \frac{2\sqrt{3}}{3} \end{bmatrix} \approx \begin{bmatrix} 1.41421 & 0.707107 & 0 \\ 0 & 1.22474 & 0.816497 \\ 0 & 0 & 1.15470 \end{bmatrix}.$$

Now, a straightforward computation shows that

$$\mathbf{Q}^T\mathbf{B} \approx \begin{bmatrix} 7.42462 \\ 9.18559 \\ 5.62917 \end{bmatrix}, \text{ and hence, } \mathbf{RX} \approx \begin{bmatrix} 1.41421 & 0.707107 & 0 \\ 0 & 1.22474 & 0.816497 \\ 0 & 0 & 1.15470 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7.42462 \\ 9.18559 \\ 5.62917 \end{bmatrix}.$$

Since $\mathbf{R}$ is upper triangular, we can quickly find the solution by using back substitution. The last equation asserts $1.15470z = 5.62917$, which leads to $z = 4.875$. From the middle equation, we have $1.22474y + 0.816497z = 9.18559$. Substituting 4.875 for $z$ and solving for $y$, we obtain $y = 4.250$. Finally, the first equation gives $1.41421x + 0.707107y = 7.42462$. Substituting 4.25 for $y$ and solving for $x$ leads to $x = 3.125$. Hence,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \approx \begin{bmatrix} 3.125 \\ 4.250 \\ 4.875 \end{bmatrix}.$$

Finally, notice that the values $x = 3.125$, $y = 4.250$, $z = 4.875$ do, in fact, come close to satisfying each equation in the original system. For example, $y + z = 9.125$ (close to 9) and $x + y = 7.375$ (close to 7.5).

Normally, back substitution is preferable when finding least-squares solutions. However, in this particular case, the roundoff error involved in finding and using the inverse of **R** is minimal, and so the result in the previous example can also be obtained by calculating

$$\mathbf{R}^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{2\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \\ 0 & 0 & \frac{3}{2\sqrt{3}} \end{bmatrix}, \text{ and then computing } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{R}^{-1}\mathbf{Q}^T\mathbf{B} \approx \begin{bmatrix} 3.125 \\ 4.250 \\ 4.875 \end{bmatrix}.$$

It should be noted that when a system $\mathbf{AX} = \mathbf{B}$ is *consistent*, the least-squares technique produces an actual solution to the system. Thus, for a consistent system $\mathbf{AX} = \mathbf{B}$, the least-squares solution $\mathbf{X} = \mathbf{R}^{-1}\mathbf{Q}^T\mathbf{B}$ in Theorem 9.3 is an actual solution.

## A More General QR Factorization

Although we do not prove it here, it can be shown that *any* $n \times k$ matrix **A**, with $n \geq k$, has a **QR** Factorization into a product of matrices **Q** and **R**, where **Q** is an $n \times k$ matrix with orthonormal columns, and where **R** is a $k \times k$ upper triangular matrix. The proof is similar to that of Theorem 9.2, but it requires a few changes: First we determine which columns of **A** are linear combinations of previous columns of **A**. We replace these columns with new vectors so that the new matrix $\mathbf{A}'$ will have all columns linearly independent. Then we use $\mathbf{A}'$ as in Theorem 9.2 to determine **Q**. As before, $\mathbf{R} = \mathbf{Q}^T\mathbf{A}$. Notice that **R** is singular when the columns of **A** are not linearly independent. Also, the main diagonal entry of any column of **R** whose corresponding column of **A** was replaced will equal zero.

## New Vocabulary

Cholesky Factorization
least-squares solutions (for a linear system)
**QR** Factorization
**QR** Factorization Method

## Highlights

- An $n \times k$ matrix **A**, with $n \geq k$, whose $k$ columns are linearly independent has a **QR** Factorization of the form $\mathbf{A} = \mathbf{QR}$, where **Q** is an $n \times k$ matrix whose columns form an orthonormal basis for the subspace of $\mathbb{R}^n$ spanned by the columns of **A**, and **R** is a nonsingular upper triangular $k \times k$ matrix. The columns of **Q** are obtained by applying the Gram-Schmidt Process to the columns of **A** and then normalizing the results (thus producing an orthonormal set of $k$ vectors), and $\mathbf{R} = \mathbf{Q}^T\mathbf{A}$.
- If **A** is square, then the matrix **Q** obtained from the **QR** Factorization Method is an orthogonal matrix.
- For an $n \times k$ matrix **A**, with $n \geq k$, whose $k$ columns are linearly independent, the least-squares solution of the linear system $\mathbf{AX} = \mathbf{B}$ is given by $\mathbf{X} = \mathbf{R}^{-1}\mathbf{Q}^T\mathbf{B}$, where **Q** and **R** are the matrices obtained from the **QR** Factorization of **A**. The least-squares solution could also be obtained by solving $\mathbf{RX} = \mathbf{Q}^T\mathbf{B}$ using back substitution.
- A **QR** Factorization is possible for any *any* $n \times k$ matrix **A**, with $n \geq k$, where the columns of **Q** form an orthonormal basis, and where **R** is an upper triangular matrix, but **R** is singular if the columns of **A** are not linearly independent.

## Exercises for Section 9.4

1. Find a **QR** Factorization for each of the following matrices. (That is, if **A** is the given $n \times k$ matrix, find an $n \times k$ matrix **Q** and a $k \times k$ matrix **R** such that $\mathbf{A} = \mathbf{QR}$, where the columns of **Q** form an orthonormal set in $\mathbb{R}^n$, and where **R** is nonsingular and upper triangular.)

★ (a) $\begin{bmatrix} 2 & 6 & -3 \\ -2 & 0 & -9 \\ 1 & 6 & -3 \end{bmatrix}$

(b) $\begin{bmatrix} 6 & 10 & -7 \\ 7 & 8 & 1 \\ 6 & 21 & 26 \end{bmatrix}$

★ (c) $\begin{bmatrix} 1 & 5 & -3 \\ -2 & -4 & -2 \\ 1 & 5 & -5 \end{bmatrix}$

(d) $\begin{bmatrix} 4 & 4 & 14 \\ 4 & -8 & 3 \\ 0 & -3 & -14 \\ 2 & -1 & -7 \end{bmatrix}$

(e) $\begin{bmatrix} 14 & 212 & 83 & 381 \\ 70 & 70 & -140 & -210 \\ 77 & 41 & -31 & 408 \\ 0 & 60 & 75 & 90 \end{bmatrix}$

2. Find a least-squares solution for each of the following inconsistent linear systems using the method of Example 2. Round your answers to three places after the decimal point.

★ (a) $\begin{cases} 3x + 10y = -8 \\ 4x - 4y = 30 \\ 12x + 27y = 10 \end{cases}$

(b) $\begin{cases} 2x + 2y + 2z = 15 \\ x + 3y - 6z = -20 \\ \quad - 2y - 11z = -50 \\ 2x + 10y + 10z = 50 \end{cases}$

★ (c) $\begin{cases} x + 15y + z = 7 \\ 4x - 4y + 18z = 11 \\ \quad 8y - 22z = -5 \\ -8x + 10y - z = 12 \end{cases}$

(d) $\begin{cases} 3x \quad + 16z = 60 \\ 2x \quad + 6z = 25 \\ 4x - 6y + 4z = -15 \\ 4x - 12y + 2z = -59 \\ 6x - 15y + 13z = -39 \end{cases}$

3. Assume **A** is a given $n \times k$ matrix, where $n \geq k$, such that the columns of **A** are linearly independent. Suppose $\mathbf{A} = \mathbf{QR}$, where **Q** is $n \times k$ and the columns of **Q** are orthonormal, and **R** is a nonsingular upper triangular $k \times k$ matrix whose main diagonal entries are positive. Show that **Q** and **R** are unique. (Hint: Prove uniqueness for both matrices simultaneously, column by column, starting with the first column.)
4. The following exercise concerns the matrix $\mathbf{A}^T\mathbf{A}$.
   (a) If **A** is square, prove that $\mathbf{A}^T\mathbf{A} = \mathbf{U}^T\mathbf{U}$, where **U** is upper triangular and has nonnegative diagonal entries. (Hint: You will need to assume the existence of the **QR** Factorization, even if the columns **A** are not linearly independent.)
   (b) If **A** is nonsingular, prove that the matrix **U** in part (a) is unique. ($\mathbf{U}^T\mathbf{U}$ is known as the **Cholesky Factorization of $\mathbf{A}^T\mathbf{A}$**.) (Hint: If $\mathbf{A} = \mathbf{QR}$ is a **QR** Factorization of **A**, and $\mathbf{A}^T\mathbf{A} = \mathbf{U}^T\mathbf{U}$, show that $(\mathbf{Q}(\mathbf{R}^T)^{-1}\mathbf{U}^T)\mathbf{U}$ is a **QR** Factorization of **A**. Then apply Exercise 3.)

★ 5. True or False:
   (a) If **A** is a nonsingular matrix, then applying the **QR** Factorization Method to **A** produces a matrix **Q** that is orthogonal.
   (b) If **A** is a singular matrix, then applying the **QR** Factorization Method to **A** produces a matrix **R** having at least one main diagonal entry equal to zero.
   (c) If **A** is an $n \times n$ upper triangular matrix with linearly independent columns, then applying the **QR** Factorization Method to **A** produces a matrix **Q** that is diagonal.
   (d) If **A** is an $n \times k$ matrix with $k$ linearly independent columns, the inconsistent system $\mathbf{AX} = \mathbf{B}$ has $\mathbf{X} = \mathbf{R}^T\mathbf{Q}^{-1}\mathbf{B}$ as a least-squares solution, where **Q** and **R** are the matrices obtained after applying the **QR** Factorization Method to **A**.
   (e) If **A** is an $n \times k$ matrix, with $n \geq k$, and $\mathbf{A} = \mathbf{QR}$ is a **QR** Factorization of **A**, then $\mathbf{R} = \mathbf{Q}^T\mathbf{A}$.

## 9.5 Singular Value Decomposition

*Prerequisite: Section 6.3: Orthogonal Diagonalization*

We have seen that for a linear operator $L$ on $\mathbb{R}^n$, finding an ordered basis $B$ such that the matrix for $L$ with respect to $B$ is diagonal makes the operator $L$ easier to handle and to understand. In this section, we consider the more general situation of linear transformations $L: \mathbb{R}^n \to \mathbb{R}^m$. We will discover that we can always find ordered orthonormal bases $B$ and $C$ for $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, so that the $m \times n$ matrix for $L$ with respect to $B$ and $C$ is, in some sense, "diagonal." In particular, we will see that every $m \times n$ matrix $\mathbf{A}$ can be expressed as $\mathbf{A} = \mathbf{QDP}^T$, where $\mathbf{D}$ is an $m \times n$ matrix with nonnegative entries on its main diagonal and zeroes elsewhere, and $\mathbf{P}$ and $\mathbf{Q}$ are orthogonal matrices. The specific product for $\mathbf{A}$ of this type introduced in this section is called a **Singular Value Decomposition** of $\mathbf{A}$.

### Singular Values and Right Singular Vectors

If $\mathbf{A}$ is any $m \times n$ matrix, then $\mathbf{A}^T\mathbf{A}$ is symmetric because $\left(\mathbf{A}^T\mathbf{A}\right)^T = \mathbf{A}^T\left(\mathbf{A}^T\right)^T = \mathbf{A}^T\mathbf{A}$. Thus, by Corollary 6.23, $\mathbf{A}^T\mathbf{A}$ is orthogonally diagonalizable. That is, there is an orthogonal matrix $\mathbf{P}$ such that $\mathbf{P}^T(\mathbf{A}^T\mathbf{A})\mathbf{P}$ is diagonal. Also, if $\lambda$ is one of the eigenvalues of $\mathbf{A}^T\mathbf{A}$ with corresponding unit eigenvector $\mathbf{v}$ (which is one of the columns of $\mathbf{P}$), then

$$\|\mathbf{Av}\|^2 = (\mathbf{Av}) \cdot (\mathbf{Av}) = (\mathbf{v}^T\mathbf{A}^T)(\mathbf{Av}) = \mathbf{v}^T(\mathbf{A}^T\mathbf{Av}) = \mathbf{v}^T(\lambda\mathbf{v}) = \lambda(\mathbf{v}\cdot\mathbf{v}) = \lambda.$$

Hence, $\lambda \geq 0$, and so all the eigenvalues of $\mathbf{A}^T\mathbf{A}$ are nonnegative.

**Example 1**

Suppose

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix}. \quad \text{Then } \mathbf{A}^T\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}.$$

Following the Orthogonal Diagonalization Method of Section 6.3, we first find the eigenvalues and eigenvectors for $\mathbf{A}^T\mathbf{A}$. Now, $p_{\mathbf{A}^T\mathbf{A}}(x) = x^2 - 6x + 8 = (x-4)(x-2)$. Solving for fundamental eigenvectors for $\lambda_1 = 4$ and $\lambda_2 = 2$, respectively, yields $\mathbf{v}_1 = [1,1]$ for $\lambda_1$ and $\mathbf{v}_2 = [-1,1]$ for $\lambda_2$. Normalizing these vectors and using these as columns for a matrix produces the orthogonal matrix $\mathbf{P} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, for which $\mathbf{P}^T(\mathbf{A}^T\mathbf{A})\mathbf{P} = \mathbf{D}$, a diagonal matrix with the eigenvalues 4 and 2 appearing on the main diagonal.

Because all eigenvalues of $\mathbf{A}^T\mathbf{A}$ are nonnegative, we can make the following definition:

**Definition** Let $\mathbf{A}$ be an $m \times n$ matrix, and suppose that $k$ of the $n$ eigenvalues of $\mathbf{A}^T\mathbf{A}$ are positive. (If all eigenvalues of $\mathbf{A}^T\mathbf{A}$ are positive, then $k = n$.) Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > \lambda_{k+1} = \cdots = \lambda_n = 0$ be the eigenvalues of $\mathbf{A}^T\mathbf{A}$, written in decreasing order. If $\sigma_i = \sqrt{\lambda_i}$, then $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k > \sigma_{k+1} = \cdots = \sigma_n = 0$ are called the **singular values** of $\mathbf{A}$. Also suppose that $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is an orthonormal set of eigenvectors for $\mathbf{A}^T\mathbf{A}$, with $\mathbf{v}_i$ corresponding to $\lambda_i$. Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is called a corresponding set of **right singular vectors** for $\mathbf{A}$.

We will assume throughout this section that the eigenvalues for $\mathbf{A}^T\mathbf{A}$ and the singular values of the matrix $\mathbf{A}$ are always labeled in nonincreasing order, as in this definition.

The singular values of the matrix $\mathbf{A}$ in Example 1 are $\sigma_1 = \sqrt{\lambda_1} = \sqrt{4} = 2$ and $\sigma_2 = \sqrt{\lambda_2} = \sqrt{2}$. A corresponding set of right singular vectors is $\left\{\frac{1}{\sqrt{2}}[1,1], \frac{1}{\sqrt{2}}[-1,1]\right\}$.

We will use the following lemma throughout this section:

**Lemma 9.4** *Suppose* $\mathbf{A}$ *is an* $m \times n$ *matrix,* $\lambda_1, \ldots, \lambda_n$ *are the eigenvalues of* $\mathbf{A}^T\mathbf{A}$, $\sigma_1, \ldots, \sigma_n$ *are the singular values of* $\mathbf{A}$, *and* $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ *is a corresponding set of right singular vectors for* $\mathbf{A}$. *Then:*

(1) *For all* $\mathbf{x} \in \mathbb{R}^n$, $(\mathbf{Ax}) \cdot (\mathbf{Av}_i) = \lambda_i(\mathbf{x} \cdot \mathbf{v}_i)$.

(2) *For* $i \neq j$, $(\mathbf{Av}_i) \cdot (\mathbf{Av}_j) = 0$.

(3) $(\mathbf{Av}_i) \cdot (\mathbf{Av}_i) = \|\mathbf{Av}_i\|^2 = \lambda_i = \sigma_i^2$.

(4) *If* $\mathbf{x} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$, *then* $(\mathbf{Ax}) \cdot (\mathbf{Av}_i) = a_i\lambda_i$.

*Proof.* Part (1): $(\mathbf{Ax})\cdot(\mathbf{Av}_i) = (\mathbf{x}^T\mathbf{A}^T)(\mathbf{Av}_i) = \mathbf{x}^T(\mathbf{A}^T\mathbf{Av}_i) = \mathbf{x}^T(\lambda_i\mathbf{v}_i) = \lambda_i(\mathbf{x}\cdot\mathbf{v}_i)$.

Part (2): By part (1), $(\mathbf{Av}_i)\cdot(\mathbf{Av}_j) = \lambda_j(\mathbf{v}_i\cdot\mathbf{v}_j) = 0$, since $\mathbf{v}_i \perp \mathbf{v}_j$. Hence, $(\mathbf{Av}_i)\cdot(\mathbf{Av}_j) = 0$.

Part (3): $(\mathbf{Av}_i)\cdot(\mathbf{Av}_i) = \|\mathbf{Av}_i\|^2$, by part (2) of Theorem 1.5. Also, by part (1), $(\mathbf{Av}_i)\cdot(\mathbf{Av}_i) = \lambda_i(\mathbf{v}_i\cdot\mathbf{v}_i) = \lambda_i$.

Part (4): If $\mathbf{x} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$, then $(\mathbf{Ax})\cdot(\mathbf{Av}_i) = (a_1\mathbf{Av}_1 + \cdots + a_n\mathbf{Av}_n)\cdot(\mathbf{Av}_i) = a_1(\mathbf{Av}_1)\cdot(\mathbf{Av}_i) + \cdots + a_n(\mathbf{Av}_n)\cdot(\mathbf{Av}_i) = a_i\lambda_i$, by parts (2) and (3). □

**Example 2**

Suppose $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix}$ is the $3\times 2$ matrix from Example 1, with $\lambda_1 = 4$, $\lambda_2 = 2$, $\mathbf{v}_1 = \frac{1}{\sqrt{2}}[1,1]$, and $\mathbf{v}_2 = \frac{1}{\sqrt{2}}[-1,1]$. Matrix multiplication gives $\mathbf{Av}_1 = \frac{1}{\sqrt{2}}[0,2,-2]$ and $\mathbf{Av}_2 = \frac{1}{\sqrt{2}}[-2,0,0]$. Note that $(\mathbf{Av}_1)\cdot(\mathbf{Av}_2) = 0$, $(\mathbf{Av}_1)\cdot(\mathbf{Av}_1) = \frac{1}{2}(0+4+4) = 4 = \lambda_1$, and $(\mathbf{Av}_2)\cdot(\mathbf{Av}_2) = \frac{1}{2}(4+0+0) = 2 = \lambda_2$, verifying parts (2) and (3) of Lemma 9.4.

Let $\mathbf{x} = [5,1]$. It is easy to check that $\mathbf{x} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2$, where $a_1 = 3\sqrt{2}$, and $a_2 = -2\sqrt{2}$. Then $\mathbf{Ax} = [4,6,-6]$, and so $(\mathbf{Ax})\cdot(\mathbf{Av}_1) = [4,6,-6]\cdot\left(\frac{1}{\sqrt{2}}[0,2,-2]\right) = \frac{1}{\sqrt{2}}(0+12+12) = 12\sqrt{2} = (3\sqrt{2})\lambda_1 = a_1\lambda_1$. Similarly, $(\mathbf{Ax})\cdot(\mathbf{Av}_2) = [4,6,-6]\cdot\left(\frac{1}{\sqrt{2}}[-2,0,0]\right) = \frac{1}{\sqrt{2}}(-8+0+0) = -4\sqrt{2} = (-2\sqrt{2})\lambda_2 = a_2\lambda_2$. This verifies part (4) for this vector $\mathbf{x}$.

## Singular Values and Left Singular Vectors

Since a set of right singular vectors $\{\mathbf{v}_1,\ldots,\mathbf{v}_n\}$ forms an orthonormal basis for $\mathbb{R}^n$, the set $\{\mathbf{Av}_1,\ldots,\mathbf{Av}_n\}$ spans the range of the linear transformation $L\colon \mathbb{R}^n \to \mathbb{R}^m$ given by $L(\mathbf{v}) = \mathbf{Av}$. Now, assuming as before that $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_k > \lambda_{k+1} = \cdots = \lambda_n = 0$, we see that, by parts (2) and (3) of Lemma 9.4, $\mathbf{Av}_{k+1} = \cdots = \mathbf{Av}_n = \mathbf{0}$, while $\{\mathbf{Av}_1,\ldots,\mathbf{Av}_k\}$ forms a nonzero orthogonal spanning set for range$(L)$, and hence an orthogonal basis for range$(L)$ by Theorem 6.1. The important role played by the vectors $\mathbf{Av}_1,\ldots,\mathbf{Av}_k$ leads to the following definition.

**Definition** Let $\mathbf{A}$ be an $m\times n$ matrix, and suppose $k$ of the $n$ eigenvalues of $\mathbf{A}^T\mathbf{A}$ are positive. Let $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_k > \sigma_{k+1} = \cdots = \sigma_n = 0$ be the singular values of $\mathbf{A}$. Also suppose that $\{\mathbf{v}_1,\ldots,\mathbf{v}_n\}$ is a set of right singular vectors for $\mathbf{A}$, with $\mathbf{v}_i$ corresponding to $\sigma_i$. If $\mathbf{u}_i = \frac{1}{\sigma_i}\mathbf{Av}_i$, for $1 \le i \le k$, and $\mathbf{u}_{k+1},\ldots,\mathbf{u}_m$ are chosen so that $\{\mathbf{u}_1,\ldots,\mathbf{u}_m\}$ is an orthonormal basis for $\mathbb{R}^m$, then $\{\mathbf{u}_1,\ldots,\mathbf{u}_m\}$ is called a set of **left singular vectors** for $\mathbf{A}$ corresponding to the set $\{\mathbf{v}_1,\ldots,\mathbf{v}_n\}$ of right singular vectors.

Notice that $\{\mathbf{u}_1,\ldots,\mathbf{u}_k\}$ is an orthonormal basis for range$(L)$ because $\{\mathbf{Av}_1,\ldots,\mathbf{Av}_k\}$ is an orthogonal basis for range$(L)$ and each $\mathbf{u}_i$ is also a unit vector by part (3) of Lemma 9.4. Therefore, to find a set of left singular vectors, we first compute $\mathbf{u}_1,\ldots,\mathbf{u}_k$, and then expand this set to an orthonormal basis for $\mathbb{R}^m$.

**Example 3**

Suppose $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix}$ is the $3\times 2$ matrix from Example 1, with $\sigma_1 = 2$, $\sigma_2 = \sqrt{2}$, $\mathbf{v}_1 = \frac{1}{\sqrt{2}}[1,1]$, and $\mathbf{v}_2 = \frac{1}{\sqrt{2}}[-1,1]$. In Example 2 we found that $\mathbf{Av}_1 = \frac{1}{\sqrt{2}}[0,2,-2]$ and $\mathbf{Av}_2 = \frac{1}{\sqrt{2}}[-2,0,0]$. Hence, $\mathbf{u}_1 = \frac{1}{\sigma_1}\mathbf{Av}_1 = \frac{1}{\sqrt{2}}[0,1,-1]$, and $\mathbf{u}_2 = \frac{1}{\sigma_2}\mathbf{Av}_2 = [-1,0,0]$. To find $\mathbf{u}_3$, we must expand the orthonormal set $\left\{\frac{1}{\sqrt{2}}[0,1,-1],[-1,0,0]\right\}$ to a basis for $\mathbb{R}^3$. Inspection (or row reduction) shows that $\left\{\frac{1}{\sqrt{2}}[0,1,-1],[-1,0,0],[0,1,0]\right\}$ is a linearly independent set. To convert this to an orthonormal basis for $\mathbb{R}^3$, we perform the Gram-Schmidt Process on this set and normalize. This does not affect the first two vectors, but changes the third vector to $\mathbf{u}_3 = \frac{1}{\sqrt{2}}[0,1,1]$. Thus, $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\} = \left\{\frac{1}{\sqrt{2}}[0,1,-1],[-1,0,0],\frac{1}{\sqrt{2}}[0,1,1]\right\}$ is a set of left singular vectors for $\mathbf{A}$ corresponding to the set $\{\mathbf{v}_1,\mathbf{v}_2\} = \left\{\frac{1}{\sqrt{2}}[1,1],\frac{1}{\sqrt{2}}[-1,1]\right\}$ of right singular vectors.

## Orthonormal Bases Derived From the Left and Right Singular Vectors

We can now prove that the sets of left and right singular vectors can each be split into two parts, with each part being an orthonormal basis for an important subspace of $\mathbb{R}^n$ or $\mathbb{R}^m$.

**Theorem 9.5** *Let* $\mathbf{A}$ *be an* $m \times n$ *matrix, and suppose that* $k$ *of the* $n$ *eigenvalues of* $\mathbf{A}^T\mathbf{A}$ *are positive. Let* $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k > \sigma_{k+1} = \cdots = \sigma_n = 0$ *be the singular values of* $\mathbf{A}$. *Also suppose that* $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ *is a set of right singular vectors for* $\mathbf{A}$, *with* $\mathbf{v}_i$ *corresponding to* $\sigma_i$ *and that* $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ *is a corresponding set of left singular vectors for* $\mathbf{A}$. *Finally, suppose that* $L\colon \mathbb{R}^n \to \mathbb{R}^m$ *and* $L_T\colon \mathbb{R}^m \to \mathbb{R}^n$ *are linear transformations given, respectively, by* $L(\mathbf{x}) = \mathbf{A}\mathbf{x}$ *and* $L_T(\mathbf{y}) = \mathbf{A}^T\mathbf{y}$. *Then,*

**(1)** $\text{rank}(\mathbf{A}) = k$,

**(2)** $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ *is an orthonormal basis for* $\text{range}(L)$,

**(3)** $\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_m\}$ *is an orthonormal basis for* $\ker(L_T) = (\text{range}(L))^\perp$,

**(4)** $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ *is an orthonormal basis for* $\text{range}(L_T) = (\ker(L))^\perp =$ *the row space of* $\mathbf{A}$, *and*

**(5)** $\{\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$ *is an orthonormal basis for* $\ker(L)$.

*Proof.* Part (2) was proven in the discussion before and after the definition of left singular vectors. This combined with part (1) of Theorem 5.9 proves that $\text{rank}(\mathbf{A}) = k$, giving us part (1) of the theorem.

The fact that $\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_m\}$ is an orthonormal basis for $(\text{range}(L))^\perp$ in part (3) follows directly from part (2) and Theorem 6.12.

To prove the set equality $\ker(L_T) = (\text{range}(L))^\perp$ in part (3), we first show that $\ker(L_T) \subseteq (\text{range}(L))^\perp$. Let $\mathbf{y} \in \ker(L_T)$. Then $L_T(\mathbf{y}) = \mathbf{A}^T\mathbf{y} = \mathbf{0}$. To show that $\mathbf{y} \in (\text{range}(L))^\perp$, we will show that $\mathbf{y}$ is orthogonal to every vector in the orthonormal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ for $\text{range}(L)$. Now, for $1 \leq i \leq k$,

$$\mathbf{y} \cdot \mathbf{u}_i = \mathbf{y} \cdot \left(\frac{1}{\sigma_i}\mathbf{A}\mathbf{v}_i\right) = \frac{1}{\sigma_i}\left(\mathbf{y} \cdot (\mathbf{A}\mathbf{v}_i)\right) = \frac{1}{\sigma_i}\mathbf{y}^T\mathbf{A}\mathbf{v}_i = \frac{1}{\sigma_i}\left(\mathbf{A}^T\mathbf{y}\right)^T\mathbf{v}_i = \frac{1}{\sigma_i}(\mathbf{0})^T\mathbf{v}_i = 0.$$

Therefore, $\ker(L_T) \subseteq (\text{range}(L))^\perp$.

We know from the basis for $(\text{range}(L))^\perp$ given above that $\dim\left((\text{range}(L))^\perp\right) = m - k$. But, by part (2) of Theorem 5.9, part (1) of this theorem, and Corollary 5.11 we see that $\dim(\ker(L_T)) = m - \text{rank}(\mathbf{A}^T) = m - \text{rank}(\mathbf{A}) = m - k$. Hence, $\ker(L_T)$ is a subspace of $(\text{range}(L))^\perp$ having the same dimension, and so $\ker(L_T) = (\text{range}(L))^\perp$.

To prove part (5), notice that for $i \geq k+1$, $\|\mathbf{A}\mathbf{v}_i\| = \sqrt{\lambda_i} = \sigma_i = 0$, by part (3) of Lemma 9.4. Hence, $\{\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$ is an orthonormal subset of $\ker(L)$. Also, $\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n$ are nonzero (eigen)vectors, and so are linearly independent by Theorem 6.1. But part (2) of Theorem 5.9 shows that $\dim(\ker(L)) = n - \text{rank}(\mathbf{A}) = n - k$. Therefore, since $\{\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$ is a linearly independent subset of $\ker(L)$ having the correct size, it is an orthonormal basis for $\ker(L)$.

Finally, to prove part (4), first note that $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an orthonormal basis for $(\ker(L))^\perp$ by part (5) and Theorem 6.12. Now, from part (3), $\ker(L_T) = (\text{range}(L))^\perp$. If we replace the $m \times n$ matrix $\mathbf{A}$ with the $n \times m$ matrix $\mathbf{A}^T$, the roles of $L$ and $L_T$ are reversed. Thus, applying part (3) of the theorem (with the matrix $\mathbf{A}^T$) shows that $\ker(L) = (\text{range}(L_T))^\perp$. Taking the orthogonal complement of both sides yields $(\ker(L))^\perp = \left((\text{range}(L_T))^\perp\right)^\perp = \text{range}(L_T)$.

To finish the proof of part (4), recall from Section 5.3 that the range of a linear transformation equals the column space of the matrix for the transformation. Therefore, $\text{range}(L_T) =$ the column space of $\mathbf{A}^T =$ the row space of $\mathbf{A}$. □

### Example 4

Once again, consider the $3 \times 2$ matrix $\mathbf{A}$ from Examples 1, 2, and 3 having right singular vectors $\{\mathbf{v}_1, \mathbf{v}_2\} = \left\{\frac{1}{\sqrt{2}}[1, 1], \frac{1}{\sqrt{2}}[-1, 1]\right\}$ and left singular vectors $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \left\{\frac{1}{\sqrt{2}}[0, 1, -1], [-1, 0, 0], \frac{1}{\sqrt{2}}[0, 1, 1]\right\}$. Let $L$ and $L_T$ be as given in Theorem 9.5. Since $\sigma_1 = 2$ and $\sigma_2 = \sqrt{2}$, we see that $k = 2$. Then part (1) of Theorem 9.5 asserts that $\text{rank}(\mathbf{A}) = 2$. We can verify this by noting that

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix} \text{ row reduces to } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

which has rank 2.

Since $k = 2$, part (2) of Theorem 9.5 asserts that $\{\mathbf{u}_1, \mathbf{u}_2\} = \left\{\frac{1}{\sqrt{2}}[0, 1, -1], [-1, 0, 0]\right\}$ is an orthonormal basis for range$(L)$, and part (3) asserts that $\{\mathbf{u}_3\}$ is an orthonormal basis for $\ker(L_T) = (\text{range}(L))^{\perp}$. We can verify these facts independently. Notice that by applying the Range Method to the row reduced matrix for $\mathbf{A}$, we see that $\dim(\text{range}(L)) = 2$. Also note that $L\left(\frac{1}{2\sqrt{2}}[1, 1]\right) = \frac{1}{\sqrt{2}}[0, 1, -1] = \mathbf{u}_1$ and $L\left(\frac{1}{2}[-1, 1]\right) = [-1, 0, 0] = \mathbf{u}_2$. Thus, $\mathbf{u}_1$ and $\mathbf{u}_2$ are in range$(L)$, and since they are orthogonal unit vectors, they form a linearly independent set of the right size, making $\{\mathbf{u}_1, \mathbf{u}_2\}$ an orthonormal basis for range$(L)$. Finally, the vector $\mathbf{u}_3$ is easily shown to be in $\ker(L_T)$ by computing

$$L_T(\mathbf{u}_3) = \mathbf{A}^T\mathbf{u}_3 = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Since $\dim(\text{range}(L)) = 2$, and the codomain of $L$ is $\mathbb{R}^3$, we have $\dim((\text{range}(L))^{\perp}) = 1$. Thus, since $\mathbf{u}_3$ is a unit vector, $\{\mathbf{u}_3\}$ is an orthonormal basis for $(\text{range}(L_T))^{\perp} = \ker(L_T)$.

Notice that the row space of $\mathbf{A}$ equals $\mathbb{R}^2$, so by part (4) of Theorem 9.5, range$(L_T) = \mathbb{R}^2$. We can confirm this by noting that the orthonormal set $\{\mathbf{v}_1, \mathbf{v}_2\}$ of right singular vectors is clearly a basis for $\mathbb{R}^2$. Finally, part (5) of Theorem 9.5 asserts $\dim(\ker(L)) = 0$, which can be verified by applying the Kernel Method to the reduced row echelon form of $\mathbf{A}$ given earlier.

## Singular Value Decomposition

We now have the machinery in place to easily prove the existence of a Singular Value Decomposition for any matrix.

**Theorem 9.6** *Let $\mathbf{A}$ be an $m \times n$ matrix, and let $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k > \sigma_{k+1} = \cdots = \sigma_n = 0$ be the singular values of $\mathbf{A}$. Also suppose that $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a set of right singular vectors for $\mathbf{A}$, with $\mathbf{v}_i$ corresponding to $\sigma_i$, and that $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ is a corresponding set of left singular vectors for $\mathbf{A}$. Let $\mathbf{U}$ be the $m \times m$ orthogonal matrix whose columns are $\mathbf{u}_1, \ldots, \mathbf{u}_m$, and let $\mathbf{V}$ be the $n \times n$ orthogonal matrix whose columns are $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Finally, let $\mathbf{\Sigma}$ represent the $m \times n$ "diagonal" matrix whose $(i, i)$ entry equals $\sigma_i$, for $i \leq k$, with all other entries equal to zero. Then*

$$\mathbf{A} = \mathbf{U\Sigma V}^T.$$

The expression of the matrix $\mathbf{A}$ as the product $\mathbf{U\Sigma V}^T$, as given in Theorem 9.6, is known as a **Singular Value Decomposition** of $\mathbf{A}$.[5]

Note that since $\text{rank}(\mathbf{A}) = k$ must be less than or equal to both $m$ and $n$, all $k$ of the nonzero singular values for $\mathbf{A}$ will appear on the main diagonal of $\mathbf{\Sigma}$, even though some of the zero-valued singular values will not appear if $m < n$. If $m > n$, there will be more main diagonal terms than there are singular values. All of these main diagonal terms will be zero. Finally, note that we have used the capital Greek letter $\mathbf{\Sigma}$ (sigma) for the diagonal matrix. This is traditional usage, and refers to the fact that the singular values, which are denoted using the lowercase $\sigma$ (sigma), appear on the main diagonal.

*Proof.* In general, we can prove that two $m \times n$ matrices $\mathbf{B}$ and $\mathbf{C}$ are equal by showing that $\mathbf{Bw}_i = \mathbf{Cw}_i$ for every $\mathbf{w}_i$ in a basis $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ for $\mathbb{R}^n$. This is because we can consider $\mathbf{B}$ and $\mathbf{C}$ to be matrices for linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$ with respect to the standard bases, and then by Theorem 5.4, since $\mathbf{Bw}_i = \mathbf{Cw}_i$ for every $\mathbf{w}_i$ in a basis, these linear transformations must be the same. Finally, by the uniqueness of the matrix for a linear transformation in Theorem 5.5, we must have $\mathbf{B} = \mathbf{C}$. We use this technique here to show that $\mathbf{A} = \mathbf{U\Sigma V}^T$.

Consider the basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ for $\mathbb{R}^n$. For each $i$, $1 \leq i \leq n$, $\mathbf{U\Sigma V}^T\mathbf{v}_i = \mathbf{U\Sigma e}_i$, because the basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is orthonormal, and the rows of $\mathbf{V}^T$ are the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$. If $i \leq k$, $\mathbf{U\Sigma e}_i = \mathbf{U}(\sigma_i\mathbf{e}_i) = \sigma_i\mathbf{Ue}_i = \sigma_i(i\text{th column of }\mathbf{U}) = \sigma_i\mathbf{u}_i = \sigma_i\frac{1}{\sigma_i}\mathbf{Av}_i = \mathbf{Av}_i$. If $i > k$, then $\mathbf{U\Sigma e}_i = \mathbf{U}(\mathbf{0}) = \mathbf{0}$. But when $i > k$, $\mathbf{Av}_i = \mathbf{0}$ by part (5) of Theorem 9.5. Hence, $\mathbf{U\Sigma V}^T\mathbf{v}_i = \mathbf{Av}_i$ for every basis vector $\mathbf{v}_i$, and so $\mathbf{A} = \mathbf{U\Sigma V}^T$. □

[5] We will see in Exercise 8 that a general decomposition of $\mathbf{A}$ of the form $\mathbf{U\Sigma V}^T$, where $\mathbf{U}$, $\mathbf{V}$ are orthogonal and with $\Sigma$ as given in Theorem 9.6 is not necessarily unique.

**Example 5**

We find a Singular Value Decomposition for the matrix $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix}$ from Examples 1 through 4. In these previous examples, we found the singular values $\sigma_1 = 2$ and $\sigma_2 = \sqrt{2}$, the set of right singular vectors $\left\{\frac{1}{\sqrt{2}}[1,1], \frac{1}{\sqrt{2}}[-1,1]\right\}$, and the corresponding set of left singular vectors $\left\{\frac{1}{\sqrt{2}}[0,1,-1], [-1,0,0], \frac{1}{\sqrt{2}}[0,1,1]\right\}$. Using the right singular vectors as the columns for $\mathbf{V}$, the left singular vectors as the columns of $\mathbf{U}$, and the singular values on the diagonal of $\mathbf{\Sigma}$ yields

$$\mathbf{V} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \mathbf{U} = \begin{bmatrix} 0 & -1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}, \text{ and } \mathbf{\Sigma} = \begin{bmatrix} 2 & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}.$$

A quick computation verifies that

$$\mathbf{U\Sigma V}^T = \begin{bmatrix} 0 & -1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix} \left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix} = \mathbf{A}.$$

We summarize the method for calculating the Singular Value Decomposition of a matrix as follows:

**Method for Decomposing a Matrix Into the Product of an Orthogonal Matrix, a "Diagonal" Matrix, and a (Second) Orthogonal Matrix (Singular Value Decomposition)**

Let $\mathbf{A}$ be an $m \times n$ matrix. To find a Singular Value Decomposition for $\mathbf{A}$:

**Step 1:** Calculate the $n$ singular values of $\mathbf{A}$, and label them in nonincreasing order as $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k > \sigma_{k+1} = \cdots = \sigma_n = 0$. (If all of the singular values are positive, then $k = n$.) Let $\mathbf{\Sigma}$ represent the $m \times n$ "diagonal" matrix whose $(i,i)$ entry equals $\sigma_i$, for $i \leq k$, with all other entries equal to zero.

**Step 2:** Calculate a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of right singular vectors for $\mathbf{A}$ (that is, an orthonormal set of eigenvectors for $\mathbf{A}^T\mathbf{A}$), with $\mathbf{v}_i$ corresponding to $\sigma_i$. Let $\mathbf{V}$ be the $n \times n$ orthogonal matrix whose columns are $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

**Step 3:** Calculate a set $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ of left singular vectors for $\mathbf{A}$ (that is, let $\mathbf{u}_i = \frac{1}{\sigma_i}\mathbf{A}\mathbf{v}_i$, for $1 \leq i \leq k$, and choose $\mathbf{u}_{k+1}, \ldots, \mathbf{u}_m$ so that $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ is an orthonormal basis for $\mathbb{R}^m$). Let $\mathbf{U}$ be the $m \times m$ orthogonal matrix whose columns are $\mathbf{u}_1, \ldots, \mathbf{u}_m$.

**Step 4:** Then $\mathbf{A} = \mathbf{U\Sigma V}^T$ is a Singular Value Decomposition.

**Example 6**

Consider the $3 \times 4$ matrix

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & 4 & 0 \\ -2 & -6 & -2 & -6 \\ 5 & 3 & 5 & 3 \end{bmatrix}, \text{ for which } \mathbf{A}^T\mathbf{A} = \begin{bmatrix} 45 & 27 & 45 & 27 \\ 27 & 45 & 27 & 45 \\ 45 & 27 & 45 & 27 \\ 27 & 45 & 27 & 45 \end{bmatrix}.$$

Now $p_{\mathbf{A}^T\mathbf{A}}(x) = x^4 - 180x^3 + 5184x^2 = x^2(x-144)(x-36)$. Hence, the eigenvalues of $\mathbf{A}^T\mathbf{A}$ are $\lambda_1 = 144$, $\lambda_2 = 36$, and $\lambda_3 = \lambda_4 = 0$. Thus, the singular values for $\mathbf{A}$ are the square roots of these eigenvalues, namely $\sigma_1 = 12$, $\sigma_2 = 6$, and $\sigma_3 = \sigma_4 = 0$. Note that $k = 2$. Thus, $\text{rank}(\mathbf{A}) = 2$.

Solving for fundamental eigenvectors for $\mathbf{A}^T\mathbf{A}$ and normalizing produces the following right singular vectors for $\mathbf{A}$: $\mathbf{v}_1 = \frac{1}{2}[1,1,1,1]$, $\mathbf{v}_2 = \frac{1}{2}[-1,1,-1,1]$, $\mathbf{v}_3 = \frac{1}{\sqrt{2}}[-1,0,1,0]$, and $\mathbf{v}_4 = \frac{1}{\sqrt{2}}[0,-1,0,1]$. Luckily, in this case, the method for finding fundamental eigenvectors happened to produce vectors $\mathbf{v}_3$ and $\mathbf{v}_4$ that are already orthogonal. Otherwise we would have had to apply the Gram-Schmidt Process to find an orthogonal basis for the eigenspace for $\lambda_3 = \lambda_4 = 0$.

Next we solve for the left singular vectors. Now, $\mathbf{u}_1 = \frac{1}{\sigma_1}\mathbf{A}\mathbf{v}_1 = \frac{1}{12}[4,-8,8] = \frac{1}{3}[1,-2,2]$. Similarly, $\mathbf{u}_2 = \frac{1}{\sigma_2}\mathbf{A}\mathbf{v}_2 = \frac{1}{6}[-4,-4,-2] = \frac{1}{3}[-2,-2,-1]$. To find $\mathbf{u}_3$, we apply the Independence Test Method to the set $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ and discover that $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{e}_1\}$ is linearly independent. Since $\mathbf{u}_1$ and $\mathbf{u}_2$ are already orthogonal, applying the Gram-Schmidt Process to this set of vectors only affects the third vector, replacing it with $\frac{1}{9}[4,-2,-4]$, which normalizes to yield $\mathbf{u}_3 = \frac{1}{3}[2,-1,-2]$.

Using all of these singular vectors and singular values, we obtain the matrices

$$\mathbf{U} = \frac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ -2 & -2 & -1 \\ 2 & -1 & -2 \end{bmatrix}, \quad \boldsymbol{\Sigma} = \begin{bmatrix} 12 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ and } \mathbf{V} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

You can verify that $\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T$.

Let us also verify the various parts of Theorem 9.5. The matrix $\mathbf{A}$ row reduces to

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and so we have independently confirmed that $\text{rank}(\mathbf{A}) = 2$, as part (1) of Theorem 9.5 claims.

Note that both $\mathbf{u}_1$ and $\mathbf{u}_2$ are in $\text{range}(L)$, since $L\left(\left[\frac{1}{12},\frac{1}{12},0,0\right]\right) = \left[\frac{1}{3},-\frac{2}{3},\frac{2}{3}\right] = \mathbf{u}_1$, and $L\left(\left[-\frac{1}{6},\frac{1}{6},0,0\right]\right) = \left[-\frac{2}{3},-\frac{2}{3},-\frac{1}{3}\right] = \mathbf{u}_2$. Since $\text{rank}(\mathbf{A}) = 2$, $\dim(\text{range}(L)) = 2$. Therefore, $\{\mathbf{u}_1,\mathbf{u}_2\}$ is an orthonormal basis for $\text{range}(L)$, as claimed in part (2) of Theorem 9.5. Checking that $\mathbf{A}^T\mathbf{u}_3 = \mathbf{0}$ verifies the claim in part (3) of Theorem 9.5 that $\{\mathbf{u}_3\} \subseteq \ker(\mathbf{A}^T)$. Since $\ker(L_T) = (\text{range}(L))^\perp \subseteq \mathbb{R}^3$, we have $\dim((\text{range}(L))^\perp) = 1$. Therefore, $\{\mathbf{u}_3\}$ is an orthonormal basis for $(\text{range}(L))^\perp = \ker(L_T)$, as claimed by part (3) of Theorem 9.5.

Also, the $2 \times 4$ matrix whose rows are the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ row reduces to the matrix whose two rows are the same as the first two rows of $\mathbf{B}$. Therefore, by the Simplified Span Method, the set $\{\mathbf{v}_1,\mathbf{v}_2\}$ spans the same subspace as the rows of $\mathbf{A}$. This verifies the claim in part (4) of Theorem 9.5 that $\{\mathbf{v}_1,\mathbf{v}_2\}$ is an orthonormal basis for the row space of $\mathbf{A}$. Finally, for part (5) of Theorem 9.5, a quick computation verifies that $\mathbf{A}\mathbf{v}_3 = \mathbf{A}\mathbf{v}_4 = \mathbf{0}$, and so $\{\mathbf{v}_3,\mathbf{v}_4\} \subseteq \ker(\mathbf{A})$.

## A Geometric Interpretation

Theorem 6.9 and Exercise 18 in Section 6.1 indicate that multiplying vectors in $\mathbb{R}^n$ by an orthogonal matrix preserves the lengths of vectors and angles between them. Such a transformation on $\mathbb{R}^n$ represents an **isometry** on $\mathbb{R}^n$. (For further information on isometries, see the corresponding web section "Isometries on Inner Product Spaces" for this text.) In $\mathbb{R}^3$, such isometries can be shown to be compositions of orthogonal reflections and rotations. (See Exercise 13 in Section 6.3.) Therefore, by expressing an $m \times n$ matrix $\mathbf{A}$ as the product $\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T$ from a Singular Value Decomposition, we are showing that the linear transformation $L: \mathbb{R}^m \to \mathbb{R}^n$ given by $L(\mathbf{v}) = \mathbf{A}\mathbf{v}$ can be thought of as the composition of an isometry on $\mathbb{R}^m$, followed by a projection onto the first $k$ axes of $\mathbb{R}^n$ that is combined with a contraction or dilation along each of these $k$ axes, followed by another isometry on $\mathbb{R}^n$.

### Example 7

Let $\mathbf{A} = \begin{bmatrix} 9 & 12 & -8 \\ 12 & 16 & 6 \end{bmatrix}$. Computing the eigenvalues and corresponding fundamental unit eigenvectors for $\mathbf{A}^T\mathbf{A}$ yields $\lambda_1 = 625$, $\lambda_2 = 100$, and $\lambda_3 = 0$, with $\mathbf{v}_1 = \frac{1}{5}[3,4,0]$, $\mathbf{v}_2 = [0,0,1]$, and $\mathbf{v}_3 = \frac{1}{5}[-4,3,0]$. Hence, $k = 2$, and the singular values for $\mathbf{A}$ are $\sigma_1 = 25$, $\sigma_2 = 10$, and $\sigma_3 = 0$. The corresponding left singular vectors are $\mathbf{u}_1 = \frac{1}{25}\mathbf{A}\mathbf{v}_1 = \frac{1}{5}[3,4]$ and $\mathbf{u}_2 = \frac{1}{10}\mathbf{A}\mathbf{v}_2 = \frac{1}{5}[-4,3]$. Hence, a Singular Value Decomposition for $\mathbf{A}$ is

$$\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}\begin{bmatrix} 25 & 0 & 0 \\ 0 & 10 & 0 \end{bmatrix}\begin{bmatrix} \frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 \\ -\frac{4}{5} & \frac{3}{5} & 0 \end{bmatrix}.$$

Using techniques from Chapter 6, the matrix $\mathbf{V}^T$ can be shown, with some effort, to represent an orthogonal reflection[6] through the plane $x - 2y + 2z = 0$, followed by a clockwise rotation about the axis $[1, -2, 2]$ through an angle of $\arccos(\frac{4}{5}) \approx 37°$. The matrix $\mathbf{\Sigma}$ then projects $\mathbb{R}^3$ on to the $xy$-plane, dilating by a factor of 25 in the $x$-direction and by a factor of 10 in the $y$-direction. Finally, multiplying the result of this transformation by $\mathbf{U}$ rotates the plane counterclockwise through an angle of $\arccos(\frac{3}{5}) \approx 53°$.

## The Outer Product Form for Singular Value Decomposition

The next theorem introduces a different form for a Singular Value Decomposition that is frequently useful.

**Theorem 9.7** *Let* $\mathbf{A}$ *be an* $m \times n$ *matrix, and let* $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k > \sigma_{k+1} = \cdots = \sigma_n = 0$ *be the singular values of* $\mathbf{A}$. *Also suppose that* $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ *is a set of right singular vectors for* $\mathbf{A}$, *with* $\mathbf{v}_i$ *corresponding to* $\sigma_i$, *and that* $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ *is a corresponding set of left singular vectors for* $\mathbf{A}$. *Then*

$$\mathbf{A} = \sigma_1\mathbf{u}_1\mathbf{v}_1^T + \sigma_2\mathbf{u}_2\mathbf{v}_2^T + \cdots + \sigma_k\mathbf{u}_k\mathbf{v}_k^T.$$

The expression for $\mathbf{A}$ in Theorem 9.7 is called the **outer product form** of the given Singular Value Decomposition for $\mathbf{A}$. In this decomposition, each $\mathbf{u}_i$ is considered to be a $m \times 1$ matrix (that is, a column vector), and each $\mathbf{v}_i$ is considered to be an $n \times 1$ matrix (that is, its transpose is a row vector). Hence, each $\mathbf{u}_i\mathbf{v}_i^T$ is an $m \times n$ matrix. (Compare the formula in Theorem 9.7 to the formula in the Spectral Theorem (Theorem 6.24) in Section 6.3.)

*Proof.* To prove that $\mathbf{A} = \sigma_1\mathbf{u}_1\mathbf{v}_1^T + \cdots + \sigma_k\mathbf{u}_k\mathbf{v}_k^T$, we use the same strategy employed to prove Theorem 9.6. In particular, we will show that the result of multiplying the matrix $(\sigma_1\mathbf{u}_1\mathbf{v}_1^T + \cdots + \sigma_k\mathbf{u}_k\mathbf{v}_k^T)$ by each vector in the basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ for $\mathbb{R}^n$ gives the same result as multiplying $\mathbf{A}$ times that vector.

For each $i$, $1 \leq i \leq k$, we have

$$\begin{aligned}(\sigma_1\mathbf{u}_1\mathbf{v}_1^T + \cdots + \sigma_k\mathbf{u}_k\mathbf{v}_k^T)\mathbf{v}_i &= \sigma_1\mathbf{u}_1\mathbf{v}_1^T\mathbf{v}_i + \cdots + \sigma_i\mathbf{u}_i\mathbf{v}_i^T\mathbf{v}_i + \cdots + \sigma_k\mathbf{u}_k\mathbf{v}_k^T\mathbf{v}_i \\ &= \mathbf{0} + \cdots + \sigma_i\mathbf{u}_i(1) + \cdots + \mathbf{0} \\ &\qquad \text{because the basis } \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \text{ is orthonormal} \\ &= \sigma_i\left(\frac{1}{\sigma_i}\mathbf{A}\mathbf{v}_i\right) = \mathbf{A}\mathbf{v}_i.\end{aligned}$$

If $i > k$, then

$$(\sigma_1\mathbf{u}_1\mathbf{v}_1^T + \cdots + \sigma_k\mathbf{u}_k\mathbf{v}_k^T)\mathbf{v}_i = \mathbf{0} = \mathbf{A}\mathbf{v}_i, \text{ by part (5) of Theorem 9.5.}$$

Therefore, for every $i$, $(\sigma_1\mathbf{u}_1\mathbf{v}_1^T + \cdots + \sigma_k\mathbf{u}_k\mathbf{v}_k^T)\mathbf{v}_i = \mathbf{A}\mathbf{v}_i$, completing the proof of the theorem. □

[6] Notice that $-1$ is an eigenvalue of $\mathbf{V}^T$ with corresponding unit eigenvector $\frac{1}{3}[1, -2, 2]$. An ordered orthonormal basis for $\mathbb{R}^3$ containing this vector is $\left(\frac{1}{3}[1, -2, 2], \frac{1}{3}[2, -1, -2], \frac{1}{3}[2, 2, 1]\right)$. The matrix for $\mathbf{V}^T$ with respect to this basis is $\begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{4}{5} & \frac{3}{5} \\ 0 & -\frac{3}{5} & \frac{4}{5} \end{bmatrix}$. This is equal to the product $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{4}{5} & \frac{3}{5} \\ 0 & -\frac{3}{5} & \frac{4}{5} \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, where the latter matrix represents an orthogonal reflection through the plane perpendicular to the first ordered basis vector (that is, the plane perpendicular to $[1, -2, 2]$) and the former matrix represents a counterclockwise rotation of angle $\arcsin(-\frac{3}{5})$ (or, a clockwise rotation of angle $\arccos(\frac{4}{5})$) about an axis in the direction of the vector $[1, -2, 2]$.

**Example 8**

Consider again the matrix $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix}$ from Examples 1 through 5. Note that

$$\begin{aligned}\sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T &= 2\left(\frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right)\left(\frac{1}{\sqrt{2}}[1, 1]\right) + \sqrt{2}\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}\left(\frac{1}{\sqrt{2}}[-1, 1]\right) \\ &= \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ -1 & -1 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix} = \mathbf{A}.\end{aligned}$$

**Example 9**

If $\mathbf{A} = \begin{bmatrix} 4 & 0 & 4 & 0 \\ -2 & -6 & -2 & -6 \\ 5 & 3 & 5 & 3 \end{bmatrix}$, the matrix from Example 6, then

$$\begin{aligned}\sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T &= 12\left(\frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}\right)\left(\frac{1}{2}[1, 1, 1, 1]\right) + 6\left(\frac{1}{3}\begin{bmatrix} -2 \\ -2 \\ -1 \end{bmatrix}\right)\left(\frac{1}{2}[-1, 1, -1, 1]\right) \\ &= \begin{bmatrix} 2 & 2 & 2 & 2 \\ -4 & -4 & -4 & -4 \\ 4 & 4 & 4 & 4 \end{bmatrix} + \begin{bmatrix} 2 & -2 & 2 & -2 \\ 2 & -2 & 2 & -2 \\ 1 & -1 & 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 0 & 4 & 0 \\ -2 & -6 & -2 & -6 \\ 5 & 3 & 5 & 3 \end{bmatrix} = \mathbf{A}.\end{aligned}$$

## Digital Images

One application of the outer product form for a Singular Value Decomposition is in the compression of digital images. For example, a black-and-white image[7] is represented by an $m \times n$ array of integers, with each entry giving a grayscale value (based on its relative lightness/darkness to the other pixels) for a single pixel in the image. If **A** represents this matrix of values, we can compute its singular values and corresponding singular vectors. In a typical photograph, many of the singular values are significantly smaller than the first few. As the values of $\sigma_i$ get smaller, the corresponding terms $\sigma_i \mathbf{u}_i \mathbf{v}_i^T$ in the outer product form of the Singular Value Decomposition have considerably less influence on the final image than the larger terms that came before them. In this way, we can still get a reasonably close rendition of the picture even if we cut out many of the nonzero terms in the outer product form. Thus, the picture can essentially be stored digitally using much less storage space.

For example, consider the black-and-white photograph in Fig. 9.2, which is 530 pixels by 779 pixels. This particular picture is represented by a $530 \times 779$ matrix **A** of grayscale values. Thus, **A** has 530 singular values. Using computer software (we used MATLAB®), we can compute the outer product form of the Singular Value Decomposition for **A**, and then truncate the sum by eliminating many of the terms corresponding to smaller singular values. In Fig. 9.3 we illustrate the resulting photograph by using just 10, 25, 50, 75, 100, and 200 of the 530 terms in the decomposition. Instructions for how to perform these computations in MATLAB can be found in the *Student Solutions Manual* for this textbook as part of the answer to Exercise 15.

## The Pseudoinverse

If **D** is a "diagonal" $m \times n$ matrix having rank $k$, whose first $k$ diagonal entries are nonzero, then the $n \times m$ "diagonal" matrix $\mathbf{D}^+$ whose first $k$ diagonal entries are the reciprocals of those of **D**, with the rest being zero, has the property that

[7] Color images can be handled by considering each of the three fundamental colors separately.

FIGURE 9.2 Grand Tetons, 1984, by Lyn Hecker. Used with permission

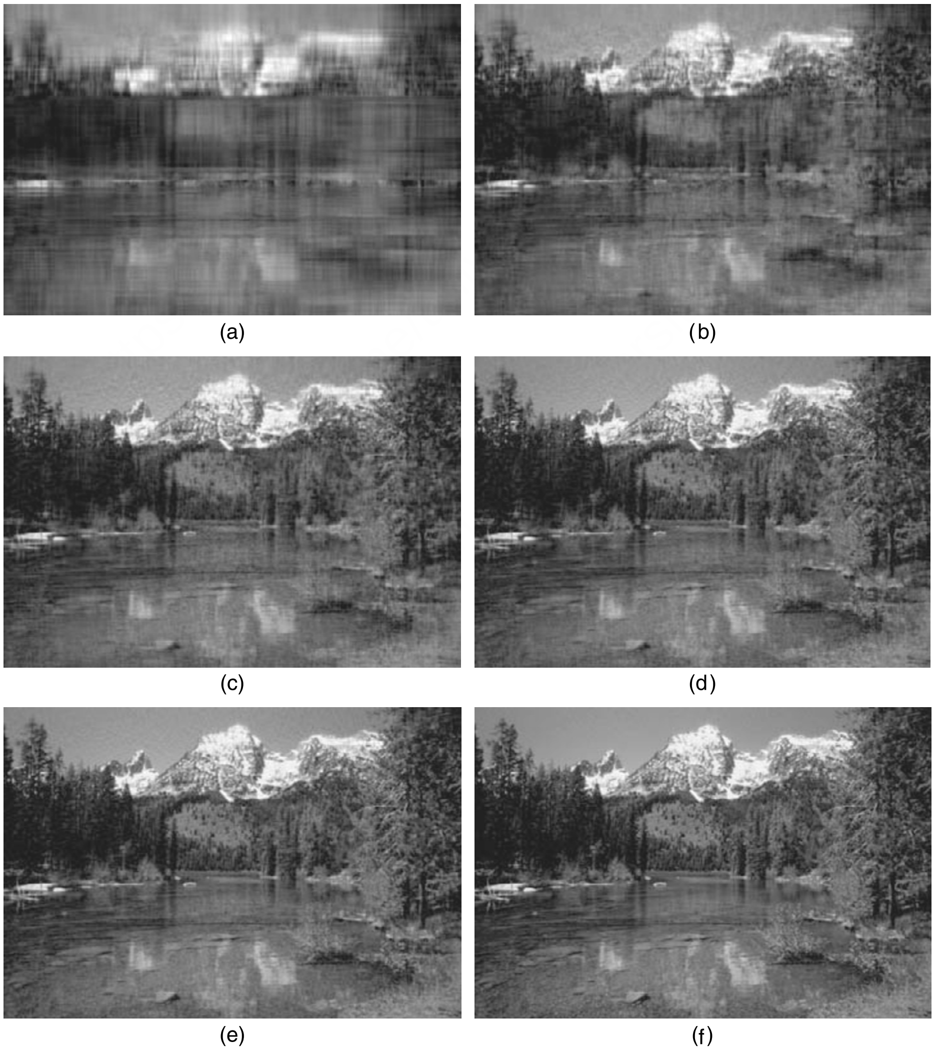


FIGURE 9.3 Compressed images of "Grand Tetons": (a) using 10 terms; (b) using 25 terms; (c) using 50 terms; (d) using 75 terms; (e) using 100 terms; (f) using 200 terms

$\mathbf{D}^+\mathbf{D}$ is the $n \times n$ diagonal matrix whose first $k$ diagonal entries are 1, and the rest are zero. Thus, $\mathbf{D}^+$ is as close as we can get to creating a left inverse for the matrix $\mathbf{D}$, considering that $\mathbf{D}$ has rank $k$. We will use the Singular Value Decomposition to find an analogous pseudoinverse for any $m \times n$ matrix $\mathbf{A}$.

**Definition** Suppose $\mathbf{A}$ is an $m \times n$ matrix of rank $k$ with Singular Value Decomposition $\mathbf{A} = \mathbf{U\Sigma V}^T$. Let $\mathbf{\Sigma}^+$ be the $n \times m$ "diagonal" matrix whose first $k$ diagonal entries are the reciprocals of those of $\mathbf{\Sigma}$, with the rest being zero. Then the $n \times m$ matrix $\mathbf{A}^+ = \mathbf{V\Sigma}^+\mathbf{U}^T$ is called a **pseudoinverse** of $\mathbf{A}$.

**Example 10**

A pseudoinverse of the matrix $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix}$ from Examples 1 through 5 is

$$\mathbf{A}^+ = \mathbf{V\Sigma}^+\mathbf{U}^T = \left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\right)\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}\begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{2} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix}.$$

Note that $\mathbf{A}^+\mathbf{A} = \mathbf{I}_2$.

**Example 11**

If $\mathbf{A} = \begin{bmatrix} 4 & 0 & 4 & 0 \\ -2 & -6 & -2 & -6 \\ 5 & 3 & 5 & 3 \end{bmatrix}$, the matrix from Example 6, then

$$\mathbf{A}^+ = \mathbf{V\Sigma}^+\mathbf{U}^T = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{12} & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \end{bmatrix} = \frac{1}{72}\begin{bmatrix} 5 & 2 & 4 \\ -3 & -6 & 0 \\ 5 & 2 & 4 \\ -3 & -6 & 0 \end{bmatrix}.$$

In this case,

$$\mathbf{A}^+\mathbf{A} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

We can see why the product $\mathbf{A}^+\mathbf{A}$ cannot equal $\mathbf{I}_4$ when we consider the linear transformation $L$ whose matrix with respect to the standard basis is $\mathbf{A}$. Since $L$ sends all vectors in $\ker(L)$ to zero, the only vectors in $\mathbb{R}^4$ that can be restored after left multiplication by $\mathbf{A}^+$ are those in $(\ker(L))^\perp$. In fact, in part (c) of Exercise 10 you are asked to prove that $\mathbf{A}^+\mathbf{A}$ is the matrix for the orthogonal projection of $\mathbb{R}^4$ onto $(\ker(L))^\perp$ with respect to the standard basis. By parts (4) and (5) of Theorem 9.5, $\{\mathbf{v}_1, \mathbf{v}_2\} = \{\frac{1}{2}[1, 1, 1, 1], \frac{1}{2}[-1, 1, -1, 1]\}$ is an orthonormal basis for $(\ker(L))^\perp$, while $\{\mathbf{v}_3, \mathbf{v}_4\} = \{\frac{1}{\sqrt{2}}[-1, 0, 1, 0], \frac{1}{\sqrt{2}}[0, -1, 0, 1]\}$ is an orthonormal basis for $\ker(L)$. You can verify that $\mathbf{A}^+\mathbf{A}$ represents the desired projection by checking that $\mathbf{A}^+\mathbf{A}\mathbf{v}_1 = \mathbf{v}_1$, $\mathbf{A}^+\mathbf{A}\mathbf{v}_2 = \mathbf{v}_2$, $\mathbf{A}^+\mathbf{A}\mathbf{v}_3 = \mathbf{0}$, and $\mathbf{A}^+\mathbf{A}\mathbf{v}_4 = \mathbf{0}$.

In Section 8.10, we studied least-squares solutions for inconsistent linear systems $\mathbf{Ax} = \mathbf{b}$. In that section we discovered that if such a system does not have a solution, we can still find a vector $\mathbf{x}$ such that $\mathbf{Ax}$ is as close as possible to $\mathbf{b}$; that is, where $\|\mathbf{Ax} - \mathbf{b}\|$ is a minimum. Such least-squares solutions are useful in many applications. The next theorem shows that a least-squares solution can be found for a linear system by using a pseudoinverse of $\mathbf{A}$.

**Theorem 9.8** *Let* $\mathbf{A}$ *be an* $m \times n$ *matrix and let* $\mathbf{A}^+$ *be a pseudoinverse for* $\mathbf{A}$. *Then* $\mathbf{x} = \mathbf{A}^+\mathbf{b}$ *is a least-squares solution to the linear system* $\mathbf{Ax} = \mathbf{b}$.

*Proof.* Let $\mathbf{A}$ be an $m \times n$ matrix, let $\mathbf{A}^+$ be a pseudoinverse for $\mathbf{A}$, and let $\mathbf{Ax} = \mathbf{b}$ be a linear system. By part (3) of Theorem 8.13, $\mathbf{x}$ is a least-squares solution to $\mathbf{Ax} = \mathbf{b}$ if and only if $(\mathbf{A}^T\mathbf{A})\mathbf{x} = \mathbf{A}^T\mathbf{b}$. We will prove that this equation holds for $\mathbf{x} = \mathbf{A}^+\mathbf{b}$.

Now, by Theorem 6.3, since the left singular vectors $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ form an orthonormal basis for $\mathbb{R}^m$, $\mathbf{b} = a_1\mathbf{u}_1 + \cdots + a_m\mathbf{u}_m$, with $a_i = \mathbf{b} \cdot \mathbf{u}_i$. Writing $\mathbf{A}$ as $\mathbf{U\Sigma V}^T$ and $\mathbf{A}^+$ as $\mathbf{V\Sigma}^+\mathbf{U}^T$, we get

$$\begin{aligned}
(\mathbf{A}^T\mathbf{A})\mathbf{x} &= (\mathbf{A}^T\mathbf{A})\mathbf{A}^+\mathbf{b} = \mathbf{A}^T(\mathbf{U\Sigma V}^T)(\mathbf{V\Sigma}^+\mathbf{U}^T)(a_1\mathbf{u}_1 + \cdots + a_m\mathbf{u}_m) \\
&= \mathbf{A}^T\mathbf{U\Sigma}(\mathbf{V}^T\mathbf{V})\mathbf{\Sigma}^+(a_1\mathbf{e}_1 + \cdots + a_m\mathbf{e}_m) && \text{since } \mathbf{U}^T\mathbf{u}_i = \mathbf{e}_i \\
&= \mathbf{A}^T\mathbf{U\Sigma}(\mathbf{I}_n)(a_1\frac{1}{\sigma_1}\mathbf{e}_1 + \cdots + a_k\frac{1}{\sigma_k}\mathbf{e}_k) \\
&= \mathbf{A}^T\mathbf{U}(a_1\mathbf{e}_1 + \cdots + a_k\mathbf{e}_k) \\
&= \mathbf{A}^T(a_1\mathbf{u}_1 + \cdots + a_k\mathbf{u}_k) \\
&= \mathbf{A}^T(a_1\mathbf{u}_1 + \cdots + a_m\mathbf{u}_m) && \text{since } \mathbf{A}^T\mathbf{u}_i = \mathbf{0} \text{ for } i > k \\
& && \text{by part (3) of Theorem 9.5} \\
&= \mathbf{A}^T\mathbf{b}.
\end{aligned}$$

□

**Example 12**

Consider the linear system

$$\begin{bmatrix} 4 & 0 & 4 & 0 \\ -2 & -6 & -2 & -6 \\ 5 & 3 & 5 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \\ 2 \end{bmatrix}.$$

Gaussian Elimination shows that this system is inconsistent. Let $\mathbf{A}$ be the given coefficient matrix. We can find a least-squares solution to this system using the pseudoinverse $\mathbf{A}^+$ for $\mathbf{A}$ that we computed in Example 11. Using $\mathbf{b} = [7, 1, 2]$ yields $\mathbf{x} = \mathbf{A}^+\mathbf{b} = \frac{1}{8}[5, -3, 5, -3]$. Note that $\mathbf{Ax} = [5, 2, 4]$. While this might not seem particularly close to $\mathbf{b} = [7, 1, 2]$, it is, in fact, the closest product of the form $\mathbf{Ax}$ to the vector $\mathbf{b}$. The reason is that vectors of the form $\mathbf{Ax}$ constitute range$(L)$, and by Theorem 6.18, the projection of $\mathbf{b}$ onto range$(L)$ is the closest vector in range$(L)$ to $\mathbf{b}$. If we express $\mathbf{b}$ as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ from Example 6, we see that $\mathbf{b} = 3\mathbf{u}_1 - 6\mathbf{u}_2 + 3\mathbf{u}_3$. By part (2) of Theorem 9.5, the projection vector equals $3\mathbf{u}_1 - 6\mathbf{u}_2 = [5, 2, 4]$, which is exactly what we have obtained.

## New Vocabulary

isometry
left singular vectors (for a matrix)
outer product form (of Singular Value Decomposition)
pseudoinverse (of a matrix)
right singular vectors (for a matrix)
Singular Value Decomposition
singular values (of a matrix)

## Highlights

- Suppose $\mathbf{A}$ is an $m \times n$ matrix, and $k$ of the $n$ eigenvalues of $\mathbf{A}^T\mathbf{A}$ are positive. (If all eigenvalues of $\mathbf{A}^T\mathbf{A}$ are positive, then $k = n$.) If $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > \lambda_{k+1} = \cdots = \lambda_n = 0$ are the eigenvalues of $\mathbf{A}^T\mathbf{A}$, written in nonincreasing order, and if $\sigma_i = \sqrt{\lambda_i}$, then $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k > \sigma_{k+1} = \cdots = \sigma_n = 0$ are called the singular values of $\mathbf{A}$.
- If $\mathbf{A}$ is an $m \times n$ matrix, and if $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is an orthonormal set of eigenvectors for $\mathbf{A}^T\mathbf{A}$, with $\mathbf{v}_i$ corresponding to $\lambda_i$ and with the $\lambda_i$ values in nonincreasing order, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a corresponding set of right singular vectors for $\mathbf{A}$. If $\mathbf{u}_i = \frac{1}{\sigma_i}\mathbf{Av}_i$, for $1 \leq i \leq k$, and $\mathbf{u}_{k+1}, \ldots, \mathbf{u}_m$ are chosen so that $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ is an orthonormal basis for $\mathbb{R}^m$, then $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ is a corresponding set of left singular vectors for $\mathbf{A}$.

- If $\mathbf{A}$ is an $m \times n$ matrix, $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $\mathbf{A}^T\mathbf{A}$ in nonincreasing order, $\sigma_1, \ldots, \sigma_n$ are the singular values of $\mathbf{A}$, and $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a corresponding set of right singular vectors for $\mathbf{A}$, then $(\mathbf{A}\mathbf{v}_i) \cdot (\mathbf{A}\mathbf{v}_j) = 0$ for $i \neq j$, and $(\mathbf{A}\mathbf{v}_i) \cdot (\mathbf{A}\mathbf{v}_i) = \|\mathbf{A}\mathbf{v}_i\|^2 = \lambda_i = \sigma_i^2$. Also, if $\mathbf{x} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$, then $(\mathbf{A}\mathbf{x}) \cdot (\mathbf{A}\mathbf{v}_i) = a_i\lambda_i$.
- If $\mathbf{A}$ is an $m \times n$ matrix, and the singular values of $\mathbf{A}$ are $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k > \sigma_{k+1} = \cdots = \sigma_n = 0$, with a corresponding set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of right singular vectors for $\mathbf{A}$, and a corresponding set $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ of left singular vectors for $\mathbf{A}$, and if $L\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation given by $L(\mathbf{x}) = \mathbf{A}\mathbf{x}$, then $\text{rank}(\mathbf{A}) = k$, $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an orthonormal basis for $(\ker(L))^\perp$, $\{\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$ is an orthonormal basis for $\ker(L)$, $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthonormal basis for $\text{range}(L)$, and $\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_m\}$ is an orthonormal basis for $(\text{range}(L))^\perp$.
- If $\mathbf{A}$ is an $m \times n$ matrix, and the singular values of $\mathbf{A}$ are $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k > \sigma_{k+1} = \cdots = \sigma_n = 0$, then a Singular Value Decomposition for $\mathbf{A}$ is given by $\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T$, where $\mathbf{V}$ is an $n \times n$ orthogonal matrix whose columns are $\mathbf{v}_1, \ldots, \mathbf{v}_n$, a corresponding set of right singular vectors for $\mathbf{A}$, and $\mathbf{U}$ is the $m \times m$ orthogonal matrix whose columns are $\mathbf{u}_1, \ldots, \mathbf{u}_m$, a corresponding set of left singular vectors for $\mathbf{A}$, and $\boldsymbol{\Sigma}$ is the $m \times n$ "diagonal" matrix whose $(i,i)$ entry equals $\sigma_i$, for $i \leq k$, with all other entries equal to zero.
- If $\mathbf{A}$ is an $m \times n$ matrix, and the singular values of $\mathbf{A}$ are $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k > \sigma_{k+1} = \cdots = \sigma_n = 0$, with a corresponding set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of right singular vectors for $\mathbf{A}$, and a corresponding set $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ of left singular vectors for $\mathbf{A}$, then the outer product form of the related Singular Value Decomposition for $\mathbf{A}$ is $\sigma_1\mathbf{u}_1\mathbf{v}_1^T + \sigma_2\mathbf{u}_2\mathbf{v}_2^T + \cdots + \sigma_k\mathbf{u}_k\mathbf{v}_k^T$.
- If $\mathbf{A}$ is an $m \times n$ matrix of rank $k$ with Singular Value Decomposition $\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T$, and $\boldsymbol{\Sigma}^+$ is the $n \times m$ "diagonal" matrix whose first $k$ diagonal entries are the reciprocals of those of $\boldsymbol{\Sigma}$, with the rest being zero, then a pseudoinverse of $\mathbf{A}$ is given by the $n \times m$ matrix $\mathbf{A}^+ = \mathbf{V}\boldsymbol{\Sigma}^+\mathbf{U}^T$.
- If $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{A}^+$ is a pseudoinverse for $\mathbf{A}$, then $\mathbf{x} = \mathbf{A}^+\mathbf{b}$ is a least-squares solution to the linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$.

## Exercises for Section 9.5

1. In each part, find a Singular Value Decomposition for the given matrix $\mathbf{A}$.

★ (a) $\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 5 & 0 \end{bmatrix}$

(b) $\mathbf{A} = \begin{bmatrix} 1 & -17 \\ 18 & -6 \end{bmatrix}$

★ (c) $\mathbf{A} = \begin{bmatrix} 7 & 20 & -17 \\ -9 & 0 & 9 \end{bmatrix}$

(d) $\mathbf{A} = \begin{bmatrix} 3 & -4 & -10 \\ 6 & -8 & 5 \end{bmatrix}$

(e) $\mathbf{A} = \frac{1}{49}\begin{bmatrix} 40 & 6 & 18 \\ 6 & 45 & -12 \\ 18 & -12 & 13 \end{bmatrix}$

★ (f) $\mathbf{A} = \frac{1}{7}\begin{bmatrix} 10 & 14 \\ 12 & 0 \\ 1 & 7 \end{bmatrix}$

(g) $\mathbf{A} = \frac{1}{11}\begin{bmatrix} 12 & 6 \\ 12 & -27 \\ 14 & 18 \end{bmatrix}$

(h) $\mathbf{A} = \frac{1}{15}\begin{bmatrix} 16 & 12 & -12 & -16 \\ 5 & -15 & 15 & -5 \\ 13 & -9 & 9 & -13 \end{bmatrix}$

2. In each part, find a pseudoinverse $\mathbf{A}^+$ for the given matrix $\mathbf{A}$. Then use the pseudoinverse to find a least-squares solution $\mathbf{v}$ for the system $\mathbf{A}\mathbf{x} = \mathbf{b}$ with the given vector $\mathbf{b}$. Finally, verify that $\mathbf{A}^T\mathbf{A}\mathbf{v} = \mathbf{A}^T\mathbf{b}$.

★ (a) $\mathbf{A} = \frac{1}{15}\begin{bmatrix} 94 & -128 \\ 95 & 110 \\ 142 & 46 \end{bmatrix}$; $\mathbf{b} = \begin{bmatrix} 3 \\ 27 \\ 28 \end{bmatrix}$

(b) $\mathbf{A} = \begin{bmatrix} 4 & -3 \\ 11 & -2 \\ 5 & -10 \end{bmatrix}$; $\mathbf{b} = \begin{bmatrix} 0 \\ 15 \\ -20 \end{bmatrix}$

★ (c) $\mathbf{A} = \frac{1}{14}\begin{bmatrix} 23 & -11 & -6 \\ 5 & 25 & 6 \\ 19 & -17 & 6 \\ 1 & 19 & 18 \end{bmatrix}$; $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ 9 \\ 4 \end{bmatrix}$

(d) $\mathbf{A} = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 2 \\ 2 & 2 & -2 \\ 2 & 1 & -3 \end{bmatrix}$; $\mathbf{b} = \begin{bmatrix} 5 \\ 2 \\ 1 \\ 3 \end{bmatrix}$

3. In each part, write out the outer product form of the Singular Value Decomposition of the given matrix $\mathbf{A}$. Note that these are all matrices from Exercise 1. In parts (a) through (c), a regular Singular Value Decomposition for $\mathbf{A}$

appears in Appendix E for Exercise 1. For these three parts, you can start from that point, using the information in Appendix E.

★ **(a)** $\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 5 & 0 \end{bmatrix}$

**(b)** $\mathbf{A} = \begin{bmatrix} 7 & 20 & -17 \\ -9 & 0 & 9 \end{bmatrix}$

★ **(c)** $\mathbf{A} = \frac{1}{7}\begin{bmatrix} 10 & 14 \\ 12 & 0 \\ 1 & 7 \end{bmatrix}$

**(d)** $\mathbf{A} = \frac{1}{11}\begin{bmatrix} 12 & 6 \\ 12 & -27 \\ 14 & 18 \end{bmatrix}$

**(e)** $\mathbf{A} = \frac{1}{15}\begin{bmatrix} 16 & 12 & -12 & -16 \\ 5 & -15 & 15 & -5 \\ 13 & -9 & 9 & -13 \end{bmatrix}$

**4.** Prove that if $\mathbf{A}$ is an $n \times n$ orthogonal matrix then two possible Singular Value Decompositions for $\mathbf{A}$ are $\mathbf{A}\mathbf{I}_n\mathbf{I}_n$ and $\mathbf{I}_n\mathbf{I}_n\mathbf{A}$.

**5.** Let $\mathbf{A}$ be an $m \times n$ matrix. Suppose that $\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T$, with $\mathbf{U}$ and $\mathbf{V}$ orthogonal matrices and $\boldsymbol{\Sigma}$ a diagonal $m \times n$ matrix. Prove that the $i$th column of $\mathbf{V}$ must be an eigenvector for $\mathbf{A}^T\mathbf{A}$ corresponding to the eigenvalue equal to the square of the $(i, i)$ entry of $\boldsymbol{\Sigma}$ if $i \leq m$, and corresponding to the eigenvalue 0 if $i > m$.

**6.** Let $\mathbf{A}$ be a symmetric matrix. Prove that the singular values of $\mathbf{A}$ equal the absolute values of its eigenvalues. (Hint: Let $\mathbf{D} = \mathbf{P}^T\mathbf{A}\mathbf{P}$ be an orthogonal diagonalization for $\mathbf{A}$, with the eigenvalues of $\mathbf{A}$ along the main diagonal of $\mathbf{D}$. Also, use Exercise 5.)

**7.** If $\mathbf{A}$ is an $m \times n$ matrix and $\sigma_1$ is the largest singular value for $\mathbf{A}$, then $\|\mathbf{A}\mathbf{v}\| \leq \sigma_1 \|\mathbf{v}\|$ for all $\mathbf{v} \in \mathbb{R}^n$. (Note that $\|\mathbf{A}\mathbf{v}_1\| = \sigma_1$ by part (3) of Lemma 9.4.)

**8.** Let $\mathbf{A}$ be an $m \times n$ matrix and let $\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T$ be a Singular Value Decomposition for $\mathbf{A}$.

▶ **(a)** Show that $\mathbf{V}$ is not unique, because a different Singular Value Decomposition for $\mathbf{A}$ could be found by multiplying any column of $\mathbf{V}$ by $-1$, and then adjusting $\mathbf{U}$ in an appropriate manner.

▶ **(b)** Show that if one of the eigenspaces of $\mathbf{A}^T\mathbf{A}$ has dimension greater than 1, there is even greater choice involved for the columns of $\mathbf{V}$ than indicated in part (a).

**(c)** Prove that $\Sigma$ is uniquely determined by $\mathbf{A}$. (Hint: Use Exercise 5.)

**(d)** If there are $k$ nonzero singular values of $\mathbf{A}$, show that the first $k$ columns of $\mathbf{U}$ are uniquely determined by the matrix $\mathbf{V}$.

**(e)** If there are $k$ nonzero singular values of $\mathbf{A}$, and if $k < m$, show that columns $k + 1$ through $m$ of $\mathbf{U}$ are not uniquely determined, with two choices if $m = k + 1$, and an infinite number of choices if $m > k + 1$.

▶ **9.** Let $\mathbf{A}$ be an $m \times n$ matrix having rank $k$, with $k < n$.

**(a)** Explain why right singular vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ for $\mathbf{A}$ can *not* be found by merely performing the Gram-Schmidt Process on the set of rows of $\mathbf{A}$, eliminating the zero vectors, and normalizing, even though part (4) of Theorem 9.5 says that $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an orthonormal basis for the row space of $\mathbf{A}$.

**(b)** Explain why right singular vectors $\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n$ *can* be found using the Kernel Method on $\mathbf{A}$, and then using the Gram-Schmidt Process and normalizing.

**10.** Let $\mathbf{A}$ be an $m \times n$ matrix having rank $k$, let $\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T$ be a Singular Value Decomposition for $\mathbf{A}$, and let $\mathbf{A}^+$ be the corresponding pseudoinverse for $\mathbf{A}$.

▶ **(a)** Compute $\mathbf{A}^+\mathbf{A}\mathbf{v}_i$ for each $i$, for $1 \leq i \leq k$.

▶ **(b)** Compute $\mathbf{A}^+\mathbf{A}\mathbf{v}_i$ for each $i$, for $k < i$.

**(c)** Let $L\colon \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation whose matrix with respect to the standard bases is $\mathbf{A}$. Use parts (a) and (b) to prove that $\mathbf{A}^+\mathbf{A}$ is the matrix for the orthogonal projection onto $(\ker(L))^\perp$ with respect to the standard basis for $\mathbb{R}^n$.

**(d)** Prove that $\mathbf{A}\mathbf{A}^+\mathbf{A} = \mathbf{A}$. (Hint: Show that multiplying by $\mathbf{A}\mathbf{A}^+\mathbf{A}$ has the same effect on $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ as multiplication by $\mathbf{A}$.)

**(e)** Show that if $\mathbf{A}$ is a nonsingular matrix, then $\mathbf{A}^+ = \mathbf{A}^{-1}$. (Hint: Use part (d).)

**11.** Let $\mathbf{A}$ be an $m \times n$ matrix having rank $k$, let $\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T$ be a Singular Value Decomposition for $\mathbf{A}$, and let $\mathbf{A}^+$ be the corresponding pseudoinverse for $\mathbf{A}$.

▶ **(a)** Compute $\mathbf{A}^+\mathbf{u}_i$ and $\mathbf{A}\mathbf{A}^+\mathbf{u}_i$ for each $i$, for $1 \leq i \leq k$.

▶ **(b)** Compute $\mathbf{A}^+\mathbf{u}_i$ and $\mathbf{A}\mathbf{A}^+\mathbf{u}_i$ for each $i$, for $k < i$.

**(c)** Let $L\colon \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation whose matrix with respect to the standard bases is $\mathbf{A}$. Use parts (a) and (b) to prove that $\mathbf{A}\mathbf{A}^+$ is the matrix for the orthogonal projection onto range$(L)$ with respect to the standard basis for $\mathbb{R}^m$.

**(d)** Prove that $\mathbf{A}^+\mathbf{A}\mathbf{A}^+ = \mathbf{A}^+$. (Hint: Show that multiplying by $\mathbf{A}^+\mathbf{A}\mathbf{A}^+$ has the same effect on $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ as multiplication by $\mathbf{A}^+$.)

**(e)** Prove that $\mathbf{A}^+$ is independent of the particular Singular Value Decomposition used for $\mathbf{A}$. That is, show that every $m \times n$ matrix $\mathbf{A}$ has a unique pseudoinverse. (Hint: Use part (c) of Exercise 10 to show that $\mathbf{A}^+\mathbf{u}$ is uniquely determined for all $\mathbf{u} \in \text{range}(L)$. Then use part (b) of this exercise to show that $\mathbf{A}^+\mathbf{u}$ is uniquely determined for all $\mathbf{u} \in (\text{range}(L))^\perp$. Combine these results to show that $\mathbf{A}^+\mathbf{u}$ is uniquely determined on a basis for $\mathbb{R}^m$.)

**12.** Let $\mathbf{A}$ be an $m \times n$ matrix having rank $k$, and let $\sigma_1, \ldots, \sigma_k$ be the nonzero singular values for $\mathbf{A}$, listed in nonincreasing order. Prove that the sum of the squares of the entries of $\mathbf{A}$ equals $\sigma_1^2 + \cdots + \sigma_k^2$. (Hint: Use the Singular Value Decomposition of $\mathbf{A}$ and parts (a) and (c) of Exercise 28 in Section 1.5.)

**13.** Let $\mathbf{A}$ be an $m \times n$ matrix having rank $k$, and suppose that $\sigma_1, \ldots, \sigma_k$ are the nonzero singular values for $\mathbf{A}$, listed in nonincreasing order, and that $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ and $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ are corresponding sets of left and right singular vectors, respectively, for $\mathbf{A}$. For $i, j$ with $1 \le i < j \le k$, suppose that $\mathbf{A}_{ij} = \sigma_i \mathbf{u}_i \mathbf{v}_i^T + \cdots + \sigma_j \mathbf{u}_j \mathbf{v}_j^T$. Prove that $\mathbf{A}_{ij}$ has rank $j - i + 1$ and that the nonzero singular values for $\mathbf{A}_{ij}$ are $\sigma_i, \ldots, \sigma_j$. (Hint: Consider the matrices $\mathbf{V}_1$ and $\mathbf{U}_1$, which are obtained, respectively, from $\mathbf{V}$ and $\mathbf{U}$ by moving columns $i$ through $j$ to the beginning of each matrix and rearranging the other columns accordingly. Also let $\mathbf{\Sigma}_1$ be the diagonal $m \times n$ matrix with $\sigma_i$ through $\sigma_j$ as its first diagonal entries, and with all other diagonal entries equal to zero. Show that $\mathbf{U}_1\mathbf{\Sigma}_1\mathbf{V}_1$ is a Singular Value Decomposition for $\mathbf{A}_{ij}$.)

★ **14.** Suppose $\mathbf{A}$ is a $5 \times 6$ matrix determined by the following singular values and left and right singular vectors: $\sigma_1 = 150$, $\sigma_2 = 30$, $\sigma_3 = 15$, $\sigma_4 = 6$, $\sigma_5 = 3$, $\mathbf{v}_1 = \frac{1}{2}[1, 0, 1, -1, 0, -1]$, $\mathbf{v}_2 = \frac{1}{2}[1, 0, -1, 1, 0, -1]$, $\mathbf{v}_3 = \frac{1}{2}[1, -1, 0, 0, 1, 1]$, $\mathbf{v}_4 = \frac{1}{2}[1, 1, 0, 0, -1, 1]$, $\mathbf{v}_5 = \frac{1}{2}[0, 1, 1, 1, 1, 0]$, $\mathbf{v}_6 = \frac{1}{2}[0, 1, -1, -1, 1, 0]$, $\mathbf{u}_1 = \frac{1}{3}[1, 0, 2, 0, 2]$, $\mathbf{u}_2 = \frac{1}{3}[2, 0, 1, 0, -2]$, $\mathbf{u}_3 = \frac{1}{3}[2, 0, -2, 0, 1]$, $\mathbf{u}_4 = \frac{1}{5}[0, 3, 0, -4, 0]$, and $\mathbf{u}_5 = \frac{1}{5}[0, 4, 0, 3, 0]$.

**(a)** Use the outer product form of the Singular Value Decomposition to find the matrix $\mathbf{A}$.

**(b)** For each $i$ with $1 \le i \le 4$, compute $\mathbf{A}_i = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \cdots + \sigma_i \mathbf{u}_i \mathbf{v}_i^T$.

**(c)** For any matrix $\mathbf{B}$, define $N(\mathbf{B})$ to be the square root of the sum of the squares of the entries of $\mathbf{B}$. (If you think of an $m \times n$ matrix as a vector with $mn$ entries in $\mathbb{R}^{mn}$, this would be its norm.) For each $i$ with $1 \le i \le 4$, compute $N(\mathbf{A} - \mathbf{A}_i)/N(\mathbf{A})$. (Hint: Use Exercises 12 and 13.)

**(d)** Explain how this exercise relates to the discussion of the compression of digital images in the textbook.

► **15.** Using a black-and-white digital image file, use appropriate software to analyze the effect of eliminating some of the smaller singular values by producing a sequence of adjusted images, starting with using only a small percentage of the singular values and progressing up to using all of them. Detailed instructions on how to do this in MATLAB are included in the *Student Solutions Manual* under this exercise.

★ **16.** True or False:

**(a)** For every matrix $\mathbf{A}$, $\mathbf{A}^T\mathbf{A} = \mathbf{A}\mathbf{A}^T$.

**(b)** All of the singular values of a matrix are nonnegative.

**(c)** If $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, with $\mathbf{v} \cdot \mathbf{w} = 0$, then $(\mathbf{A}\mathbf{v}) \cdot (\mathbf{A}\mathbf{w}) = 0$.

**(d)** If $\mathbf{A}$ is an $m \times n$ matrix, then a set of left singular vectors for $\mathbf{A}$ is completely determined by $\mathbf{A}$ and the corresponding set of right singular vectors.

**(e)** The right singular vectors $\{\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$ form an orthonormal basis for $(\ker(L))^\perp$.

**(f)** If $\mathbf{A}$ and $\mathbf{B}$ are $m \times n$ matrices such that $\mathbf{A}\mathbf{v} = \mathbf{B}\mathbf{v}$ for every vector $\mathbf{v}$ in a basis for $\mathbb{R}^n$, then $\mathbf{A} = \mathbf{B}$.

**(g)** Every $m \times n$ matrix has a unique Singular Value Decomposition.

**(h)** If $\mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$ is a Singular Value Decomposition for a matrix $\mathbf{A}$, then $\mathbf{V}\mathbf{\Sigma}^T\mathbf{U}^T$ is a Singular Value Decomposition for $\mathbf{A}^T$.

**(i)** Only nonsingular square matrices have pseudoinverses.

**(j)** For a nonsingular matrix, its pseudoinverse must equal its inverse.

**(k)** The outer product form of the Singular Value Decomposition for a matrix might not use all of the right singular vectors.

# Appendix A

# Miscellaneous Proofs

In this appendix, we present some proofs of theorems that were omitted from the text.

## Proof of Theorem 1.16, Part (1)

Part (1) of Theorem 1.16 can be restated as follows:

**Theorem 1.16, Part (1)** *If* $\mathbf{A}$ *is an* $m \times n$ *matrix,* $\mathbf{B}$ *is an* $n \times p$ *matrix, and* $\mathbf{C}$ *is a* $p \times r$ *matrix, then* $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$.

*Proof.* We must show that the $(i, j)$ entry of $\mathbf{A}(\mathbf{BC})$ is the same as the $(i, j)$ entry of $(\mathbf{AB})\mathbf{C}$. Now,

$$\begin{aligned}
(i, j) \text{ entry of } \mathbf{A}(\mathbf{BC}) &= [i\text{th row of } \mathbf{A}] \cdot [j\text{th column of } \mathbf{BC}] \\
&= [i\text{th row of } \mathbf{A}] \cdot \left[\sum_{k=1}^{p} b_{1k}c_{kj}, \sum_{k=1}^{p} b_{2k}c_{kj}, \ldots, \sum_{k=1}^{p} b_{nk}c_{kj}\right] \\
&= a_{i1}\left(\sum_{k=1}^{p} b_{1k}c_{kj}\right) + a_{i2}\left(\sum_{k=1}^{p} b_{2k}c_{kj}\right) + \cdots + a_{in}\left(\sum_{k=1}^{p} b_{nk}c_{kj}\right) \\
&= \sum_{k=1}^{p}\left(a_{i1}b_{1k}c_{kj} + a_{i2}b_{2k}c_{kj} + \cdots + a_{in}b_{nk}c_{kj}\right).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
(i, j) \text{ entry of } (\mathbf{AB})\mathbf{C} &= [i\text{th row of } \mathbf{AB}] \cdot [j\text{th column of } \mathbf{C}] \\
&= \left[\sum_{k=1}^{n} a_{ik}b_{k1}, \sum_{k=1}^{n} a_{ik}b_{k2}, \ldots, \sum_{k=1}^{n} a_{ik}b_{kp}\right] \cdot [j\text{th column of } \mathbf{C}] \\
&= \left(\sum_{k=1}^{n} a_{ik}b_{k1}\right)c_{1j} + \left(\sum_{k=1}^{n} a_{ik}b_{k2}\right)c_{2j} + \cdots + \left(\sum_{k=1}^{n} a_{ik}b_{kp}\right)c_{pj} \\
&= \sum_{k=1}^{n}\left(a_{ik}b_{k1}c_{1j} + a_{ik}b_{k2}c_{2j} + \cdots + a_{ik}b_{kp}c_{pj}\right).
\end{aligned}$$

It then follows that the final sums for the $(i, j)$ entries of $\mathbf{A}(\mathbf{BC})$ and $(\mathbf{AB})\mathbf{C}$ are equal, because both are equal to the giant sum of terms

$$\left\{\begin{array}{c}
a_{i1}b_{11}c_{1j} + a_{i1}b_{12}c_{2j} + a_{i1}b_{13}c_{3j} + \cdots + a_{i1}b_{1p}c_{pj} \\
a_{i2}b_{21}c_{1j} + a_{i2}b_{22}c_{2j} + a_{i2}b_{23}c_{3j} + \cdots + a_{i2}b_{2p}c_{pj} \\
\vdots \\
a_{in}b_{n1}c_{1j} + a_{in}b_{n2}c_{2j} + a_{in}b_{n3}c_{3j} + \cdots + a_{in}b_{np}c_{pj}
\end{array}\right\}.$$

Notice that the $i$th term in the sum for $\mathbf{A}(\mathbf{BC})$ represents the $i$th column of terms in the giant sum, whereas the $i$th term in the sum for $(\mathbf{AB})\mathbf{C}$ represents the $i$th row of terms in the giant sum. Hence, the $(i, j)$ entries of $\mathbf{A}(\mathbf{BC})$ and $(\mathbf{AB})\mathbf{C}$ agree. □

## Proof of Theorem 2.6

**Theorem 2.6** *Every matrix is row equivalent to a unique matrix in reduced row echelon form.*

The proof of this theorem uses Theorem 2.9, which states that two row equivalent matrices have the same row space. Please note that, although Theorem 2.9 appears later in the text than Theorem 2.6, the proof of Theorem 2.9 given in the text is independent of Theorem 2.6, so we are not employing a circular argument here.

*Proof.* First, given any matrix, the Gauss-Jordan Method will produce a matrix in reduced row echelon form that is row equivalent to the given matrix. This handles the existence part of the proof of Theorem 2.6. Next, we address the more interesting part of the theorem: the uniqueness assertion.

Suppose $\mathbf{A}$ and $\mathbf{B}$ are two $m \times n$ matrices in reduced row echelon form, both row equivalent to an $m \times n$ matrix $\mathbf{C}$. We will prove that $\mathbf{A} = \mathbf{B}$.

We begin by showing that the pivots in $\mathbf{A}$ and $\mathbf{B}$ are in the same locations. Suppose that $\mathbf{a}_1, \ldots, \mathbf{a}_m$ are the rows of $\mathbf{A}$ and $\alpha_1, \ldots, \alpha_m$ are defined so that if there is a pivot in row $i$, then $\alpha_i$ is the column in which the pivot appears, and otherwise $\alpha_i = n + 1$. Note that $\alpha_1 \le \alpha_2 \le \ldots \le \alpha_m$, with $\alpha_i = \alpha_{i+1}$ only if both equal $n + 1$. Similarly define $\mathbf{b}_1, \ldots, \mathbf{b}_m$ and $\beta_1, \ldots, \beta_m$ for the matrix $\mathbf{B}$. We need to prove that $\alpha_i = \beta_i$ for all $i$.

If not, let $j$ be the smallest subscript such that $\alpha_j \neq \beta_j$. That is, $\alpha_i = \beta_i$ for all $i < j$. Without loss of generality, assume that $\alpha_j < \beta_j$. Now, because $\mathbf{A}$ and $\mathbf{B}$ are both row equivalent to $\mathbf{C}$, we know that $\mathbf{A}$ is row equivalent to $\mathbf{B}$. (By part (1) of Theorem 2.4, $\mathbf{C}$ is row equivalent to $\mathbf{B}$. Then, part (2) of Theorem 2.4 shows that $\mathbf{A}$ is row equivalent to $\mathbf{B}$.) Hence, by Theorem 2.9, $\mathbf{A}$ and $\mathbf{B}$ have the same row spaces. In particular, the $j$th row of $\mathbf{A}$ is in the row space of $\mathbf{B}$. That is, there are real numbers $d_1, \ldots, d_m$ such that

$$\mathbf{a}_j = d_1\mathbf{b}_1 + \cdots + d_j\mathbf{b}_j + \cdots + d_m\mathbf{b}_m.$$

Since $\mathbf{B}$ is in reduced row echelon form, the entries in columns $\beta_1, \ldots, \beta_{j-1}$ of $(d_1\mathbf{b}_1 + \cdots + d_j\mathbf{b}_j + \cdots + d_m\mathbf{b}_m)$ must equal $d_1, \ldots, d_{j-1}$. But, because $\alpha_i = \beta_i$ for all $i < j$, $\mathbf{a}_j$ has a zero in all of these columns, and so $d_1 = d_2 = \cdots = d_{j-1} = 0$. Also, since $\alpha_j < \beta_j$, $(d_j\mathbf{b}_j + \cdots + d_m\mathbf{b}_m)$ equals zero in the $\alpha_j$ column, while $\mathbf{a}_j$ equals 1 in this column. (Note that $\alpha_j \neq n + 1$, since we must have $\alpha_j < \beta_j \le n + 1$.) This contradiction shows that we can not have any $\alpha_j \neq \beta_j$. Therefore, the reduced row echelon form matrices $\mathbf{A}$ and $\mathbf{B}$ have pivots in exactly the same columns.

Finally, we prove that $\mathbf{a}_i = \mathbf{b}_i$ for all $i$. For a given $i$, if $\alpha_i = \beta_i = n + 1$, then $\mathbf{a}_i = \mathbf{b}_i = \mathbf{0}$. If $\alpha_i = \beta_i < n + 1$, then, again, since $\mathbf{A}$ and $\mathbf{B}$ have the same row spaces, there are real numbers $d_1, \ldots, d_m$ such that

$$\mathbf{a}_i = d_1\mathbf{b}_1 + \cdots + d_i\mathbf{b}_i + \cdots + d_m\mathbf{b}_m.$$

But the entries in columns $\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_m$ of $\mathbf{a}_i$ equal zero, implying that $d_1 = \cdots = d_{i-1} = d_{i+1} = \cdots = d_m = 0$, since the same columns contain the pivots for $\mathbf{B}$. Similarly, the entry in the $\alpha_i$ column of both $\mathbf{a}_i$ and $\mathbf{b}_i$ equals 1. Hence, $d_i = 1$, and so $\mathbf{a}_i = \mathbf{b}_i$. □

## Proof of Theorem 2.10

**Theorem 2.10** *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be* $n \times n$ *matrices. If either product* $\mathbf{AB}$ *or* $\mathbf{BA}$ *equals* $\mathbf{I}_n$*, then the other product also equals* $\mathbf{I}_n$*, and* $\mathbf{A}$ *and* $\mathbf{B}$ *are inverses of each other.*

We say that $\mathbf{B}$ is a **left inverse** of $\mathbf{A}$ and $\mathbf{A}$ is a **right inverse** of $\mathbf{B}$ whenever $\mathbf{BA} = \mathbf{I}_n$.

*Proof.* We need to show that any left inverse of a matrix is also a right inverse, and vice versa.

First, suppose that $\mathbf{B}$ is a left inverse of $\mathbf{A}$; that is, $\mathbf{BA} = \mathbf{I}_n$. We will show that $\mathbf{AB} = \mathbf{I}_n$. To do this, we show that $\text{rank}(\mathbf{A}) = n$, then use this to find a right inverse $\mathbf{C}$ of $\mathbf{A}$, and finally show $\mathbf{B} = \mathbf{C}$.

Consider the homogeneous system $\mathbf{AX} = \mathbf{O}$ of $n$ equations and $n$ unknowns. This system has only the trivial solution, because multiplying both sides of $\mathbf{AX} = \mathbf{O}$ by $\mathbf{B}$ on the left, we obtain

$$\mathbf{B(AX)} = \mathbf{BO} \implies \mathbf{(BA)X} = \mathbf{O} \implies \mathbf{I}_n\mathbf{X} = \mathbf{O} \implies \mathbf{X} = \mathbf{O}.$$

By Theorem 2.7, rank$(\mathbf{A}) = n$, and every column of $\mathbf{A}$ becomes a pivot column during the Gauss-Jordan Method. Therefore, each of the augmented matrices

$$\left[\begin{array}{c|c}\mathbf{A} & \begin{array}{c}\text{1st}\\ \text{column}\\ \text{of } \mathbf{I}_n\end{array}\end{array}\right], \left[\begin{array}{c|c}\mathbf{A} & \begin{array}{c}\text{2nd}\\ \text{column}\\ \text{of } \mathbf{I}_n\end{array}\end{array}\right], \ldots, \left[\begin{array}{c|c}\mathbf{A} & \begin{array}{c}n\text{th}\\ \text{column}\\ \text{of } \mathbf{I}_n\end{array}\end{array}\right]$$

represents a system with a unique solution. Consider the matrix $\mathbf{C}$, whose $i$th column is the solution to the $i$th of these systems. Then $\mathbf{C}$ is a right inverse for $\mathbf{A}$, because the product $\mathbf{AC} = \mathbf{I}_n$. But then

$$\mathbf{B} = \mathbf{B}(\mathbf{I}_n) = \mathbf{B}(\mathbf{AC}) = (\mathbf{BA})\mathbf{C} = \mathbf{I}_n\mathbf{C} = \mathbf{C}.$$

Hence, $\mathbf{B}$ is also a right inverse for $\mathbf{A}$.

Conversely, suppose that $\mathbf{B}$ is a right inverse for $\mathbf{A}$; that is, $\mathbf{AB} = \mathbf{I}_n$. We must show that $\mathbf{B}$ is also a left inverse for $\mathbf{A}$. By assumption, $\mathbf{A}$ is a left inverse for $\mathbf{B}$. However, we have already shown that any left inverse is also a right inverse. Therefore, $\mathbf{A}$ must be a (full) inverse for $\mathbf{B}$, and $\mathbf{AB} = \mathbf{BA} = \mathbf{I}_n$. Hence, $\mathbf{B}$ is a left (and a full) inverse for $\mathbf{A}$. □

## Proof of Theorem 3.3, Part (3), Case 2

**Theorem 3.3, Part (3), Case 2** *Let* $\mathbf{A}$ *be an* $n \times n$ *matrix with* $n > 2$. *If* $R$ *is the row operation* $\langle n-1\rangle \leftrightarrow \langle n\rangle$, *then* $|R(\mathbf{A})| = -|\mathbf{A}|$.

*Proof.* Suppose $R$ is the row operation $\langle n-1\rangle \leftrightarrow \langle n\rangle$, switching the last two rows of $\mathbf{A}$. Let $\mathbf{B} = R(\mathbf{A})$. Define the notation $\mathbf{A}^{i,j}$ to represent the $(n-2)\times(n-2)$ submatrix formed by deleting rows $n-1$ and $n$, as well as deleting columns $i$ and $j$ from $\mathbf{A}$. Define $\mathbf{B}^{i,j}$ similarly. Notice that because the first $n-2$ rows of $\mathbf{A}$ and $\mathbf{B}$ are identical, $\mathbf{A}^{i,j} = \mathbf{B}^{i,j}$, for $1 \le i, j \le n$.

The following observation is useful in what follows: Since the $i$th column of $\mathbf{B}$ is removed from the submatrix $\mathbf{B}_{ni}$, any element of the form $b_{kj}$ is in the $j$th column of $\mathbf{B}_{ni}$ if $j < i$, but $b_{kj}$ is in the $(j-1)$st column of $\mathbf{B}_{ni}$ if $j > i$. Similarly, since the $j$th column of $\mathbf{A}$ is removed from $\mathbf{A}_{nj}$, any element of the form $a_{ki}$ is in the $i$th column of $\mathbf{A}_{nj}$ if $i < j$, but $a_{ki}$ is in the $(i-1)$st column of $\mathbf{A}_{nj}$ if $i > j$.

Now,

$$\begin{aligned}
|\mathbf{B}| &= \sum_{i=1}^{n} b_{ni}\mathcal{B}_{ni} = \sum_{i=1}^{n}(-1)^{n+i} b_{ni}|\mathbf{B}_{ni}| \\
&= \sum_{i=1}^{n}(-1)^{n+i} b_{ni}\left(\sum_{j=1}^{i-1}(-1)^{(n-1)+j} b_{(n-1)j}|\mathbf{B}^{i,j}| + \sum_{j=i+1}^{n}(-1)^{(n-1)+(j-1)} b_{(n-1)j}|\mathbf{B}^{i,j}|\right) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{i-1}(-1)^{2n+i+j-1} b_{ni}b_{(n-1)j}|\mathbf{B}^{i,j}| + \sum_{i=1}^{n}\sum_{j=i+1}^{n}(-1)^{2n+i+j-2} b_{ni}b_{(n-1)j}|\mathbf{B}^{i,j}| \\
&= \sum_{\substack{i,j\\ j<i}}(-1)^{2n+i+j-1} b_{ni}b_{(n-1)j}|\mathbf{B}^{i,j}| + \sum_{\substack{i,j\\ j>i}}(-1)^{2n+i+j-2} b_{ni}b_{(n-1)j}|\mathbf{B}^{i,j}|.
\end{aligned}$$

But, $b_{ni} = a_{(n-1)i}$, and $b_{(n-1)j} = a_{nj}$, because we are switching rows $n$ and $n-1$. Also recall that $\mathbf{A}^{i,j} = \mathbf{B}^{i,j}$. Making these substitutions and then reversing the previous steps, we have

$$\begin{aligned}
|\mathbf{B}| &= \sum_{\substack{i,j\\ j<i}}(-1)^{2n+i+j-1} a_{(n-1)i}a_{nj}|\mathbf{A}^{i,j}| + \sum_{\substack{i,j\\ j>i}}(-1)^{2n+i+j-2} a_{(n-1)i}a_{nj}|\mathbf{A}^{i,j}| \\
&= (-1)\sum_{\substack{i,j\\ j<i}}(-1)^{2n+i+j-2} a_{nj}a_{(n-1)i}|\mathbf{A}^{i,j}| + (-1)\sum_{\substack{i,j\\ j>i}}(-1)^{2n+i+j-1} a_{nj}a_{(n-1)i}|\mathbf{A}^{i,j}| \\
&= -\left(\sum_{j=1}^{n}\sum_{i=j+1}^{n}(-1)^{2n+i+j-2} a_{nj}a_{(n-1)i}|\mathbf{A}^{i,j}| + \sum_{j=1}^{n}\sum_{i=1}^{j-1}(-1)^{2n+i+j-1} a_{nj}a_{(n-1)i}|\mathbf{A}^{i,j}|\right)
\end{aligned}$$

$$= -\left(\sum_{j=1}^{n}(-1)^{n+j}a_{nj}\left(\sum_{i=j+1}^{n}(-1)^{n+i-2}a_{(n-1)i}|\mathbf{A}^{i,j}| + \sum_{i=1}^{j-1}(-1)^{n+i-1}a_{(n-1)i}|\mathbf{A}^{i,j}|\right)\right)$$

$$= -\sum_{j=1}^{n}(-1)^{n+j}a_{nj}\left(\sum_{i=j+1}^{n}(-1)^{(n-1)+(i-1)}a_{(n-1)i}|\mathbf{A}^{i,j}| + \sum_{i=1}^{j-1}(-1)^{(n-1)+i}a_{(n-1)i}|\mathbf{A}^{i,j}|\right)$$

$$= -\sum_{j=1}^{n}(-1)^{n+j}a_{nj}|\mathbf{A}_{nj}| = -\sum_{j=1}^{n}a_{nj}\mathcal{A}_{nj} = -|\mathbf{A}|.$$

This completes Case 2. □

## Proof of Theorem 5.31

**Theorem 5.31 (Cayley-Hamilton Theorem)** *Let* $\mathbf{A}$ *be an* $n \times n$ *matrix, and let* $p_{\mathbf{A}}(x)$ *be its characteristic polynomial. Then* $p_{\mathbf{A}}(\mathbf{A}) = \mathbf{O}_n$.

*Proof.* Let $\mathbf{A}$ be an $n \times n$ matrix with characteristic polynomial $p_{\mathbf{A}}(x) = |x\mathbf{I}_n - \mathbf{A}| = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$, for some real numbers $a_0, \ldots, a_{n-1}$.

Recall the definition of the classical adjoint of an $n \times n$ matrix from Exercise 17 in Section 3.3. For a square matrix $\mathbf{D}$, its classical adjoint $\mathbf{C}$ is the *transpose* of the matrix whose $(i, j)$ entry equals $\mathcal{D}_{ij}$, the $(i, j)$ cofactor of $\mathbf{D}$. That is, the $(i, j)$ entry of $\mathbf{C}$ equals $\mathcal{D}_{ji}$, the $(j, i)$ cofactor of $\mathbf{D}$. It was proved in Exercise 17 of Section 3.3 that $\mathbf{DC} = |\mathbf{D}|\,\mathbf{I}_n$.

We let $\mathbf{B}(x)$ represent the classical adjoint of $x\mathbf{I}_n - \mathbf{A}$. Then from the last equation, we have

$$(x\mathbf{I}_n - \mathbf{A})\,\mathbf{B}(x) = p_{\mathbf{A}}(x)\mathbf{I}_n,$$

for every $x \in \mathbb{R}$. We will find an expanded form for $\mathbf{B}(x)$ and then use the preceding equation to show that $p_{\mathbf{A}}(\mathbf{A})$ reduces to $\mathbf{O}_n$.

Now, each entry of $\mathbf{B}(x)$ is defined as a cofactor for $x\mathbf{I}_n - \mathbf{A}$, which equals $\pm$ the determinant of an $(n-1) \times (n-1)$ minor of $x\mathbf{I}_n - \mathbf{A}$. Hence, each entry of $\mathbf{B}(x)$ is a polynomial in $x$ of degree $\le n-1$ (see Exercise 22 in Section 3.4). For each $k$, $0 \le k \le n-1$, create the matrix $\mathbf{B}_k$ whose $(i, j)$ entry is the coefficient of $x^k$ in the $(i, j)$ entry of $\mathbf{B}(x)$. Thus,

$$\mathbf{B}(x) = x^{n-1}\mathbf{B}_{n-1} + x^{n-2}\mathbf{B}_{n-2} + \cdots + x\mathbf{B}_1 + \mathbf{B}_0.$$

Therefore,

$$\begin{aligned}(x\mathbf{I}_n - \mathbf{A})\mathbf{B}(x) &= (x^n\mathbf{B}_{n-1} - x^{n-1}\mathbf{AB}_{n-1}) + (x^{n-1}\mathbf{B}_{n-2} - x^{n-2}\mathbf{AB}_{n-2}) + \cdots + (x^2\mathbf{B}_1 - x\mathbf{AB}_1) + (x\mathbf{B}_0 - \mathbf{AB}_0)\\ &= x^n\mathbf{B}_{n-1} + x^{n-1}(-\mathbf{AB}_{n-1} + \mathbf{B}_{n-2}) + x^{n-2}(-\mathbf{AB}_{n-2} + \mathbf{B}_{n-3}) + \cdots + x(-\mathbf{AB}_1 + \mathbf{B}_0) + (-\mathbf{AB}_0).\end{aligned}$$

Setting the coefficient of $x^k$ in this expression equal to the coefficient of $x^k$ in $p_{\mathbf{A}}(x)\mathbf{I}_n$ yields

$$\begin{cases} \mathbf{B}_{n-1} = \mathbf{I}_n \\ -\mathbf{AB}_k + \mathbf{B}_{k-1} = a_k\mathbf{I}_n, \text{ for } 1 \le k \le n-1 \\ -\mathbf{AB}_0 = a_0\mathbf{I}_n \end{cases}.$$

Hence,

$$\begin{aligned}p_{\mathbf{A}}(\mathbf{A}) &= \mathbf{A}^n + a_{n-1}\mathbf{A}^{n-1} + a_{n-2}\mathbf{A}^{n-2} + \cdots + a_1\mathbf{A} + a_0\mathbf{I}_n\\ &= \mathbf{A}^n\mathbf{I}_n + \mathbf{A}^{n-1}(a_{n-1}\mathbf{I}_n) + \mathbf{A}^{n-2}(a_{n-2}\mathbf{I}_n) + \cdots + \mathbf{A}(a_1\mathbf{I}_n) + a_0\mathbf{I}_n\\ &= \mathbf{A}^n(\mathbf{B}_{n-1}) + \mathbf{A}^{n-1}(-\mathbf{AB}_{n-1} + \mathbf{B}_{n-2}) + \mathbf{A}^{n-2}(-\mathbf{AB}_{n-2} + \mathbf{B}_{n-3}) + \cdots + \mathbf{A}(-\mathbf{AB}_1 + \mathbf{B}_0) + (-\mathbf{AB}_0)\\ &= \mathbf{A}^n\mathbf{B}_{n-1} + (-\mathbf{A}^n\mathbf{B}_{n-1} + \mathbf{A}^{n-1}\mathbf{B}_{n-2}) + (-\mathbf{A}^{n-1}\mathbf{B}_{n-2} + \mathbf{A}^{n-2}\mathbf{B}_{n-3}) + \cdots + (-\mathbf{A}^2\mathbf{B}_1 + \mathbf{AB}_0) + (-\mathbf{AB}_0)\\ &= \mathbf{A}^n(\mathbf{B}_{n-1} - \mathbf{B}_{n-1}) + \mathbf{A}^{n-1}(\mathbf{B}_{n-2} - \mathbf{B}_{n-2}) + \mathbf{A}^{n-2}(\mathbf{B}_{n-3} - \mathbf{B}_{n-3}) + \cdots + \mathbf{A}^2(\mathbf{B}_1 - \mathbf{B}_1) + \mathbf{A}(\mathbf{B}_0 - \mathbf{B}_0)\\ &= \mathbf{O}_n.\end{aligned}$$

□

## Proof of Theorem 6.19

**Theorem 6.19** *Let $\mathcal{V}$ be a nontrivial subspace of $\mathbb{R}^n$, and let $L$ be a linear operator on $\mathcal{V}$. Let $B$ be an ordered orthonormal basis for $\mathcal{V}$, and let $\mathbf{A}_{BB}$ be the matrix for $L$ with respect to $B$. Then $L$ is a symmetric operator if and only if $\mathbf{A}_{BB}$ is a symmetric matrix.*

*Proof.* Let $\mathcal{V}$, $L$, $B$, and $\mathbf{A}_{BB}$ be given as in the statement of the theorem, and let $k = \dim(\mathcal{V})$. Also, suppose that $B = (\mathbf{v}_1, \ldots, \mathbf{v}_k)$.

First we claim that, for all $\mathbf{w}_1, \mathbf{w}_2 \in \mathcal{V}$, $[\mathbf{w}_1]_B \cdot [\mathbf{w}_2]_B = \mathbf{w}_1 \cdot \mathbf{w}_2$, where the first dot product is in $\mathbb{R}^k$ and the second is in $\mathbb{R}^n$. To prove this statement, suppose that $[\mathbf{w}_1]_B = [a_1, \ldots, a_k]$ and $[\mathbf{w}_2]_B = [b_1, \ldots, b_k]$. Then,

$$\begin{aligned}
\mathbf{w}_1 \cdot \mathbf{w}_2 &= (a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k) \cdot (b_1\mathbf{v}_1 + \cdots + b_k\mathbf{v}_k) \\
&= \sum_{i=1}^{k}\sum_{j=1}^{k}\left(a_i b_j\right)\mathbf{v}_i \cdot \mathbf{v}_j = \sum_{i=1}^{k}\left(a_i b_i\right)\mathbf{v}_i \cdot \mathbf{v}_i && \text{since } \mathbf{v}_i \cdot \mathbf{v}_j = 0 \text{ if } i \neq j \\
&= \sum_{i=1}^{k} a_i b_i && \text{since } \mathbf{v}_i \cdot \mathbf{v}_i = 1 \\
&= [\mathbf{w}_1]_B \cdot [\mathbf{w}_2]_B\,.
\end{aligned}$$

Now suppose that $L$ is a symmetric operator on $\mathcal{V}$. We will prove that $\mathbf{A}_{BB}$ is symmetric by showing that its $(i, j)$ entry equals its $(j, i)$ entry. We have

$$\begin{aligned}
(i, j) \text{ entry of } \mathbf{A}_{BB} &= \mathbf{e}_i \cdot (\mathbf{A}_{BB}\mathbf{e}_j) \\
&= [\mathbf{v}_i]_B \cdot \left(\mathbf{A}_{BB}[\mathbf{v}_j]_B\right) \\
&= [\mathbf{v}_i]_B \cdot [L(\mathbf{v}_j)]_B \\
&= \mathbf{v}_i \cdot L(\mathbf{v}_j) && \text{by the claim verified earlier in this proof} \\
&= L(\mathbf{v}_i) \cdot \mathbf{v}_j && \text{since } L \text{ is symmetric} \\
&= [L(\mathbf{v}_i)]_B \cdot [\mathbf{v}_j]_B && \text{by the claim} \\
&= (\mathbf{A}_{BB}[\mathbf{v}_i]_B) \cdot [\mathbf{v}_j]_B \\
&= (\mathbf{A}_{BB}\mathbf{e}_i) \cdot \mathbf{e}_j = (j, i) \text{ entry of } \mathbf{A}_{BB}.
\end{aligned}$$

Conversely, if $\mathbf{A}_{BB}$ is a symmetric matrix and $\mathbf{w}_1, \mathbf{w}_2 \in \mathcal{V}$, we have

$$\begin{aligned}
L(\mathbf{w}_1) \cdot \mathbf{w}_2 &= [L(\mathbf{w}_1)]_B \cdot [\mathbf{w}_2]_B && \text{by the claim} \\
&= (\mathbf{A}_{BB}[\mathbf{w}_1]_B) \cdot [\mathbf{w}_2]_B \\
&= (\mathbf{A}_{BB}[\mathbf{w}_1]_B)^T [\mathbf{w}_2]_B && \text{changing vector dot product to matrix multiplication} \\
&= [\mathbf{w}_1]_B^T \mathbf{A}_{BB}^T [\mathbf{w}_2]_B \\
&= [\mathbf{w}_1]_B^T \mathbf{A}_{BB} [\mathbf{w}_2]_B && \text{since } \mathbf{A}_{BB} \text{ is symmetric} \\
&= [\mathbf{w}_1]_B \cdot (\mathbf{A}_{BB}[\mathbf{w}_2]_B) && \text{changing matrix multiplication to vector dot product} \\
&= [\mathbf{w}_1]_B \cdot [L(\mathbf{w}_2)]_B \\
&= \mathbf{w}_1 \cdot L(\mathbf{w}_2) && \text{by the claim}
\end{aligned}$$

Thus, $L$ is a symmetric operator on $\mathcal{V}$, and the proof is complete. □

# Appendix B

# Functions

In this appendix, we define some basic terminology associated with functions: *domain*, *codomain*, *range*, *image*, *pre-image*, *one-to-one*, *onto*, *composition*, and *inverses*. It is a good idea to review this material thoroughly before beginning Chapter 5.

## Functions: Domain, Codomain, and Range

**Definition** A **function** $f$ from a set $X$ to a set $Y$, expressed as $f\colon X \to Y$, is a mapping (assignment) of the elements of $X$ (called the **domain**) to elements of $Y$ (called the **codomain**) in such a way that each element of $X$ is assigned to some (single) chosen element of $Y$.

That is, for a function $f\colon X \to Y$, every element of $X$ must be assigned to *some* element of $Y$ and to *only one* element of $Y$.

### Example 1

The assignment $f\colon \mathbb{Z} \to \mathbb{R}$ (where $\mathbb{Z}$ represents the set $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ of all integers) given by $f(x) = x^2$ is a function, since each integer in $\mathbb{Z}$ is assigned by $f$ to one and only one element of $\mathbb{R}$.

Notice that the definition of a function allows two different elements of $X$ to map (be assigned) to the same element of $Y$, as in the function $f\colon \mathbb{Z} \to \mathbb{R}$ given by $f(x) = x^2$, where $f(3) = f(-3) = 9$. However, no function allows any member of the domain to map to more than one element of the codomain.

### Example 2

The rule $x \to \pm\sqrt{x}$, for $x \in \mathbb{R}^+$ (positive real numbers) is *not* a function, since, for example, 4 would have to map to both 2 and $-2$.

**Definition** Let $f\colon X \to Y$ be a function. If $x \in X$, the **image** of $x$, written as $f(x)$, is the unique element of $Y$ to which $x$ is mapped under $f$. If $y \in Y$, the **pre-images** of $y$ are the elements of $X$ that map to $y$ under $f$.

### Example 3

For the function $f\colon \mathbb{Z} \to \mathbb{R}$ given by $f(x) = x^2$, the image of 2 is 4 (that is, $f(2) = 4$), and the pre-images of 4 are 2 and $-2$, since $2^2 = (-2)^2 = 4$.

If $f\colon X \to Y$ is a function, not every element of $Y$ necessarily has a pre-image.

### Example 4

For the function $f\colon \mathbb{Z} \to \mathbb{R}$ given by $f(x) = x^2$, the element 5 in the codomain $\mathbb{R}$ has no pre-image in the domain $\mathbb{Z}$, because no integer squared equals 5.

**Definition** Let $f\colon X \to Y$ be a function. The **image of a subset** $S$ of $X$ under a function $f$, written as $f(S)$, is the set of all elements in $Y$ that are mapped to under $f$ by elements of $S$. The **pre-image of a subset** $T$ of $Y$ under $f$, written as $f^{-1}(T)$, is the set of *all* elements in $X$ that map to elements of $T$ under $f$.

**Example 5**

For the function $f: \mathbb{Z} \to \mathbb{R}$ given by $f(x) = x^2$, the image of the subset $\{-5, -3, 3, 5\}$ of the domain $\mathbb{Z}$ is the subset $\{9, 25\}$ of the codomain $\mathbb{R}$. Also, the pre-image of the subset $\{15, 16, 17\}$ of the codomain $\mathbb{R}$ is the subset $\{4, -4\}$ of the domain $\mathbb{Z}$.

**Definition** Let $f: X \to Y$ be a function. The image of the (entire) set $X$ (the domain) is the **range** of $f$.

**Example 6**

For the function $f: \mathbb{Z} \to \mathbb{R}$ given by $f(x) = x^2$, the range is the subset of $\mathbb{R}$ consisting of all squares of integers. In this case, the range is a proper subset of the codomain. This situation is depicted in Fig. B.1.

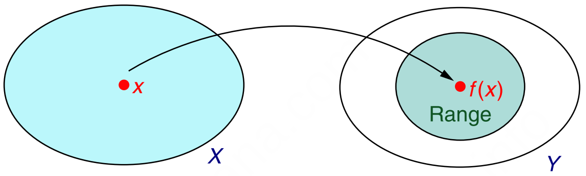



FIGURE B.1 The domain $X$, codomain $Y$, and range of a function $f: X \to Y$

For some functions, however, the range is the whole codomain, as we will see shortly.

## One-to-One and Onto Functions

We now consider two very important types of functions: one-to-one and onto functions.

**Definition** A function $f: X \to Y$ is **one-to-one** if and only if distinct elements of $X$ map to distinct elements of $Y$. That is, $f$ is one-to-one if and only if no two different elements of $X$ map to the same element of $Y$.

**Example 7**

The function $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ is one-to-one, since no two distinct real numbers (in the domain) have the same cube.

A standard method of proving that a function $f$ is one-to-one is as follows:

**To show that $f: X \to Y$ is one-to-one**: Prove that for arbitrary elements $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

In other words, if $f: X \to Y$ is one-to-one and $x_1, x_2 \in X$, then the only way $x_1$ and $x_2$ can have the same image is if they are really not distinct elements of $X$.

**Example 8**

We show that $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x - 7$ is one-to-one. Suppose that $f(x_1) = f(x_2)$, for some $x_1, x_2 \in \mathbb{R}$. Then $3x_1 - 7 = 3x_2 - 7$. Hence, $3x_1 = 3x_2$, which implies $x_1 = x_2$. Thus, $f$ is one-to-one.

On the other hand, we sometimes need to show that a function is *not* one-to-one. The usual method for doing this is as follows:

**To show that $f: X \to Y$ is not one-to-one**: Find two different elements $x_1$ and $x_2$ in the domain $X$ such that $f(x_1) = f(x_2)$.

**Example 9**

The function $g\colon \mathbb{R} \to \mathbb{R}$ given by $g(x) = x^2$ is not one-to-one, because $g(3) = g(-3) = 9$. That is, both elements 3 and $-3$ in the domain $\mathbb{R}$ of $g$ have the same image 9, so $g$ is not one-to-one.

**Definition** A function $f\colon X \to Y$ is **onto** if and only if every element of $Y$ is an image of some element in $X$. That is, $f$ is onto if and only if the range of $f$ equals the codomain of $f$.

**Example 10**

The function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x$ is onto, since every real number $y_1$ in the codomain $\mathbb{R}$ is the image of the real number $x_1 = \frac{1}{2}y_1$; that is, $f(x_1) = f\left(\frac{1}{2}y_1\right) = y_1$.

In the previous example, we used the standard method of proving that a given function is onto:

**To show that** $f\colon X \to Y$ **is onto**: Choose an arbitrary element $y_1 \in Y$, and show that there is some $x_1 \in X$ such that $y_1 = f(x_1)$.

On the other hand, we sometimes need to show that a function is *not* onto. The usual method for doing this is as follows:

**To show that** $f\colon X \to Y$ **is not onto**: Find an element $y_1$ in the codomain $Y$ that is not the image of any element $x_1$ in the domain $X$.

**Example 11**

The function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is not onto, since the real number $-4$ in the codomain $\mathbb{R}$ is never the image of any real number in the domain; that is, for all $x \in \mathbb{R}$, $f(x) \neq -4$.

## Composition and Inverses of Functions

**Definition** If $f\colon X \to Y$ and $g\colon Y \to Z$ are functions, the **composition** of $f$ and $g$ is the function $g \circ f\colon X \to Z$ given by $(g \circ f)(x) = g(f(x))$.

The composition of two functions $f\colon X \to Y$ and $g\colon Y \to Z$ is pictured in Fig. B.2.

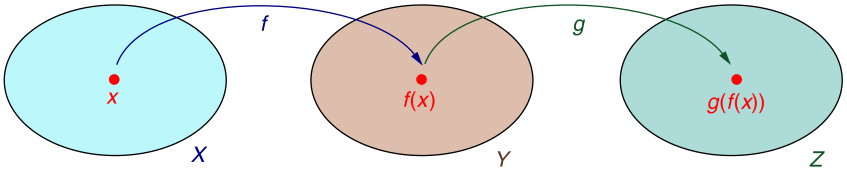



FIGURE B.2 Composition $g \circ f$ of $f\colon X \to Y$ and $g\colon Y \to Z$

**Example 12**

Consider $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = 1 - x^2$ and $g\colon \mathbb{R} \to \mathbb{R}$ given by $g(x) = 5\cos x$. Then $(g \circ f)(x) = g(f(x)) = g(1 - x^2) = 5\cos(1 - x^2)$. In particular, $(g \circ f)(2) = g(f(2)) = g(1 - 2^2) = g(-3) = 5\cos(-3) \approx -4.95$.

The following result shows that composition preserves the one-to-one and onto properties of functions.

**Theorem B.1**
**(1)** *If $f: X \to Y$ and $g: Y \to Z$ are both one-to-one, then $g \circ f: X \to Z$ is one-to-one.*
**(2)** *If $f: X \to Y$ and $g: Y \to Z$ are both onto, then $g \circ f: X \to Z$ is onto.*

*Proof.* **Part (1)**: Assume that $f$ and $g$ are both one-to-one. To prove $g \circ f$ is one-to-one, we assume that $(g \circ f)(x_1) = (g \circ f)(x_2)$, for two elements $x_1, x_2 \in X$, and prove that $x_1 = x_2$. However, $(g \circ f)(x_1) = (g \circ f)(x_2)$ implies that $g(f(x_1)) = g(f(x_2))$. Hence, $f(x_1)$ and $f(x_2)$ have the same image under $g$. Since $g$ is one-to-one, we must have $f(x_1) = f(x_2)$. Then $x_1$ and $x_2$ have the same image under $f$. Since $f$ is one-to-one, $x_1 = x_2$. Hence, $g \circ f$ is one-to-one.

**Part (2)**: Assume that $f$ and $g$ are both onto. To prove that $g \circ f: X \to Z$ is onto, we choose an arbitrary element $z_1 \in Z$ and try to find some element in $X$ that $g \circ f$ maps to $z_1$. Now, since $g$ is onto, there is some $y_1 \in Y$ for which $g(y_1) = z_1$. Also, since $f$ is onto, there is some $x_1 \in X$ for which $f(x_1) = y_1$. Therefore, $(g \circ f)(x_1) = g(f(x_1)) = g(y_1) = z_1$, and so $g \circ f$ maps $x_1$ to $z_1$. Hence, $g \circ f$ is onto. □

**Definition** Two functions $f: X \to Y$ and $g: Y \to X$ are **inverses** of each other if and only if $(g \circ f)(x) = x$ and $(f \circ g)(y) = y$, for every $x \in X$ and $y \in Y$.

### Example 13

The functions $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ and $g: \mathbb{R} \to \mathbb{R}$ given by $g(x) = \sqrt[3]{x}$ are inverses of each other because $(g \circ f)(x) = g(f(x)) = g(x^3) = \sqrt[3]{x^3} = x$, and $(f \circ g)(x) = f(g(x)) = f\left(\sqrt[3]{x}\right) = (\sqrt[3]{x})^3 = x$.

Not every function can be paired with an inverse function. The next theorem characterizes those functions that do have an inverse.

**Theorem B.2** *The function $f: X \to Y$ has an inverse $g: Y \to X$ if and only if $f$ is both one-to-one and onto.*

*Proof.* First, suppose that $f: X \to Y$ has an inverse $g: Y \to X$. We show that $f$ is one-to-one and onto. To prove $f$ is one-to-one, we assume that $f(x_1) = f(x_2)$, for some $x_1, x_2 \in X$, and try to prove $x_1 = x_2$. Since $f(x_1) = f(x_2)$, we have $g(f(x_1)) = g(f(x_2))$. However, since $g$ is an inverse for $f$, $x_1 = (g \circ f)(x_1) = g(f(x_1)) = g(f(x_2)) = x_2$, and so $x_1 = x_2$. Hence, $f$ is one-to-one. To prove $f$ is onto, we choose an arbitrary $y_1 \in Y$. We must show that $y_1$ is the image of some $x_1 \in X$. Now, $g$ maps $y_1$ to an element $x_1$ of $X$; that is, $g(y_1) = x_1$. However, $f(x_1) = f(g(y_1)) = (f \circ g)(y_1) = y_1$, since $f$ and $g$ are inverses. Hence, $f$ maps $x_1$ to $y_1$, and $f$ is onto.

Conversely, we assume that $f: X \to Y$ is one-to-one and onto and show that $f$ has an inverse $g: Y \to X$. Let $y_1$ be an arbitrary element of $Y$. Since $f$ is onto, the element $y_1$ in $Y$ is the image of some element in $X$. Since $f$ is one-to-one, $y_1$ is the image of precisely one element, say $x_1$, in $X$. Hence, $y_1$ has a *unique* pre-image under $f$. Now consider the mapping $g: Y \to X$, which maps each element $y_1$ in $Y$ to its unique pre-image $x_1$ in $X$ under $f$. Then $(f \circ g)(y_1) = f(g(y_1)) = f(x_1) = y_1$.

To finish the proof, we must show that $(g \circ f)(x_1) = x_1$, for any $x_1 \in X$. But $(g \circ f)(x_1) = g(f(x_1))$ is defined to be the unique pre-image of $f(x_1)$ under $f$. Since $x_1$ is this pre-image, we have $(g \circ f)(x_1) = x_1$. Thus, $g$ and $f$ are inverses. □

### Example 14

The functions $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ and $g: \mathbb{R} \to \mathbb{R}$ given by $g(x) = \sqrt[3]{x}$ are inverses of each other. Both of these functions are easily shown to be one-to-one and onto, as predicted by Theorem B.2.

**Example 15**

The function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ has no inverse, since it is not one-to-one. (We could also have shown that $f$ has no inverse since it is not onto.)

The next result assures us that when inverses exist, they are unique.

**Theorem B.3** *If $f\colon X \to Y$ has an inverse $g\colon Y \to X$, then $g$ is the only inverse of $f$.*

*Proof.* Suppose that $g_1\colon Y \to X$ and $g_2\colon Y \to X$ are both inverse functions for $f$. Our goal is to show that $g_1(y) = g_2(y)$, for all $y \in Y$, for then $g_1$ and $g_2$ are identical functions, and the inverse of $f$ is unique.

Now, $(g_2 \circ f)(x) = x$, for every $x \in X$, since $f$ and $g_2$ are inverses. Thus, since $g_1(y) \in X$, $g_1(y) = (g_2 \circ f)(g_1(y)) = g_2(f(g_1(y))) = g_2((f \circ g_1)(y)) = g_2(y)$, since $f$ and $g_1$ are inverses. □

**Example 16**

From Example 13, we know that $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ and $g\colon \mathbb{R} \to \mathbb{R}$ given by $g(x) = \sqrt[3]{x}$ are inverses of each other. Theorem B.3 now tells us that $g$ is the only possible inverse of $f$, and that $f$ is the only possible inverse of $g$.

Whenever a function $f\colon X \to Y$ has an inverse, we denote this unique inverse by $f^{-1}\colon Y \to X$.

**Theorem B.4** *If $f\colon X \to Y$ and $g\colon Y \to Z$ both have inverses, then $g \circ f\colon X \to Z$ has an inverse, and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.*

*Proof.* Because $g^{-1}\colon Z \to Y$ and $f^{-1}\colon Y \to X$, it follows that $f^{-1} \circ g^{-1}$ is a well-defined function from $Z$ to $X$. We need to show that the inverse of $g \circ f$ is $f^{-1} \circ g^{-1}$. If we can show that both

$$\left((g \circ f) \circ \left(f^{-1} \circ g^{-1}\right)\right)(z) = z, \qquad \text{for all } z \in Z,$$

$$\text{and} \qquad ((f^{-1} \circ g^{-1}) \circ (g \circ f))(x) = x, \qquad \text{for all } x \in X,$$

then by definition, $g \circ f$ and $f^{-1} \circ g^{-1}$ are inverses. Now,

$$\begin{aligned}\left((g \circ f) \circ \left(f^{-1} \circ g^{-1}\right)\right)(z) &= g\left(f\left(f^{-1}\left(g^{-1}(z)\right)\right)\right) \\ &= g\left(g^{-1}(z)\right) && \text{since } f \text{ and } f^{-1} \text{ are inverses} \\ &= z. && \text{since } g \text{ and } g^{-1} \text{ are inverses}\end{aligned}$$

A similar argument establishes the other statement. □

**Example 17**

Consider $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ and $g\colon \mathbb{R} \to \mathbb{R}^+$ given by $g(x) = e^x$. Then, $g \circ f\colon \mathbb{R} \to \mathbb{R}^+$ is given by $(g \circ f)(x) = e^{x^3}$. Since $f^{-1}(x) = \sqrt[3]{x}$ and $g^{-1}(x) = \ln x$, Theorem B.4 asserts that $(g \circ f)^{-1}\colon \mathbb{R}^+ \to \mathbb{R}$ is given by

$$(g \circ f)^{-1}(x) = (f^{-1} \circ g^{-1})(x) = f^{-1}(g^{-1}(x)) = \sqrt[3]{\ln x}.$$

## New Vocabulary

codomain of a function
composition (of functions)
domain of a function
function
image of an element (under a function)
image of a subset (under a function)
inverse of a function
one-to-one function
onto function
pre-image of an element (under a function)
pre-image of a subset (under a function)
range of a function

## Highlights

- A function $f: X \to Y$ is a mapping of the elements of $X$ (the domain of $f$) to elements of $Y$ (the codomain of $f$) in such a way that each element of $X$ is assigned to one and only one element of $Y$.
- Let $f: X \to Y$ be a function. If $x \in X$, then the image of $x$, $f(x)$, is the unique element of $Y$ to which $x$ is mapped under $f$. If $S$ is a subset of $X$, then the image of $S$, $f(S)$, is the set of all elements in $Y$ that are mapped to under $f$ by elements of $S$. The range of $f$ is the image of $X$ under $f$.
- Let $f: X \to Y$ is a function. If $y \in Y$, then the pre-images of $y$ are the elements of $X$ which map to $y$ under $f$. If $T$ is a subset of $Y$, then the pre-image of $T$, $f^{-1}(T)$, is the set of *all* elements in $X$ that map under $f$ to elements of $T$.
- A function $f: X \to Y$ is one-to-one if and only if distinct elements of $X$ map to distinct elements of $Y$. That is, $f$ is one-to-one if and only if no two different elements of $X$ map to the same element of $Y$.
- To show that $f: X \to Y$ is one-to-one, prove that for arbitrary elements $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.
- To show that $f: X \to Y$ is *not* one-to-one, find two different elements $x_1$ and $x_2$ in the domain $X$ such that $f(x_1) = f(x_2)$.
- A function $f: X \to Y$ is onto if and only if every element of $Y$ is an image of some element in $X$. That is, $f$ is onto if and only if the range of $f$ equals the codomain of $f$.
- To show that a function $f: X \to Y$ is onto, choose an arbitrary element $y_1 \in Y$, and show that there is some $x_1 \in X$ such that $y_1 = f(x_1)$.
- To show that a function $f: X \to Y$ is *not* onto, find an element $y_1$ in $Y$ that is not the image of any element $x_1$ in $X$.
- For functions $f: X \to Y$ and $g: Y \to Z$, the composition of $f$ and $g$ is the function $g \circ f: X \to Z$ given by $(g \circ f)(x) = g(f(x))$.
- If the functions $f: X \to Y$ and $g: Y \to Z$ are both one-to-one, then $g \circ f: X \to Z$ is one-to-one.
- If the functions $f: X \to Y$ and $g: Y \to Z$ are both onto, then $g \circ f: X \to Z$ is onto.
- Two functions $f: X \to Y$ and $g: Y \to X$ are inverses of each other if and only if $(g \circ f)(x) = x$ and $(f \circ g)(y) = y$, for every $x \in X$ and $y \in Y$.
- A function $f: X \to Y$ has an inverse $g: Y \to X$ if and only if $f$ is both one-to-one and onto. If an inverse for $f$ exists, it is unique.
- If $f: X \to Y$ and $g: Y \to Z$ both have inverses, then $g \circ f: X \to Z$ has an inverse, and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

## Exercises for Appendix B

**1.** Which of the following are functions? For those that are functions, determine the range, as well as the image and all pre-images of the value 2. For those that are not functions, explain why with a precise reason. (Note: $\mathbb{N}$ represents the set $\{0, 1, 2, 3, \ldots\}$ of natural numbers, and $\mathbb{Z}$ represents the set $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$ of integers.)

★ **(a)** $f: \mathbb{R} \to \mathbb{R}$, given by $f(x) = \sqrt{x-1}$

**(b)** $g: \mathbb{R} \to \mathbb{R}$, given by $g(x) = \sqrt{|x-1|}$

★ **(c)** $h: \mathbb{R} \to \mathbb{R}$, given by $h(x) = \pm\sqrt{|x-1|}$

**(d)** $j: \mathbb{N} \to \mathbb{Z}$, given by $j(a) = \begin{cases} a-5, & \text{if } a \text{ is odd} \\ a-4, & \text{if } a \text{ is even} \end{cases}$

★ **(e)** $k: \mathbb{R} \to \mathbb{R}$, given by $k(\theta) = \tan\theta$ (where $\theta$ is in radians)

★ **(f)** $l: \mathbb{N} \to \mathbb{N}$, where $l(t)$ is the smallest prime number $\geq t$

**(g)** $m: \mathbb{R} \to \mathbb{R}$, given by $m(x) = \begin{cases} x-3 & \text{if } x \leq 2 \\ x+4 & \text{if } x \geq 2 \end{cases}$

**2.** Let $f\colon \mathbb{Z} \to \mathbb{N}$ (with $\mathbb{Z}$ and $\mathbb{N}$ as in Exercise 1) be given by $f(x) = 2|x|$.

★ **(a)** Find the pre-image of the set $\{10, 20, 30\}$.

**(b)** Find the pre-image of the set $\{10, 11, 12, \ldots, 19\}$.

★ **(c)** Find the pre-image of the multiples of 4 in $\mathbb{N}$.

★ **3.** Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = (5x-1)/4$ and $g(x) = \sqrt{3x^2+2}$. Find $g \circ f$ and $f \circ g$.

★ **4.** Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3 & -2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. Let $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $g\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -4 & 4 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. Describe $g \circ f$ and $f \circ g$.

**5.** Let $A = \{1, 2, 3\}$, $B = \{4, 5, 6, 7\}$, and $C = \{8, 9, 10\}$.

**(a)** Give an example of functions $f\colon A \to B$ and $g\colon B \to C$ such that $g \circ f$ is onto but $f$ is not onto.

**(b)** Give an example of functions $f\colon A \to B$ and $g\colon B \to C$ such that $g \circ f$ is one-to-one but $g$ is not one-to-one.

**6.** For $n \geq 2$, show that $f\colon \mathcal{M}_{nn} \to \mathbb{R}$ given by $f(\mathbf{A}) = |\mathbf{A}|$ is onto but not one-to-one.

**7.** Show that $f\colon \mathcal{M}_{33} \to \mathcal{M}_{33}$ given by $f(\mathbf{A}) = \mathbf{A} + \mathbf{A}^T$ is neither one-to-one nor onto.

★ **8.** For $n \geq 1$, show that the function $f\colon \mathcal{P}_n \to \mathcal{P}_n$ given by $f(\mathbf{p}) = \mathbf{p}'$ is neither one-to-one nor onto. When $n \geq 3$, what is the pre-image of the subset $\mathcal{P}_2$ of the codomain?

**9.** Prove that $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x^3 - 5$ has an inverse by showing that it is both one-to-one and onto. Give a formula for $f^{-1}\colon \mathbb{R} \to \mathbb{R}$.

★ **10.** Let $\mathbf{B}$ be a fixed nonsingular matrix in $\mathcal{M}_{nn}$. Show that the map $f\colon \mathcal{M}_{nn} \to \mathcal{M}_{nn}$ given by $f(\mathbf{A}) = \mathbf{B}^{-1}\mathbf{A}\mathbf{B}$ is both one-to-one and onto. What is the inverse of $f$?

**11.** Let $f\colon A \to B$ and $g\colon B \to C$ be functions.

**(a)** Prove that if $g \circ f$ is onto, then $g$ is onto. (Compare this exercise with Exercise 5(a).)

**(b)** Prove that if $g \circ f$ is one-to-one, then $f$ is one-to-one. (Compare this exercise with Exercise 5(b).)

★ **12.** True or False:

**(a)** If $f$ assigns elements of $X$ to elements of $Y$, and two different elements of $X$ are assigned by $f$ to the same element of $Y$, then $f$ is not a function.

**(b)** If $f$ assigns elements of $X$ to elements of $Y$, and each element of $X$ is assigned to exactly one element of $Y$, but not every element of $Y$ corresponds to an element of $X$, then $f$ is a function.

**(c)** If $f\colon \mathbb{R} \to \mathbb{R}$ is a function, and $f(5) = f(6)$, then $f^{-1}(5) = 6$.

**(d)** If $f\colon X \to Y$ and the domain of $f$ equals the codomain of $f$, then $f$ must be onto.

**(e)** If $f\colon X \to Y$ then $f$ is one-to-one if $x_1 = x_2$ implies $f(x_1) = f(x_2)$.

**(f)** If $f\colon X \to Y$ and $g\colon Y \to Z$ are functions, and $g \circ f\colon X \to Z$ is one-to-one, then both $f$ and $g$ are one-to-one.

**(g)** If $f\colon X \to Y$ is a function, then $f$ has an inverse if $f$ is either one-to-one or onto.

**(h)** If $f\colon X \to Y$ and $g\colon Y \to Z$ both have inverses, and $g \circ f\colon X \to Z$ has an inverse, then $(g \circ f)^{-1} = g^{-1} \circ f^{-1}$.

Appendix C

# Complex Numbers

In this appendix, we define complex numbers and, for reference, list their most important operations and properties. Complex numbers employ the use of the number $i$, which is outside the real number system, and has the property that $i^2 = -1$.

**Definition** The set $\mathbb{C}$ of **complex numbers** is the set of all numbers of the form $a + bi$, where $i^2 = -1$ and where $a$ and $b$ are real numbers. The **real part** of $a + bi$ is $a$, and the **imaginary part** of $a + bi$ is $b$. A complex number of the form $0 + bi = bi$ is called a **pure imaginary** complex number.

Some examples of complex numbers are $2 + 3i$, $-\frac{1}{2} + \frac{1}{4}i$, and $\sqrt{3} - i$. Any real number $a$ can be expressed as $a + 0i$, so the real numbers are a subset of the complex numbers; that is, $\mathbb{R} \subset \mathbb{C}$.

**Definition** Two complex numbers $a + bi$ and $c + di$ are **equal** if and only if $a = c$ and $b = d$.

For example, if $3 + bi = c - 4i$, then $b = -4$ and $c = 3$.

**Definition** The **magnitude**, or **absolute value,** of the complex number $a + bi$ is $|a + bi| = \sqrt{a^2 + b^2}$, a nonnegative real number.

**Example 1**

The magnitude of $3 - 2i$ is $|3 - 2i| = \sqrt{3^2 + (-2)^2} = \sqrt{13}$.

We define **addition** of complex numbers $a + bi$ and $c + di$ by

$$(a + bi) + (c + di) = (a + c) + (b + d)i,$$

and complex number **multiplication** by

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i,$$

where $a, b, c, d \in \mathbb{R}$.

**Example 2**

From the rules for complex addition and multiplication, we have

$$\begin{aligned}(3 - 2i)[(2 - i) + (-3 + 5i)] &= (3 - 2i)(-1 + 4i) \\ &= [(3)(-1) - (-2)(4)] + [(3)(4) + (-2)(-1)]\,i \\ &= 5 + 14i.\end{aligned}$$

If $z = a + bi$, we let $-z$ denote the special product $-1z = -a - bi$.

**Definition** The **complex conjugate** of a complex number $a + bi$ is

$$\overline{a+bi} = a - bi.$$

Notice that a complex number $a + bi$ and its complex conjugate $a - bi$ have the same magnitude (absolute value) because $|a - bi| = |a + (-b)i| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2} = |a + bi|$.

**Example 3**

From the definition of complex conjugate, $\overline{6+5i} = 6 - 5i$ and $\overline{-4-3i} = -4 + 3i$. Also, $|\overline{6+5i}| = |6 - 5i| = \sqrt{6^2 + (-5)^2} = \sqrt{61} = \sqrt{6^2 + 5^2} = |6 + 5i|$.

Notice that if $z = a + bi$, then $\overline{z} = a - bi$, and so

$$z\overline{z} = (a + bi)(a - bi) = a^2 + b^2 = |a + bi|^2 = |z|^2,$$

a real number. We can use this property to calculate the **multiplicative inverse,** or **reciprocal,** of a complex number, as follows:

If $z = a + bi \neq 0$, then

$$\frac{1}{z} = \frac{1}{a+bi} = \frac{1}{a+bi} \cdot \frac{a-bi}{a-bi} = \frac{a-bi}{a^2+b^2} = \frac{\overline{z}}{|z|^2}.$$

**Example 4**

The reciprocal of $z = 8 + 15i$ is

$$\frac{1}{z} = \frac{\overline{z}}{|z|^2} = \frac{8-15i}{8^2+15^2} = \frac{8-15i}{289} = \frac{8}{289} - \frac{15}{289}i.$$

It is a straightforward matter to show that the operations of complex addition and multiplication satisfy the **commutative** ($z_1 + z_2 = z_2 + z_1$ and $z_1z_2 = z_2z_1$), **associative** ($(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ and $(z_1z_2)z_3 = z_1(z_2z_3)$), and **distributive** ($z_1(z_2 + z_3) = z_1z_2 + z_1z_3$ and $(z_1 + z_2)z_3 = z_1z_3 + z_2z_3$) laws, for all $z_1, z_2, z_3 \in \mathbb{C}$. Some other useful properties are listed in the next theorem, whose proof is left as Exercise 3. You are asked to prove further properties in Exercise 4.

**Theorem C.1** *Let $z_1, z_2, z_3 \in \mathbb{C}$. Then*

| | | |
|---|---|---|
| **(1)** | $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ | *Additive Conjugate Law* |
| **(2)** | $\overline{(z_1z_2)} = \overline{z_1}\ \overline{z_2}$ | *Multiplicative Conjugate Law* |
| **(3)** | *If $z_1z_2 = 0$, then either $z_1 = 0$ or $z_2 = 0$* | *Zero Product Property* |
| **(4)** | *$z_1 = \overline{z_1}$ if and only if $z_1$ is real* | *Condition for complex number to be real* |
| **(5)** | *$z_1 = -\overline{z_1}$ if and only if $z_1$ is pure imaginary* | *Condition for complex number to be pure imaginary* |

## New Vocabulary

absolute value (magnitude) of a complex number
addition of complex numbers
Additive Conjugate Law
associative laws of addition and multiplication for complex numbers
commutative laws of addition and multiplication for complex numbers
complex conjugate of a complex number
complex number
distributive laws of multiplication over addition for complex numbers
equal complex numbers
imaginary part of a complex number
multiplication of complex numbers

Multiplicative Conjugate Law
multiplicative inverse (reciprocal) of a complex number
pure imaginary complex number
real part of a complex number
Zero Product Property for complex number multiplication

## Highlights

- The set $\mathbb{C}$ of complex numbers consists of all numbers of the form $a + bi$, where $i^2 = -1$ and where $a$ (the real part of $a + bi$) and $b$ (the imaginary part of $a + bi$) are real numbers. A complex number of the form $0 + bi = bi$ is a pure imaginary complex number.
- Two complex numbers $a + bi$ and $c + di$ are equal if and only if $a = c$ and $b = d$.
- The magnitude, or absolute value, of $a + bi$ is $|a + bi| = \sqrt{a^2 + b^2}$.
- For complex numbers $a + bi$, $c + di$, we have $(a + bi) + (c + di) = (a + c) + (b + d)i$, and $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$.
- The complex conjugate of $a + bi$ is equal to $\overline{a + bi} = a - bi$. Also, $|\overline{a + bi}| = |a + bi|$.
- If $z = a + bi$, then $z\overline{z} = a^2 + b^2 = |z|^2$.
- If $z = a + bi \neq 0$, then $\dfrac{1}{z} = \dfrac{a - bi}{a^2 + b^2} = \dfrac{\overline{z}}{|z|^2}$.
- For complex addition and multiplication, the commutative, associative, and distributive laws hold for all $z_1, z_2, z_3 \in \mathbb{C}$.
- If $z_1, z_2 \in \mathbb{C}$, then $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$, and $\overline{(z_1 z_2)} = \overline{z_1}\,\overline{z_2}$. Also, if $z_1 z_2 = 0$, then either $z_1 = 0$ or $z_2 = 0$.
- If $z \in \mathbb{C}$, then $z = \overline{z}$ if and only if $z$ is real, and $z = -\overline{z}$ if and only if $z$ is pure imaginary.

## Exercises for Appendix C

1. Perform the following computations involving complex numbers:
   - ★ (a) $(6 - 3i) + (5 + 2i)$
   - (b) $8(3 - 4i)$
   - ★ (c) $4((8 - 2i) - (3 + i))$
   - (d) $-3((-2 + i) - (4 - 2i))$
   - ★ (e) $(5 + 3i)(3 + 2i)$
   - (f) $(-6 + 4i)(3 - 5i)$
   - ★ (g) $(7 - i)(-2 - 3i)$
   - (h) $\overline{5 + 4i}$
   - ★ (i) $\overline{9 - 2i}$
   - (j) $\overline{-6}$
   - ★ (k) $\overline{(6 + i)(2 - 4i)}$
   - (l) $|8 - 3i|$
   - ★ (m) $|-2 + 7i|$
   - (n) $|\overline{3 + 4i}|$
2. Find the multiplicative inverse (reciprocal) of each of the following:
   - ★ (a) $6 - 2i$
   - (b) $3 + 4i$
   - ★ (c) $-4 + i$
   - (d) $-5 - 3i$
3. ▶ This exercise asks for proofs for various parts of Theorem C.1.
   - (a) Prove parts (1) and (2) of Theorem C.1.
   - (b) Prove part (3) of Theorem C.1.
   - (c) Prove parts (4) and (5) of Theorem C.1.
4. Let $z_1$ and $z_2$ be complex numbers.
   - (a) Prove that $|z_1 z_2| = |z_1||z_2|$.
   - (b) If $z_1 \neq 0$, prove that $\left|\frac{1}{z_1}\right| = \frac{1}{|z_1|}$.
   - (c) If $z_2 \neq 0$, prove that $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$.
5. ★ True or False:
   - (a) The magnitude (absolute value) of a complex number is the product of the number and its conjugate.
   - (b) A complex number equals its conjugate if and only if it is zero.
   - (c) The conjugate of a pure imaginary number is equal to its negative.
   - (d) Every complex number has an additive inverse.
   - (e) Every complex number has a multiplicative inverse.

Appendix D

# Elementary Matrices

### *Prerequisite: Section 2.4, Inverses of Matrices*

In this appendix, we introduce elementary matrices and show that performing a row operation on a matrix is equivalent to multiplying it by an elementary matrix. We conclude with some useful properties of elementary matrices.

## Elementary Matrices

**Definition** An $n \times n$ matrix is an **elementary matrix of Type (I), (II)**, or **(III)** if and only if it is obtained by performing a single row operation of Type (I), (II), or (III), respectively, on the identity matrix $\mathbf{I}_n$.

That is, an elementary matrix is a matrix that is one step away from an identity matrix in terms of row operations.

**Example 1**

The Type (I) row operation $\langle 2 \rangle \leftarrow -3\langle 2 \rangle$ converts the identity matrix

$$\mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{into} \quad \mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence, $\mathbf{A}$ is an elementary matrix of Type (I) because it is the result of a single row operation of that type on $\mathbf{I}_3$. Next, consider

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since $\mathbf{B}$ is obtained from $\mathbf{I}_3$ by performing the single Type (II) row operation $\langle 1 \rangle \leftarrow -2\langle 3 \rangle + \langle 1 \rangle$, $\mathbf{B}$ is an elementary matrix of Type (II). Finally,

$$\mathbf{C} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is an elementary matrix of Type (III) because it is obtained by performing the single Type (III) row operation $\langle 1 \rangle \leftrightarrow \langle 2 \rangle$ on $\mathbf{I}_2$.

## Representing a Row Operation as Multiplication by an Elementary Matrix

The next theorem shows that there is a connection between row operations and matrix multiplication.

**Theorem D.1** *Let $\mathbf{A}$ and $\mathbf{B}$ be $m \times n$ matrices. If $\mathbf{B}$ is obtained from $\mathbf{A}$ by performing a single row operation and if $\mathbf{E}$ is the $m \times m$ elementary matrix obtained by performing that same row operation on $\mathbf{I}_m$, then $\mathbf{B} = \mathbf{EA}$.*

In other words, the effect of a single row operation on $\mathbf{A}$ can be obtained by multiplying $\mathbf{A}$ on the left by the appropriate elementary matrix.

*Proof.* Suppose $\mathbf{B}$ is obtained from $\mathbf{A}$ by performing the row operation $R$. Then $\mathbf{E} = R(\mathbf{I}_m)$. Hence, by Theorem 2.1, $\mathbf{B} = R(\mathbf{A}) = R(\mathbf{I}_m\mathbf{A}) = (R(\mathbf{I}_m))\mathbf{A} = \mathbf{EA}$. □

**Example 2**

Consider the matrices

$$\mathbf{A}=\begin{bmatrix}2&-3&0&1\\1&6&-2&-2\\0&5&3&4\end{bmatrix}\quad\text{and}\quad\mathbf{B}=\begin{bmatrix}2&-3&0&1\\1&6&-2&-2\\-3&-13&9&10\end{bmatrix}.$$

Notice that $\mathbf{B}$ is obtained from $\mathbf{A}$ by performing the operation (II): $\langle 3\rangle \leftarrow -3\langle 2\rangle + \langle 3\rangle$. The elementary matrix

$$\mathbf{E}=\begin{bmatrix}1&0&0\\0&1&0\\0&-3&1\end{bmatrix}$$

is obtained by performing this same row operation on $\mathbf{I}_3$. Notice that

$$\mathbf{EA}=\begin{bmatrix}1&0&0\\0&1&0\\0&-3&1\end{bmatrix}\begin{bmatrix}2&-3&0&1\\1&6&-2&-2\\0&5&3&4\end{bmatrix}=\begin{bmatrix}2&-3&0&1\\1&6&-2&-2\\-3&-13&9&10\end{bmatrix}=\mathbf{B}.$$

That is, $\mathbf{B}$ can also be obtained from $\mathbf{A}$ by multiplying $\mathbf{A}$ on the left by the appropriate elementary matrix.

## Inverses of Elementary Matrices

Recall that every row operation has a corresponding inverse row operation. The exact form for the inverse of a row operation of each type was given in Table 2.1 in Section 2.3. These inverse row operations can be used to find inverses of elementary matrices, as we see in the next theorem.

**Theorem D.2** *Every elementary matrix* $\mathbf{E}$ *is nonsingular, and its inverse* $\mathbf{E}^{-1}$ *is an elementary matrix of the same Type ((I), (II), or (III)).*

*Proof.* Any $n \times n$ elementary matrix $\mathbf{E}$ is formed by performing a single row operation (of Type (I), (II), or (III)) on $\mathbf{I}_n$. If we then perform its inverse operation on $\mathbf{E}$, the result is $\mathbf{I}_n$ again. But the inverse row operation has the same type as the original row operation, and so its corresponding $n \times n$ elementary matrix $\mathbf{F}$ has the same type as $\mathbf{E}$. Now by Theorem D.1, the product $\mathbf{FE}$ must equal $\mathbf{I}_n$. Hence $\mathbf{F}$ and $\mathbf{E}$ are inverses and have the same type. □

**Example 3**

Suppose we want the inverse of the elementary matrix

$$\mathbf{B}=\begin{bmatrix}1&0&-2\\0&1&0\\0&0&1\end{bmatrix}.$$

The row operation corresponding to $\mathbf{B}$ is (II): $\langle 1\rangle \leftarrow -2\langle 3\rangle + \langle 1\rangle$. Hence, the inverse operation is (II): $\langle 1\rangle \leftarrow 2\langle 3\rangle + \langle 1\rangle$, whose elementary matrix is

$$\mathbf{B}^{-1}=\begin{bmatrix}1&0&2\\0&1&0\\0&0&1\end{bmatrix}.$$

## Using Elementary Matrices to Show Row Equivalence

If two matrices $\mathbf{A}$ and $\mathbf{B}$ are row equivalent, there is some finite sequence of, say, $k$ row operations that converts $\mathbf{A}$ into $\mathbf{B}$. But according to Theorem D.1, performing each of these row operations is equivalent to multiplying (on the left) by an appropriate elementary matrix. Hence, there must be a sequence of $k$ elementary matrices $\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_k$, such that $\mathbf{B} = \mathbf{E}_k(\cdots(\mathbf{E}_3(\mathbf{E}_2(\mathbf{E}_1\mathbf{A})))\cdots)$. In fact, the converse is true as well since if $\mathbf{B} = \mathbf{E}_k(\cdots(\mathbf{E}_3(\mathbf{E}_2(\mathbf{E}_1\mathbf{A})))\cdots)$ for some collection of elementary matrices $\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_k$, then $\mathbf{B}$ can be obtained from $\mathbf{A}$ through a sequence of $k$ row operations. Hence, we have the following result:

**Theorem D.3** *Two* $m \times n$ *matrices* $\mathbf{A}$ *and* $\mathbf{B}$ *are row equivalent if and only if there is a (finite) sequence* $\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_k$ *of elementary matrices such that* $\mathbf{B} = \mathbf{E}_k \cdots \mathbf{E}_2\mathbf{E}_1\mathbf{A}$.

### Example 4

Consider the matrix $\mathbf{A} = \begin{bmatrix} 0 & 1 & -4 \\ 2 & 5 & 9 \end{bmatrix}$. We perform a series of row operations to obtain a row equivalent matrix $\mathbf{B}$. Next to each operation we give its corresponding elementary matrix.

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & -4 \\ 2 & 5 & 9 \end{bmatrix}$$

(III): $\langle 1 \rangle \leftrightarrow \langle 2 \rangle$ $\quad \begin{bmatrix} 2 & 5 & 9 \\ 0 & 1 & -4 \end{bmatrix} \quad \mathbf{E}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(I): $\langle 1 \rangle \leftarrow \frac{1}{2}\langle 1 \rangle$ $\quad \begin{bmatrix} 1 & \frac{5}{2} & \frac{9}{2} \\ 0 & 1 & -4 \end{bmatrix} \quad \mathbf{E}_2 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$

(II): $\langle 1 \rangle \leftarrow -\frac{5}{2}\langle 2 \rangle + \langle 1 \rangle$ $\quad \begin{bmatrix} 1 & 0 & \frac{29}{2} \\ 0 & 1 & -4 \end{bmatrix} = \mathbf{B}. \quad \mathbf{E}_3 = \begin{bmatrix} 1 & -\frac{5}{2} \\ 0 & 1 \end{bmatrix}$

Alternatively, the same result $\mathbf{B}$ is obtained if we multiply $\mathbf{A}$ on the left by the product of the elementary matrices $\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1$:

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & \frac{29}{2} \\ 0 & 1 & -4 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -\frac{5}{2} \\ 0 & 1 \end{bmatrix}}_{\mathbf{E}_3} \underbrace{\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{E}_2} \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\mathbf{E}_1} \underbrace{\begin{bmatrix} 0 & 1 & -4 \\ 2 & 5 & 9 \end{bmatrix}}_{\mathbf{A}}.$$

(Verify that the final product really does equal $\mathbf{B}$.) Note that the product is written in the *reverse* of the order in which the row operations were performed.

## Nonsingular Matrices Expressed as a Product of Elementary Matrices

Suppose that we can convert a matrix $\mathbf{A}$ to a matrix $\mathbf{B}$ using row operations. Then, by Theorem D.3, $\mathbf{B} = \mathbf{E}_k \cdots \mathbf{E}_2\mathbf{E}_1\mathbf{A}$, for some elementary matrices $\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_k$. But we can multiply both sides by $\mathbf{E}_k^{-1}, \ldots, \mathbf{E}_2^{-1}, \mathbf{E}_1^{-1}$ (in that order) to obtain $\mathbf{E}_1^{-1}\mathbf{E}_2^{-1} \cdots \mathbf{E}_k^{-1}\mathbf{B} = \mathbf{A}$. Now, by Theorem D.2, each of the inverses $\mathbf{E}_1^{-1}, \mathbf{E}_2^{-1}, \ldots, \mathbf{E}_k^{-1}$ is also an elementary matrix. Therefore, we have found a product of elementary matrices that converts $\mathbf{B}$ back into the original matrix $\mathbf{A}$. We can use this fact to express a nonsingular matrix as a product of elementary matrices, as in the next example.

### Example 5

Suppose that we want to express the nonsingular matrix $\mathbf{A} = \begin{bmatrix} -5 & -2 \\ 7 & 3 \end{bmatrix}$ as a product of elementary matrices. We begin by row reducing $\mathbf{A}$, keeping track of the row operations used.

$$\mathbf{A} = \begin{bmatrix} -5 & -2 \\ 7 & 3 \end{bmatrix}$$

(I): $\langle 1 \rangle \leftarrow -\frac{1}{5}\langle 1 \rangle$ $\quad \begin{bmatrix} 1 & \frac{2}{5} \\ 7 & 3 \end{bmatrix}$

(II): $\langle 2 \rangle \leftarrow -7\langle 1 \rangle + \langle 2 \rangle$ $\quad \begin{bmatrix} 1 & \frac{2}{5} \\ 0 & \frac{1}{5} \end{bmatrix}$

(I): $\langle 2 \rangle \leftarrow 5\langle 2 \rangle$ $\quad \begin{bmatrix} 1 & \frac{2}{5} \\ 0 & 1 \end{bmatrix}$

(II): $\langle 1 \rangle \leftarrow -\frac{2}{5}\langle 2 \rangle + \langle 1 \rangle$ $\quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}_2.$

Reversing this process, we get a series of row operations that start with $\mathbf{I}_2$ and end with $\mathbf{A}$. The inverse of each of these row operations, in reverse order, is listed here along with its corresponding elementary matrix.

$$\text{(II): } \langle 1\rangle \leftarrow \tfrac{2}{5}\langle 2\rangle + \langle 1\rangle \qquad \mathbf{F}_1 = \begin{bmatrix} 1 & \frac{2}{5} \\ 0 & 1 \end{bmatrix}$$

$$\text{(I): } \langle 2\rangle \leftarrow \tfrac{1}{5}\langle 2\rangle \qquad \mathbf{F}_2 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{5} \end{bmatrix}$$

$$\text{(II): } \langle 2\rangle \leftarrow 7\langle 1\rangle + \langle 2\rangle \qquad \mathbf{F}_3 = \begin{bmatrix} 1 & 0 \\ 7 & 1 \end{bmatrix}$$

$$\text{(I): } \langle 1\rangle \leftarrow -5\langle 1\rangle \qquad \mathbf{F}_4 = \begin{bmatrix} -5 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore, we can express $\mathbf{A}$ as the product

$$\mathbf{A} = \underbrace{\begin{bmatrix} -5 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{F}_4}\underbrace{\begin{bmatrix} 1 & 0 \\ 7 & 1 \end{bmatrix}}_{\mathbf{F}_3}\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{5} \end{bmatrix}}_{\mathbf{F}_2}\underbrace{\begin{bmatrix} 1 & \frac{2}{5} \\ 0 & 1 \end{bmatrix}}_{\mathbf{F}_1}\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{I}_2}.$$

You should verify that this product is really equal to $\mathbf{A}$.

Example 5 motivates the following corollary of Theorem D.3. We leave the proof for you to do in Exercise 7.

**Corollary D.4** *An $n \times n$ matrix $\mathbf{A}$ is nonsingular if and only if $\mathbf{A}$ is the product of a finite collection of $n \times n$ elementary matrices.*

## New Vocabulary

elementary matrix (of Type (I), (II), or (III))

## Highlights

- An elementary matrix $\mathbf{E}$ is any matrix obtained by performing a single row operation of Type (I), (II), or (III) on $\mathbf{I}_n$.
- If a row operation (of Type (I), (II), or (III)) is performed on a matrix $\mathbf{A}$ to obtain a matrix $\mathbf{B}$, then $\mathbf{B} = \mathbf{EA}$, where $\mathbf{E}$ is the elementary matrix corresponding to that row operation.
- If $\mathbf{E}$ is an elementary matrix (of Type (I), (II), or (III)), then $\mathbf{E}^{-1}$ is an elementary matrix of the same type. The row operations corresponding to $\mathbf{E}$ and $\mathbf{E}^{-1}$ are inverses of each other.
- Two matrices $\mathbf{A}$ and $\mathbf{B}$ are row equivalent if and only if $\mathbf{B} = \mathbf{E}_k \cdots \mathbf{E}_2\mathbf{E}_1\mathbf{A}$, for some sequence $\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_k$ of elementary matrices.
- A matrix $\mathbf{A}$ is nonsingular if and only if $\mathbf{A}$ is a (finite) product of elementary matrices.

# Exercises for Appendix D

1. For each elementary matrix below, determine its corresponding row operation. Also, use the inverse operation to find the inverse of the given matrix.

★ (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

★ (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

★ **(e)** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

**(f)** $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

**2.** Express each of the following as a product of elementary matrices (if possible), in the manner of Example 5.

★ **(a)** $\begin{bmatrix} 4 & 9 \\ 3 & 7 \end{bmatrix}$

**(b)** $\begin{bmatrix} 1 & -3 & 2 \\ -3 & 9 & 7 \\ 5 & 1 & 18 \end{bmatrix}$

**(c)** $\begin{bmatrix} -3 & 2 & 1 \\ 13 & -8 & -9 \\ 1 & -1 & 2 \end{bmatrix}$

★ **(d)** $\begin{bmatrix} 0 & 0 & 5 & 0 \\ -3 & 0 & 0 & -2 \\ 0 & 6 & -10 & -1 \\ 3 & 0 & 0 & 3 \end{bmatrix}$

**(e)** $\begin{bmatrix} -2 & -4 & 7 & 3 \\ 3 & 6 & -3 & 18 \\ 4 & 13 & -13 & -7 \\ -5 & 5 & 13 & 3 \end{bmatrix}$

**3.** Let $\mathbf{A}$ and $\mathbf{B}$ be $m \times n$ matrices. Prove that $\mathbf{A}$ and $\mathbf{B}$ are row equivalent if and only if $\mathbf{B} = \mathbf{PA}$, for some nonsingular $m \times m$ matrix $\mathbf{P}$.

**4.** Prove that if $\mathbf{U}$ is an upper triangular matrix with all main diagonal entries nonzero, then $\mathbf{U}^{-1}$ exists and is upper triangular. (Hint: Show that the method for calculating the inverse of a matrix does not produce a row of zeroes on the left side of the augmented matrix. Also, show that for each row reduction step, the corresponding elementary matrix is upper triangular. Conclude that $\mathbf{U}^{-1}$ is the product of upper triangular matrices, and is therefore upper triangular (see Exercise 18(b) in Section 1.5).)

**5.** If $\mathbf{E}$ is an elementary matrix, show that $\mathbf{E}^T$ is also an elementary matrix. What is the relationship between the row operation corresponding to $\mathbf{E}$ and the row operation corresponding to $\mathbf{E}^T$?

**6.** Let $\mathbf{F}$ be an elementary $n \times n$ matrix. Show that the product $\mathbf{AF}^T$ is the matrix obtained by performing a "column" operation on $\mathbf{A}$ analogous to one of the three types of row operations. (Hint: What is $(\mathbf{AF}^T)^T$?)

▶ **7.** Prove Corollary D.4.

**8.** Consider the homogeneous system $\mathbf{AX} = \mathbf{O}$, where $\mathbf{A}$ is an $n \times n$ matrix. Show that this system has a nontrivial solution if and only if $\mathbf{A}$ cannot be expressed as the product of elementary $n \times n$ matrices.

**9.** Let $\mathbf{A}$ and $\mathbf{B}$ be $m \times n$ and $n \times p$ matrices, respectively.

- **(a)** Let $\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_k$ be $m \times m$ elementary matrices. Prove that $\text{rank}(\mathbf{E}_k \cdots \mathbf{E}_2\mathbf{E}_1\mathbf{A}) = \text{rank}(\mathbf{A})$.
- **(b)** Show that if $\mathbf{A}$ has $k$ rows of all zeroes, then $\text{rank}(\mathbf{A}) \leq m - k$.
- **(c)** Show that if $\mathbf{A}$ is in reduced row echelon form, then $\text{rank}(\mathbf{AB}) \leq \text{rank}(\mathbf{A})$. (Use part (b).)
- **(d)** Use parts (a) and (c) to prove that for a general matrix $\mathbf{A}$, $\text{rank}(\mathbf{AB}) \leq \text{rank}(\mathbf{A})$.
- **(e)** Compare this exercise with Exercise 18 in Section 2.3.

★ **10.** True or False:

- **(a)** Every elementary matrix is square.
- **(b)** If $\mathbf{A}$ and $\mathbf{B}$ are row equivalent matrices, then there must be an elementary matrix $\mathbf{E}$ such that $\mathbf{B} = \mathbf{EA}$.
- **(c)** If $\mathbf{E}_1, \ldots, \mathbf{E}_k$ are $n \times n$ elementary matrices, then the inverse of $\mathbf{E}_1\mathbf{E}_2 \cdots \mathbf{E}_k$ is $\mathbf{E}_k \cdots \mathbf{E}_2\mathbf{E}_1$.
- **(d)** If $\mathbf{A}$ is a nonsingular matrix, then $\mathbf{A}^{-1}$ can be expressed as a product of elementary matrices.
- **(e)** If $R$ is a row operation, $\mathbf{E}$ is its corresponding $m \times m$ matrix, and $\mathbf{A}$ is any $m \times n$ matrix, then the reverse row operation $R^{-1}$ has the property $R^{-1}(\mathbf{A}) = \mathbf{E}^{-1}\mathbf{A}$.

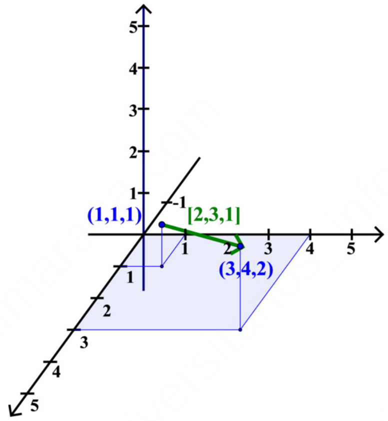
Appendix E

# Answers to Selected Exercises

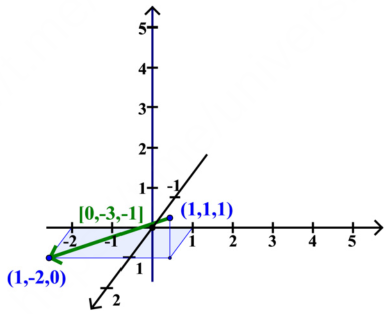
## Section 1.1 (p. 10–12)

**1.** **(a)** $[9, -4]$; distance $= \sqrt{97}$ **(c)** $[-1, -1, 2, -3, -4]$; distance $= \sqrt{31}$

**2.** **(a)** $(3, 4, 2)$ (see accompanying figure)




**(c)** $(1, -2, 0)$ (see accompanying figure)




**3.** **(a)** $(7, -13)$ **(c)** $(-1, 3, -1, 4, 6)$

**4.** **(a)** $(\frac{16}{3}, -\frac{13}{3}, 8)$

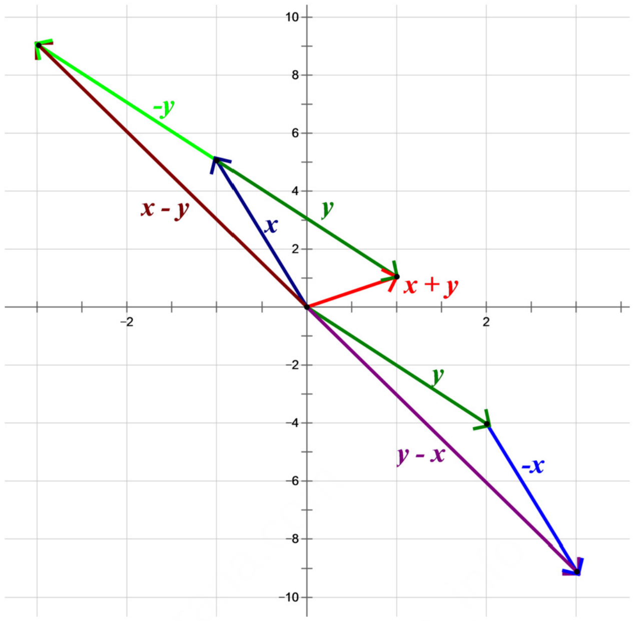
**5.** **(a)** $\left[\frac{3}{\sqrt{70}}, -\frac{5}{\sqrt{70}}, \frac{6}{\sqrt{70}}\right]$; shorter, since length of original vector is $> 1$

**(c)** $[0.6, -0.8]$; neither, since given vector is a unit vector

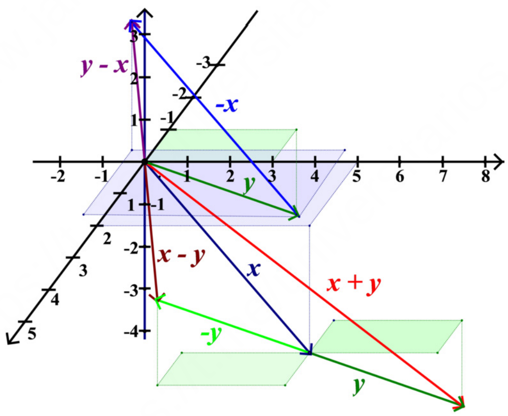
**6.** **(a)** Parallel **(c)** Not parallel

**7.** **(a)** $[-6, 12, 15]$ **(c)** $[-7, 1, 11]$ **(e)** $[-10, -32, -1]$

**8. (a)** $\mathbf{x} + \mathbf{y} = [1, 1]$; $\mathbf{x} - \mathbf{y} = [-3, 9]$; $\mathbf{y} - \mathbf{x} = [3, -9]$ (see accompanying figure)




**(c)** $\mathbf{x} + \mathbf{y} = [1, 8, -5]$; $\mathbf{x} - \mathbf{y} = [3, 2, -1]$; $\mathbf{y} - \mathbf{x} = [-3, -2, 1]$ (see accompanying figure)




**10. (a)** $[10, -10]$ **(b)** $[-5\sqrt{3}, -15]$

**13.** $[0.5 - 0.6\sqrt{2}, -0.4\sqrt{2}] \approx [-0.3485, -0.5657]$ (with units for both in meters)

**14.** Net velocity $= [-2\sqrt{2}, -3 + 2\sqrt{2}]$ (with units in m/sec$^2$); resultant speed $\approx 2.83$ km/hr

**16.** $[-8 - \sqrt{2}, -\sqrt{2}]$ (with units in km/hr)

**17.** Acceleration $= \frac{1}{20}\left[\frac{12}{13}, -\frac{344}{65}, \frac{392}{65}\right] \approx [0.0462, -0.2646, 0.3015]$ (with units for both in m/sec$^2$)

**20.** $\mathbf{a} = \left[\frac{-mg}{1+\sqrt{3}}, \frac{mg}{1+\sqrt{3}}\right]$; $\mathbf{b} = \left[\frac{mg}{1+\sqrt{3}}, \frac{mg\sqrt{3}}{1+\sqrt{3}}\right]$ (with units for **a**, **b** in newtons)

**25. (a)** F **(b)** T **(c)** T **(d)** F **(e)** T **(f)** F **(g)** F **(h)** F

## Section 1.2 (p. 20–22)

**1. (a)** $\cos^{-1}(-\frac{27}{5\sqrt{37}}) \approx 152.6°$, or 2.66 radians **(c)** $\cos^{-1}(0) = 90°$, or $\frac{\pi}{2}$ radians

**4. (b)** $\frac{1040\sqrt{5}}{9} \approx 258.4$ joules

**7.** No; consider $\mathbf{x} = [1, 0]$, $\mathbf{y} = [0, 1]$, and $\mathbf{z} = [1, 1]$.

**16.** **(a)** Length of diagonal $= \sqrt{3}s$  **(b)** Angle $= \cos^{-1}(\frac{\sqrt{3}}{3}) \approx 54.7°$, or 0.955 radians

**17.** **(a)** $\left[-\frac{3}{5}, -\frac{3}{10}, -\frac{3}{2}\right]$  **(c)** $\left[\frac{1}{6}, 0, -\frac{1}{6}, \frac{1}{3}\right]$

**19.** $a\mathbf{i}, b\mathbf{j}, c\mathbf{k}$

**20.** **(a)** Parallel: $\left[\frac{20}{29}, -\frac{30}{29}, \frac{40}{29}\right]$; orthogonal: $\left[-\frac{194}{29}, \frac{88}{29}, \frac{163}{29}\right]$ (with units for both in newtons)

**(c)** Parallel: $\left[\frac{60}{49}, -\frac{40}{49}, \frac{120}{49}\right]$; orthogonal: $\left[-\frac{354}{49}, \frac{138}{49}, \frac{223}{49}\right]$ (with units for both in newtons)

**23.** **(a)** $c = \frac{2}{5}$; $\mathbf{w} = \left[\frac{7}{5}, -\frac{24}{5}, -4\right]$

**25.** **(a)** Two acute angles, each measuring 58.41°, or about 1.02 radians; two obtuse angles, each measuring 121.59°, or about 2.12 radians.

**(b)** Two acute angles, each measuring 57.42°, or about 1.00 radians; two obtuse angles, each measuring 122.58°, or about 2.14 radians.

**26.** **(a)** T **(b)** T **(c)** F **(d)** F **(e)** T **(f)** F

## Section 1.3 (p. 32–34)

**1.** **(b)** Let $m = \max\{|c|, |d|\}$. Then $\|c\mathbf{x} \pm d\mathbf{y}\| \leq m(\|\mathbf{x}\| + \|\mathbf{y}\|)$.

**2.** **(b)** Consider the number 8.

**6.** **(a)** Consider $\mathbf{x} = [1, 0, 0]$ and $\mathbf{y} = [1, 1, 0]$.  **(c)** Yes

**(b)** If $\mathbf{x} \neq \mathbf{y}$, then $\mathbf{x} \cdot \mathbf{y} \neq \|\mathbf{x}\|^2$.

**7.** Assuming $\mathbf{x}$ and $\mathbf{y}$ are vectors in $\mathbb{R}^n$, the contrapositive of the given statement is: "If $\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$, then $\mathbf{x} \cdot \mathbf{y} = 0$." To prove the contrapositive, we assume the premise $\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$. Then by applying part (2) of Theorem 1.5 to both sides of this equation, we have $(\mathbf{x}+\mathbf{y}) \cdot (\mathbf{x}+\mathbf{y}) = \mathbf{x} \cdot \mathbf{x} + \mathbf{y} \cdot \mathbf{y}$. Expanding the left side (using the distributive laws from Theorem 1.5), we find $\mathbf{x} \cdot \mathbf{x} + 2(\mathbf{x} \cdot \mathbf{y}) + \mathbf{y} \cdot \mathbf{y} = \mathbf{x} \cdot \mathbf{x} + \mathbf{y} \cdot \mathbf{y}$, which reduces to $\mathbf{x} \cdot \mathbf{y} = 0$, the desired conclusion. Since the contrapositive of the original statement has been proven, the original statement is proven as well.

**9.** **(a)** Contrapositive: If $\mathbf{x} = \mathbf{0}$, then $\mathbf{x}$ is not a unit vector.
Converse: If $\mathbf{x}$ is nonzero, then $\mathbf{x}$ is a unit vector.
Inverse: If $\mathbf{x}$ is not a unit vector, then $\mathbf{x} = \mathbf{0}$.

**(c)** (Let $\mathbf{x}$, $\mathbf{y}$ be nonzero vectors.)
Contrapositive: If $\mathbf{proj_y x} \neq \mathbf{0}$, then $\mathbf{proj_x y} \neq \mathbf{0}$.
Converse: If $\mathbf{proj_y x} = \mathbf{0}$, then $\mathbf{proj_x y} = \mathbf{0}$.
Inverse: If $\mathbf{proj_x y} \neq \mathbf{0}$, then $\mathbf{proj_y x} \neq \mathbf{0}$.

**11.** **(b)** Converse: Let $\mathbf{x}$ and $\mathbf{y}$ be vectors in $\mathbb{R}^n$. If $\|\mathbf{x}+\mathbf{y}\| \geq \|\mathbf{y}\|$, then $\mathbf{x} \cdot \mathbf{y} = 0$. The original statement is true, but the converse is false in general. Proof of the original statement follows from

$$\begin{aligned}\|\mathbf{x}+\mathbf{y}\|^2 &= (\mathbf{x}+\mathbf{y}) \cdot (\mathbf{x}+\mathbf{y}) \\ &= \|\mathbf{x}\|^2 + 2(\mathbf{x} \cdot \mathbf{y}) + \|\mathbf{y}\|^2 \\ &= \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 \geq \|\mathbf{y}\|^2.\end{aligned}$$

Counterexample to converse: let $\mathbf{x} = [1, 0]$, $\mathbf{y} = [1, 1]$.

**12.** Assume each coordinate of both $\mathbf{x}$ and $\mathbf{y}$ is equal to either 1 or $-1$. Suppose $\mathbf{x}$ is orthogonal to $\mathbf{y}$ and $n$ is odd. Then $\mathbf{x} \cdot \mathbf{y} = 0$. Now $\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i$. But each product $x_i y_i$ equals either 1 or $-1$. If exactly $k$ of these products equal 1, then $\mathbf{x} \cdot \mathbf{y} = k - (n-k) = -n + 2k$. Hence $-n + 2k = 0$, and so $n = 2k$, contradicting $n$ odd.

**17.** Base Step ($k = 1$): $\|\mathbf{x}_1\|^2 = \|\mathbf{x}_1\|^2$.
Inductive Step: Assume $\|\mathbf{x}_1 + \cdots + \mathbf{x}_k\|^2 = \|\mathbf{x}_1\|^2 + \cdots + \|\mathbf{x}_k\|^2$.
Prove: $\|\mathbf{x}_1 + \cdots + \mathbf{x}_k + \mathbf{x}_{k+1}\|^2 = \|\mathbf{x}_1\|^2 + \cdots + \|\mathbf{x}_k\|^2 + \|\mathbf{x}_{k+1}\|^2$.
We have

$$\begin{aligned}&\|(\mathbf{x}_1 + \cdots + \mathbf{x}_k) + \mathbf{x}_{k+1}\|^2 \\ &= \|\mathbf{x}_1 + \cdots + \mathbf{x}_k\|^2 + 2((\mathbf{x}_1 + \cdots + \mathbf{x}_k) \cdot \mathbf{x}_{k+1}) + \|\mathbf{x}_{k+1}\|^2 \\ &= \|\mathbf{x}_1 + \cdots + \mathbf{x}_k\|^2 + \|\mathbf{x}_{k+1}\|^2 \qquad \text{since } \mathbf{x}_{k+1} \text{ is orthogonal to all of } \mathbf{x}_1, \ldots, \mathbf{x}_k \\ &= \|\mathbf{x}_1\|^2 + \cdots + \|\mathbf{x}_k\|^2 + \|\mathbf{x}_{k+1}\|^2,\end{aligned}$$

by the inductive hypothesis.

**20.** Step 1 cannot be reversed, because $y$ could equal $\pm(x^2+2)$.
Step 2 cannot be reversed, because $y^2$ could equal $x^4+4x^2+c$.
Step 4 cannot be reversed, because in general $y$ does not have to equal $x^2+2$.
Step 6 cannot be reversed, since $\frac{dy}{dx}$ could equal $2x+c$.
All other steps remain true when reversed.

**21. (a)** For every unit vector $\mathbf{x}$ in $\mathbb{R}^3$, $\mathbf{x}\cdot[1,-2,3]\neq 0$.
**(c)** $\mathbf{x}=\mathbf{0}$ or $\|\mathbf{x}+\mathbf{y}\|\neq\|\mathbf{y}\|$, for all $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$.
**(e)** There is an $\mathbf{x}\in\mathbb{R}^3$ such that for every nonzero $\mathbf{y}\in\mathbb{R}^3$, $\mathbf{x}\cdot\mathbf{y}\neq 0$.

**22. (a)** Contrapositive: If $\mathbf{x}\neq\mathbf{0}$ and $\|\mathbf{x}-\mathbf{y}\|\leq\|\mathbf{y}\|$, then $\mathbf{x}\cdot\mathbf{y}\neq 0$.
Converse: If $\mathbf{x}=\mathbf{0}$ or $\|\mathbf{x}-\mathbf{y}\|>\|\mathbf{y}\|$, then $\mathbf{x}\cdot\mathbf{y}=0$.
Inverse: If $\mathbf{x}\cdot\mathbf{y}\neq 0$, then $\mathbf{x}\neq\mathbf{0}$ and $\|\mathbf{x}-\mathbf{y}\|\leq\|\mathbf{y}\|$.

**26. (a)** Let $\mathbf{x}$ be a vector in $\mathbb{R}^n$, and let $c$ be a scalar. If $c\neq 0$ and $\mathbf{x}\neq\mathbf{0}$, then $c\mathbf{x}\neq\mathbf{0}$.
**(c)** Using the rephrasing of Result 7 given in the hint, the contrapositive is: Let $S=\{\mathbf{x}_1,\ldots,\mathbf{x}_k\}$ be a set of vectors in $\mathbb{R}^n$. If some vector in $S$ can be expressed as a linear combination of the other vectors in $S$, then $S$ is not a mutually orthogonal set or some vector in $S$ is the zero vector.

**27. (a)** F **(b)** T **(c)** T **(d)** F **(e)** F **(f)** F **(g)** F **(h)** T **(i)** F

## Section 1.4 (p. 41–43)

**1. (a)** $\begin{bmatrix} 2 & 1 & 3 \\ 2 & 7 & -5 \\ 9 & 0 & -1 \end{bmatrix}$

**(c)** $\begin{bmatrix} -16 & 8 & 12 \\ 0 & 20 & -4 \\ 24 & 4 & -8 \end{bmatrix}$

**(e)** Impossible

**(g)** $\begin{bmatrix} -23 & 14 & -9 \\ -5 & 8 & 8 \\ -9 & -18 & 1 \end{bmatrix}$

**(i)** $\begin{bmatrix} -1 & 1 & 12 \\ -1 & 5 & 8 \\ 8 & -3 & -4 \end{bmatrix}$

**(l)** Impossible

**(n)** $\begin{bmatrix} 13 & -6 & 2 \\ 3 & -3 & -5 \\ 3 & 5 & 1 \end{bmatrix}$

**2.** Square: **B**, **C**, **E**, **F**, **G**, **H**, **J**, **K**, **L**, **M**, **N**, **P**, **Q**
Diagonal: **B**, **G**, **N**
Upper triangular: **B**, **G**, **L**, **N**
Lower triangular: **B**, **G**, **M**, **N**, **Q**
Symmetric: **B**, **F**, **G**, **J**, **N**, **P**
Skew-symmetric: **H** (but not **C**, **E**, **K**)
Transposes: $\mathbf{A}^T=\begin{bmatrix} -1 & 0 & 6 \\ 4 & 1 & 0 \end{bmatrix}$, $\mathbf{B}^T=\mathbf{B}$, $\mathbf{C}^T=\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$, and so on

**3. (a)** $\begin{bmatrix} 3 & -\frac{1}{2} & \frac{5}{2} \\ -\frac{1}{2} & 2 & 1 \\ \frac{5}{2} & 1 & 2 \end{bmatrix}+\begin{bmatrix} 0 & -\frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & 0 & 4 \\ -\frac{3}{2} & -4 & 0 \end{bmatrix}$

**5. (d)** The matrix must be a square zero matrix.

**13. (a)** Trace (**B**) = 1; trace (**C**) = 0; trace (**E**) = −6; trace (**F**) = 2; trace (**G**) = 18; trace (**H**) = 0; trace (**J**) = 1; trace (**K**) = 4; trace (**L**) = 3; trace (**M**) = 0; trace (**N**) = 3; trace (**P**) = 0; trace (**Q**) = 1
**(c)** No; consider matrices **L** and **N** in Exercise 2. (Note: If $n=1$, the statement is true.)

**14. (a)** F **(b)** T **(c)** F **(d)** T **(e)** T

## Section 1.5 (p. 50–54)

**1.** **(b)** $\begin{bmatrix} 34 & -24 \\ 42 & 49 \\ 8 & -22 \end{bmatrix}$

**(c)** Impossible

**(e)** $[-38]$

**(f)** $\begin{bmatrix} -24 & 48 & -16 \\ 3 & -6 & 2 \\ -12 & 24 & -8 \end{bmatrix}$

**(g)** Impossible

**(j)** Impossible

**(l)** $\begin{bmatrix} 5 & 3 & 2 & 5 \\ 4 & 1 & 3 & 1 \\ 1 & 1 & 0 & 2 \\ 4 & 1 & 3 & 1 \end{bmatrix}$

**(n)** $\begin{bmatrix} 146 & 5 & -603 \\ 154 & 27 & -560 \\ 38 & -9 & -193 \end{bmatrix}$

**2.** **(a)** No

**(c)** No

**(d)** Yes

**3.** **(a)** $[15, -13, -8]$

**(c)** $[4]$

**4.** **(a)** Valid, by Theorem 1.16, part (1)

**(b)** Invalid

**(c)** Valid, by Theorem 1.16, part (1)

**(d)** Valid, by Theorem 1.16, part (2)

**(e)** Valid, by Theorem 1.18

**(f)** Invalid

**(g)** Valid, by Theorem 1.16, part (3)

**(h)** Valid, by Theorem 1.16, part (2)

**(i)** Invalid

**(j)** Valid, by Theorem 1.16, part (3), and Theorem 1.18

**5.**

| | Salary | Fringe Benefits |
|---|---|---|
| Outlet 1 | \$367500 | \$78000 |
| Outlet 2 | \$225000 | \$48000 |
| Outlet 3 | \$765000 | \$162000 |
| Outlet 4 | \$360000 | \$76500 |

**7.**

| | Field 1 | Field 2 | Field 3 |
|---|---|---|---|
| Nitrogen | 1.00 | 0.45 | 0.65 |
| Phosphate | 0.90 | 0.35 | 0.75 |
| Potash | 0.95 | 0.35 | 0.85 |

(in tons)

**9.** **(a)** One example: $\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$

**(b)** One example: $\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

**(c)** Consider $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

**10.** **(a)** Third row, fourth column entry of **AB**

**(c)** Third row, second column entry of **BA**

**11.** **(a)** $\sum_{k=1}^{n} a_{3k} b_{k2}$

**12.** **(a)** $[-27, 43, -56]$

**(b)** $\begin{bmatrix} 56 \\ -57 \\ 18 \end{bmatrix}$

**29.** **(a)** Consider any matrix of the form $\begin{bmatrix} 1 & 0 \\ x & 0 \end{bmatrix}$.

**30.** **(b)** Consider $\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$.

**31.** Any such matrix must have the form $c\mathbf{I}_2$ for some scalar $c$.

**32.** **(a)** T **(b)** T **(c)** T **(d)** F **(e)** F **(f)** F **(g)** F

## Chapter 1 Review Exercises (p. 54–55)

**2.** $\mathbf{u} = \left[\frac{5}{\sqrt{394}}, -\frac{12}{\sqrt{394}}, \frac{15}{\sqrt{394}}\right] \approx [0.2481, -0.5955, 0.7444]$; slightly longer.

**4.** $\mathbf{a} = [-10, 9, 10]$ (with units in m/sec$^2$)

**6.** $\theta \approx 136°$

**9.** $-1782$ joules

**11.** First, $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{y} \neq \mathbf{0}$ (why?). Assume $\mathbf{x} \parallel \mathbf{y}$. Then, there is a scalar $c \neq 0$ such that $\mathbf{y} = c\mathbf{x}$. Hence, $\mathbf{proj_x y} = \left(\frac{\mathbf{x}\cdot c\mathbf{x}}{\|\mathbf{x}\|^2}\right)\mathbf{x} = \left(\frac{c\|\mathbf{x}\|^2}{\|\mathbf{x}\|^2}\right)\mathbf{x} = c\mathbf{x} = \mathbf{y}$, a contradiction.

**12.** **(a)** $3\mathbf{A} - 4\mathbf{C}^T = \begin{bmatrix} 3 & 2 & 13 \\ -11 & -19 & 0 \end{bmatrix}$; $\mathbf{AB} = \begin{bmatrix} 15 & -21 & -4 \\ 22 & -30 & 11 \end{bmatrix}$; $\mathbf{BA}$ is not defined; $\mathbf{AC} = \begin{bmatrix} 23 & 14 \\ -5 & 23 \end{bmatrix}$;

$\mathbf{CA} = \begin{bmatrix} 30 & -11 & 17 \\ 2 & 0 & 18 \\ -11 & 5 & 16 \end{bmatrix}$; $\mathbf{A}^3$ is not defined; $\mathbf{B}^3 = \begin{bmatrix} 97 & -128 & 24 \\ -284 & 375 & -92 \\ 268 & -354 & 93 \end{bmatrix}$

**(b)** Third row of $\mathbf{BC} = [5, 8]$.

**14.** **(a)** $(3(\mathbf{A}-\mathbf{B})^T)^T = 3((\mathbf{A}-\mathbf{B})^T)^T = 3(\mathbf{A}-\mathbf{B}) = 3(-\mathbf{A}^T - (-\mathbf{B}^T))$ (since $\mathbf{A}$, $\mathbf{B}$ are skew-symmetric) $= -3(\mathbf{A}^T - \mathbf{B}^T) = (-1)(3(\mathbf{A}-\mathbf{B})^T)$.

**16.**

| | Price | Shipping Cost |
|---|---|---|
| **Company I** | \$168500 | \$24200 |
| **Company II** | \$202500 | \$29100 |
| **Company III** | \$155000 | \$22200 |

**17.** Take transpose of both sides of $\mathbf{A}^T\mathbf{B}^T = \mathbf{B}^T\mathbf{A}^T$ to get $\mathbf{BA} = \mathbf{AB}$. Then, $(\mathbf{AB})^2 = (\mathbf{AB})(\mathbf{AB}) = \mathbf{A}(\mathbf{BA})\mathbf{B} = \mathbf{A}(\mathbf{AB})\mathbf{B} = \mathbf{A}^2\mathbf{B}^2$.

**19.** If $\mathbf{A} \neq \mathbf{O}_{22}$, then some row of $\mathbf{A}$, say the $i$th row, is nonzero. Apply Result 5 in Section 1.3 with $\mathbf{x} = (i\text{th row of } \mathbf{A})$.

**21.** **(a)** F **(c)** F **(e)** F **(g)** F **(i)** F **(k)** T **(m)** T **(o)** F **(q)** F
**(b)** T **(d)** F **(f)** T **(h)** F **(j)** T **(l)** T **(n)** F **(p)** F **(r)** T

## Section 2.1 (p. 69–70)

**1.** **(a)** Consistent; solution set $= \{(-2, 3, 6)\}$

**(c)** Inconsistent; solution set $= \{\}$

**(e)** Consistent; solution set $= \{(2b - d - 4, b, 2d + 5, d, 2) \mid b, d \in \mathbb{R}\}$; three particular solutions are: $(-4, 0, 5, 0, 2)$ (with $b = d = 0$), $(-2, 1, 5, 0, 2)$ (with $b = 1, d = 0$), and $(-5, 0, 7, 1, 2)$ (with $b = 0, d = 1$)

**(g)** Consistent; solution set $= \{(6, -1, 3)\}$

**2.** **(a)** Solution set $= \{(3c + 11e + 46, c + e + 13, c, -2e + 5, e) \mid c, e \in \mathbb{R}\}$

**(c)** Solution set $= \{(-20c + 9d - 153f - 68, 7c - 2d + 37f + 15, c, d, 4f + 2, f) \mid c, d, f \in \mathbb{R}\}$

**3.** 52 nickels, 64 dimes, 32 quarters

**4.** $y = 2x^2 - x + 5$

**6.** $x^2 + y^2 - 6x - 8y = 0$, or, $(x-3)^2 + (y-4)^2 = 25$

**7.** **(a)** $R(\mathbf{AB}) = (R(\mathbf{A}))\mathbf{B} = \begin{bmatrix} 26 & 15 & -6 \\ 6 & 4 & 1 \\ 0 & -6 & 12 \\ 10 & 4 & -14 \end{bmatrix}$.

**11.** **(a)** T **(b)** F **(c)** F **(d)** F **(e)** T **(f)** T **(g)** F

## Section 2.2 (p. 77–80)

**1.** Matrices in (a), (b), (c), (d), and (f) are not in reduced row echelon form.
Matrix in (a) fails condition 2 of the definition.
Matrix in (b) fails condition 4 of the definition.
Matrix in (c) fails condition 1 of the definition.

Matrix in (d) fails conditions 1, 2, and 3 of the definition.
Matrix in (f) fails condition 3 of the definition.

**2. (a)** $\left[\begin{array}{ccc|c} 1 & 4 & 0 & -13 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$

**(c)** $\left[\begin{array}{ccccc} 1 & -2 & 0 & 11 & -23 \\ 0 & 0 & 1 & -2 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

**(b)** $\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] = \mathbf{I}_4$

**(e)** $\left[\begin{array}{ccccc|c} 1 & -2 & 0 & 2 & -1 & 1 \\ 0 & 0 & 1 & -1 & 3 & 2 \end{array}\right]$

**3. (a)** $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 6 \end{array}\right]$; Solution set $= \{(-2, 3, 6)\}$

**(e)** $\left[\begin{array}{ccccc|c} 1 & -2 & 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -2 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right]$; Solution set $= \{(2b - d - 4, b, 2d + 5, d, 2) \mid b, d \in \mathbb{R}\}$

**(g)** $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$; Solution set $= \{(6, -1, 3)\}$

**4. (a)** Solution set $= \{(c - 2d, -3d, c, d) \mid c, d \in \mathbb{R}\}$; one particular solution $= (-3, -6, 1, 2)$
**(c)** Solution set $= \{(-4b + 2d - f, b, -3d + 2f, d, -2f, f) \mid b, d, f \in \mathbb{R}\}$;
one particular solution $= (-3, 1, 0, 2, -6, 3)$

**5. (a)** Solution set $= \{(2c, -4c, c) \mid c \in \mathbb{R}\} = \{c(2, -4, 1) \mid c \in \mathbb{R}\}$
**(c)** Solution set $= \{(0, 0, 0, 0)\}$

**6. (a)** $a = 2, b = 15, c = 12, d = 6$ **(c)** $a = 4, b = 2, c = 4, d = 1, e = 4$

**7. (a)** $A = 3, B = 4, C = -2$

**8.** Solution for system $\mathbf{AX} = \mathbf{B}_1$: $(6, -51, 21)$;
Solution for system $\mathbf{AX} = \mathbf{B}_2$: $(\frac{35}{3}, -98, \frac{79}{2})$

**12. (b)** Any nonhomogeneous system with two equations and two unknowns that has a unique solution will serve as a counterexample. For instance, consider

$$\begin{cases} x + y = 1 \\ x - y = 1 \end{cases}.$$

This system has a unique solution: $(1, 0)$. Let $(s_1, s_2)$ and $(t_1, t_2)$ both equal $(1, 0)$. Then the sum of solutions is not a solution in this case. Also, if $c \neq 1$, the scalar multiple of a solution by $c$ is not a solution in this case.

**17. (a)** T **(b)** T **(c)** F **(d)** T **(e)** F **(f)** F

## Section 2.3 (p. 86–89)

**1. (a)** A row operation of type (I) converts **A** to **B**: $\langle 2\rangle \leftarrow -5\,\langle 2\rangle$.
**(c)** A row operation of type (II) converts **A** to **B**: $\langle 2\rangle \leftarrow \langle 3\rangle + \langle 2\rangle$.

**2. (b)** The sequence of row operations converting **B** to **A** is:

(II): $\langle 1\rangle \leftarrow -5\,\langle 3\rangle + \langle 1\rangle$
(III): $\langle 2\rangle \leftrightarrow \langle 3\rangle$
(II): $\langle 3\rangle \leftarrow 3\,\langle 1\rangle + \langle 3\rangle$
(II): $\langle 2\rangle \leftarrow -2\,\langle 1\rangle + \langle 2\rangle$
(I): $\langle 1\rangle \leftarrow 4\,\langle 1\rangle$

**3. (b)** The common reduced row echelon form is $\mathbf{I}_3$.

**(c)** The sequence of row operations is:

(II): $\langle 3\rangle \leftarrow 2\,\langle 2\rangle + \langle 3\rangle$
(I): $\langle 3\rangle \leftarrow -1\,\langle 3\rangle$
(II): $\langle 1\rangle \leftarrow -9\,\langle 3\rangle + \langle 1\rangle$
(II): $\langle 2\rangle \leftarrow 3\,\langle 3\rangle + \langle 2\rangle$
(II): $\langle 3\rangle \leftarrow -\frac{9}{5}\,\langle 2\rangle + \langle 3\rangle$
(II): $\langle 1\rangle \leftarrow -\frac{3}{5}\,\langle 2\rangle + \langle 1\rangle$
(I): $\langle 2\rangle \leftarrow -\frac{1}{5}\,\langle 2\rangle$
(II): $\langle 3\rangle \leftarrow -3\,\langle 1\rangle + \langle 3\rangle$
(II): $\langle 2\rangle \leftarrow -2\,\langle 1\rangle + \langle 2\rangle$
(I): $\langle 1\rangle \leftarrow -5\,\langle 1\rangle$

**5. (a)** 2 **(e)** 3

**(c)** 2

**6. (a)** Rank = 3. Thus, Theorem 2.7 predicts the system has only the trivial solution. In fact, the complete solution set = $\{(0, 0, 0)\}$.

**7.** In the following answers, the asterisk represents any real entry:

**(a)** Smallest rank = 1: $\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$, where the $*$ in the last column must be nonzero, so that the system is nonhomogeneous;

largest rank = 4: $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

**(c)** Smallest rank = 2: $\left[\begin{array}{cccc|c} 1 & * & * & * & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$; largest rank = 3: $\left[\begin{array}{cccc|c} 1 & 0 & * & * & 0 \\ 0 & 1 & * & * & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$

**8. (a)** $\mathbf{x} = -\frac{21}{11}\mathbf{a}_1 + \frac{6}{11}\mathbf{a}_2$

**(c)** Not possible

**(e)** The answer is not unique; one possible answer is $\mathbf{x} = -3\mathbf{a}_1 + 2\mathbf{a}_2 + 0\mathbf{a}_3$.

**(g)** $\mathbf{x} = 2\mathbf{a}_1 - \mathbf{a}_2 - \mathbf{a}_3$

**9. (a)** Yes: 5(row 1) − 3(row 2) − 1(row 3)

**(c)** Not in row space

**(e)** Yes, but the linear combination of the rows is not unique; one possible expression for the given vector is −3(row 1) + 1(row 2) + 0(row 3).

**10. (a)** $[13, -23, 60] = -2\mathbf{q}_1 + \mathbf{q}_2 + 3\mathbf{q}_3$ **(c)** $[13, -23, 60] = -\mathbf{r}_1 - 14\mathbf{r}_2 + 11\mathbf{r}_3$

**(b)** $\mathbf{q}_1 = 3\mathbf{r}_1 - \mathbf{r}_2 - 2\mathbf{r}_3$
$\mathbf{q}_2 = 2\mathbf{r}_1 + 2\mathbf{r}_2 - 5\mathbf{r}_3$
$\mathbf{q}_3 = \mathbf{r}_1 - 6\mathbf{r}_2 + 4\mathbf{r}_3$

**11. (a)** **(i)** $\mathbf{B} = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$;

**(ii)** $[1, 0, -1, 2] = -\frac{7}{8}[0, 4, 12, 8] + \frac{1}{2}[2, 7, 19, 18] + 0[1, 2, 5, 6]$;
$[0, 1, 3, 2] = \frac{1}{4}[0, 4, 12, 8] + 0[2, 7, 19, 18] + 0[1, 2, 5, 6]$
(other solutions are possible for $[1, 0, -1, 2]$ and $[0, 1, 3, 2]$);

**(iii)** $[0, 4, 12, 8] = 0[1, 0, -1, 2] + 4[0, 1, 3, 2]$;
$[2, 7, 19, 18] = 2[1, 0, -1, 2] + 7[0, 1, 3, 2]$;
$[1, 2, 5, 6] = 1[1, 0, -1, 2] + 2[0, 1, 3, 2]$.

**14.** The zero vector is a solution to $\mathbf{AX} = \mathbf{0}$, but it is not a solution for $\mathbf{AX} = \mathbf{B}$.

**16.** Consider the systems

$$\begin{cases} x + y = 1 \\ x + y = 0 \end{cases} \quad \text{and} \quad \begin{cases} x - y = 1 \\ x - y = 2 \end{cases}.$$

The reduced row echelon matrices for these inconsistent systems are, respectively,

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 1 \end{array}\right].$$

Thus, the original augmented matrices are not row equivalent, since their reduced row echelon forms are different.

**17.** Because $\mathbf{A}$ and $\mathbf{B}$ are row equivalent, $\mathbf{A} = R_n(\cdots(R_2(R_1(\mathbf{B})))\cdots)$ for some row operations $R_1, \ldots, R_n$. Now, if $\mathbf{D}$ is the unique reduced row echelon form matrix to which $\mathbf{A}$ is row equivalent, then for some additional row operations $R_{n+1}, \ldots, R_{n+k}$,

$$\begin{aligned} \mathbf{D} &= R_{n+k}(\cdots(R_{n+2}(R_{n+1}(\mathbf{A})))\cdots) \\ &= R_{n+k}(\cdots(R_{n+2}(R_{n+1}(R_n(\cdots(R_2(R_1(\mathbf{B})))\cdots))))\cdots), \end{aligned}$$

showing that $\mathbf{B}$ also has $\mathbf{D}$ as its reduced echelon form matrix. Therefore, by the definition of rank, rank($\mathbf{B}$) = the number of nonzero rows in $\mathbf{D}$ = rank($\mathbf{A}$).

**22.** **(a)** T **(b)** T **(c)** F **(d)** F **(e)** F **(f)** T

## Section 2.4 (p. 96–98)

**2.** **(a)** Rank = 2; nonsingular
**(c)** Rank = 3; nonsingular
**(e)** Rank = 3; singular

**3.** **(a)** $\begin{bmatrix} \frac{1}{10} & \frac{1}{15} \\ \frac{3}{10} & -\frac{2}{15} \end{bmatrix}$

**(c)** $\begin{bmatrix} -\frac{2}{21} & -\frac{5}{84} \\ \frac{1}{7} & -\frac{1}{28} \end{bmatrix}$

**(e)** No inverse exists.

**4.** **(a)** $\begin{bmatrix} 1 & 3 & 2 \\ -1 & 0 & 2 \\ 2 & 2 & -1 \end{bmatrix}$

**(c)** $\begin{bmatrix} \frac{3}{2} & 0 & \frac{1}{2} \\ -3 & \frac{1}{2} & -\frac{1}{2} \\ -\frac{8}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$

**(e)** No inverse exists.

**5.** **(c)** $\begin{bmatrix} \frac{1}{a_{11}} & 0 & \cdots & 0 \\ 0 & \frac{1}{a_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{a_{nn}} \end{bmatrix}$

**6.** **(a)** The general inverse is $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.

When $\theta = \frac{\pi}{6}$, matrix = $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$; inverse = $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$.

When $\theta = \frac{\pi}{4}$, matrix $= \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$; inverse $= \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$.

When $\theta = \frac{\pi}{2}$, matrix $= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$; inverse $= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

**(b)** The general inverse is $\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

When $\theta = \frac{\pi}{6}$, matrix $= \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$; inverse $= \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

When $\theta = \frac{\pi}{4}$, matrix $= \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$; inverse $= \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

When $\theta = \frac{\pi}{2}$, matrix $= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; inverse $= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

**7. (a)** Inverse $= \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{7}{3} & \frac{5}{3} \end{bmatrix}$; solution set $= \{(3, -5)\}$

**(c)** Inverse $= \begin{bmatrix} 1 & -13 & -15 & 5 \\ -3 & 3 & 0 & -7 \\ -1 & 2 & 1 & -3 \\ 0 & -4 & -5 & 1 \end{bmatrix}$; solution set $= \{(5, -8, 2, -1)\}$

**8. (a)** Consider $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

**(b)** Consider $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

**(c)** $\mathbf{A} = \mathbf{A}^{-1}$ if $\mathbf{A}$ is involutory.

**10. (a)** $\mathbf{B}$ must be the zero matrix.

**(b)** No, since $\mathbf{A}^{-1} = \mathbf{B}$ exists, $\mathbf{AC} = \mathbf{O}_n \Longrightarrow \mathbf{A}^{-1}\mathbf{AC} = \mathbf{A}^{-1}\mathbf{O}_n \Longrightarrow \mathbf{C} = \mathbf{O}_n$.

**11.** $\ldots, \mathbf{A}^{-9}, \mathbf{A}^{-5}, \mathbf{A}^{-1}, \mathbf{A}^{3}, \mathbf{A}^{7}, \mathbf{A}^{11}, \ldots$

**12.** $\mathbf{B}^{-1}\mathbf{A}$ is the inverse of $\mathbf{A}^{-1}\mathbf{B}$.

**14. (a)** If the matrix contains a column of zeroes, no row operations can alter such a column, so its unique reduced row echelon form must also contain a column of zeroes, and thus cannot equal $\mathbf{I}_n$. Hence the matrix has rank $< n$, and must be singular by Theorem 2.15.

Suppose instead the matrix contains a row of all zeroes. If it were nonsingular, its transpose would also be nonsingular by part (4) of Theorem 2.12. But the transpose contains a column of all zeroes, contradicting the first paragraph. Thus the matrix must be singular.

**21.** Suppose $\mathbf{AB} = \mathbf{I}_n$ and $n > k$. Corollary 2.3 shows that there is a nontrivial vector $\mathbf{X}$ such that $\mathbf{BX} = \mathbf{0}$. But then $\mathbf{X} = \mathbf{I}_n\mathbf{X} = (\mathbf{AB})\mathbf{X} = \mathbf{A}(\mathbf{BX}) = \mathbf{A0} = \mathbf{0}$, a contradiction.

**22. (a)** F **(b)** T **(c)** T **(d)** F **(e)** F **(f)** T

## Chapter 2 Review Exercises (p. 98–100)

**1.** **(a)** $x_1 = -6, x_2 = 8, x_3 = -5$
**(b)** No solutions
**(c)** $\{[-5 - c + e, 1 - 2c - e, c, 1 + 2e, e] \mid c, e \in \mathbb{R}\}$

**2.** $y = -2x^3 + 5x^2 - 6x + 3$

**4.** $a = 4, b = 7, c = 4, d = 6$

**8.** **(a)** $\text{rank}(\mathbf{A}) = 2$, $\text{rank}(\mathbf{B}) = 4$, $\text{rank}(\mathbf{C}) = 3$
**(b)** $\mathbf{AX} = \mathbf{0}$ and $\mathbf{CX} = \mathbf{0}$: infinite number of solutions; $\mathbf{BX} = \mathbf{0}$: one solution

**10.** **(a)** Yes. $[-34, 29, -21] = 5[2, 3, -1] + 2[5, -2, -1] - 6[9, -8, 3]$
**(b)** Yes. $[-34, 29, -21]$ is a linear combination of the rows of the matrix.

**12.** **(b)** Singular

**15.** The inverse of the coefficient matrix is $\begin{bmatrix} 3 & -1 & -4 \\ 2 & -1 & -3 \\ 1 & -2 & -2 \end{bmatrix}$; solution set is $x_1 = -27, x_2 = -21, x_3 = -1$.

**17.** **(a)** F **(b)** F **(c)** F **(d)** T **(e)** F **(f)** T **(g)** T **(h)** T **(i)** F **(j)** T **(k)** F **(l)** F **(m)** T **(n)** T **(o)** F **(p)** T **(q)** T **(r)** T **(s)** T

## Section 3.1 (p. 106–109)

**1.** **(a)** $-17$
**(c)** 0
**(e)** $-108$
**(g)** $-40$
**(i)** 0
**(j)** $-3$

**2.** **(a)** $\begin{vmatrix} 4 & 3 \\ -2 & 4 \end{vmatrix} = 22$
**(c)** $\begin{vmatrix} -3 & 0 & 5 \\ 2 & -1 & 4 \\ 6 & 4 & 0 \end{vmatrix} = 118$

**3.** **(a)** $(-1)^{2+2}\begin{vmatrix} 4 & -3 \\ 9 & -7 \end{vmatrix} = -1$
**(c)** $(-1)^{4+3}\begin{vmatrix} -5 & 2 & 13 \\ -8 & 2 & 22 \\ -6 & -3 & -16 \end{vmatrix} = 222$
**(d)** $(-1)^{1+2}\begin{vmatrix} x-4 & x-3 \\ x-1 & x+2 \end{vmatrix} = -2x + 11$

**4.** See answers to Exercise 1.

**5.** **(a)** 0
**(d)** 352

**7.** Let $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, and let $\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

**9.** **(a)** 7
**(c)** 12

**11.** **(a)** 18
**(c)** 63

**15.** **(a)** $x = -5$ or $x = 2$
**(c)** $x = 3, x = 1$, or $x = 2$

**16.** **(b)** 20

**18.** **(a)** F **(b)** T **(c)** F **(d)** F **(e)** T

## Section 3.2 (p. 113–116)

**1.** **(a)** (II): $\langle 1\rangle \leftarrow -3\,\langle 2\rangle + \langle 1\rangle$; determinant $= 1$
**(c)** (I): $\langle 3\rangle \leftarrow -4\,\langle 3\rangle$; determinant $= -4$
**(f)** (III): $\langle 1\rangle \leftrightarrow \langle 2\rangle$; determinant $= -1$

**2.** **(a)** 30
**(c)** $-4$
**(e)** 35

**3.** **(a)** Determinant $= -2$; matrix is nonsingular because determinant is nonzero
**(c)** Determinant $= -79$; matrix is nonsingular
**4.** **(a)** Determinant $= -1$; system has only the trivial solution
**6.** $-a_{16}a_{25}a_{34}a_{43}a_{52}a_{61}$
**16.** By Corollary 3.6 and part (1) of Theorem 2.7, the homogeneous system $\mathbf{AX} = \mathbf{0}$ has nontrivial solutions. Let $\mathbf{B}$ be any $n \times n$ matrix such that every column of $\mathbf{B}$ is a nontrivial solution for $\mathbf{AX} = \mathbf{0}$.
Then the $i$th column of $\mathbf{AB} = \mathbf{A}(i\text{th column of } \mathbf{B}) = \mathbf{0}$ for every $i$. Hence, $\mathbf{AB} = \mathbf{O}_n$.
**17.** **(a)** F **(b)** T **(c)** F **(d)** F **(e)** F **(f)** T

## Section 3.3 (p. 121–123)

**1.** **(a)** $a_{31}|\mathbf{A}_{31}| - a_{32}|\mathbf{A}_{32}| + a_{33}|\mathbf{A}_{33}| - a_{34}|\mathbf{A}_{34}|$ **(c)** $-a_{14}|\mathbf{A}_{14}| + a_{24}|\mathbf{A}_{24}| - a_{34}|\mathbf{A}_{34}| + a_{44}|\mathbf{A}_{44}|$

**2.** **(a)** $-76$ **(c)** $102$

**3.** **(a)** $\{(-4, 3, -7)\}$ **(d)** $\{(4, -1, -3, 6)\}$

**8.** **(b)** Consider $\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

**9.** **(b)** Consider $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

**13.** **(b)** For example, consider $\mathbf{B} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} \mathbf{A} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -6 & -4 \\ 16 & 11 \end{bmatrix}$, and $\mathbf{B} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}^{-1} \mathbf{A} \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 10 & -12 \\ 4 & -5 \end{bmatrix}$.

**20.** **(a)** T **(b)** T **(c)** F **(d)** T **(e)** T **(f)** T

## Section 3.4 (p. 132–136)

**1.** **(a)** $x^2 - 7x + 14$ **(e)** $x^4 - 3x^3 - 4x^2 + 12x$
**(c)** $x^3 - 8x^2 + 21x - 18$

**2.** **(a)** $E_2 = \{b[1, 1] \mid b \in \mathbb{R}\}$ **(c)** $E_{-1} = \{b[1, 2, 0] + c[0, 0, 1] \mid b, c \in \mathbb{R}\}$

**3.** **(a)** $\lambda = 1$; $E_1 = \{a[1, 0] \mid a \in \mathbb{R}\}$; algebraic multiplicity of $\lambda$ is 2
**(c)** $\lambda_1 = 1$; $E_1 = \{a[1, 0, 0] \mid a \in \mathbb{R}\}$; algebraic multiplicity of $\lambda_1$ is 1; $\lambda_2 = 2$; $E_2 = \{b[0, 1, 0] \mid b \in \mathbb{R}\}$; algebraic multiplicity of $\lambda_2$ is 1; $\lambda_3 = -5$; $E_{-5} = \{c[-\frac{1}{6}, \frac{3}{7}, 1] \mid c \in \mathbb{R}\} = \{c[-7, 18, 42] \mid c \in \mathbb{R}\}$; algebraic multiplicity of $\lambda_3$ is 1
**(e)** $\lambda_1 = 0$; $E_0 = \{c[1, 3, 2] \mid c \in \mathbb{R}\}$; algebraic multiplicity of $\lambda_1$ is 1; $\lambda_2 = 2$; $E_2 = \{b[1, 0, 1] + c[0, 1, 0] \mid b, c \in \mathbb{R}\}$; algebraic multiplicity of $\lambda_2$ is 2
**(h)** $\lambda_1 = 0$; $E_0 = \{c[-1, 1, 1, 0] + d[0, -1, 0, 1] \mid c, d \in \mathbb{R}\}$; algebraic multiplicity of $\lambda_1$ is 2; $\lambda_2 = -3$; $E_{-3} = \{d[-1, 0, 2, 2] \mid d \in \mathbb{R}\}$; algebraic multiplicity of $\lambda_2$ is 2

**4.** **(a)** $\mathbf{P} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$; $\mathbf{D} = \begin{bmatrix} 3 & 0 \\ 0 & -5 \end{bmatrix}$

**(c)** Not diagonalizable

**(d)** $\mathbf{P} = \begin{bmatrix} 6 & 1 & 1 \\ 2 & 2 & 1 \\ 5 & 1 & 1 \end{bmatrix}$; $\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

**(f)** Not diagonalizable

**(g)** $\mathbf{P} = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 0 & -1 \\ 0 & 3 & 1 \end{bmatrix}$; $\mathbf{D} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

**(i)** $\mathbf{P} = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 2 & 0 & 2 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$; $\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

**5.** **(a)** $\begin{bmatrix} 32770 & -65538 \\ 32769 & -65537 \end{bmatrix}$

**(c)** $\mathbf{A}^{49} = \mathbf{A}$

**(e)** $\begin{bmatrix} 4188163 & 6282243 & -9421830 \\ 4192254 & 6288382 & -9432060 \\ 4190208 & 6285312 & -9426944 \end{bmatrix}$

**7.** **(b)** **A** has a square root if **A** has all eigenvalues nonnegative.

**8.** One possible answer: $\begin{bmatrix} 3 & -2 & -2 \\ -7 & 10 & 11 \\ 8 & -10 & -11 \end{bmatrix}$

**10.** **(b)** Consider the matrix $\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, which represents a rotation about the origin in $\mathbb{R}^2$ through an angle of $\frac{\pi}{2}$ radians, or 90°. Although **A** has no eigenvalues, $\mathbf{A}^4 = \mathbf{I}_2$ has 1 as an eigenvalue.

**26.** **(a)** T **(b)** F **(c)** T **(d)** T **(e)** F **(f)** T **(g)** T **(h)** F

## Chapter 3 Review Exercises (p. 136–138)

**1.** **(b)** $\mathcal{A}_{34} = -|\mathbf{A}_{34}| = 30$

**(d)** $|\mathbf{A}| = -830$

**3.** $|\mathbf{A}| = -42$

**5.** **(a)** $|\mathbf{B}| = 60$

**(b)** $|\mathbf{B}| = -15$

**(c)** $|\mathbf{B}| = 15$

**7.** 378

**8.** **(a)** $|\mathbf{A}| = 0$

**(b)** No. A nontrivial solution is $\begin{bmatrix} 1 \\ -8 \end{bmatrix}$.

**10.** $x_1 = -4,\ x_2 = -3,\ x_3 = 5$

**11.** **(a)** The determinant of the given matrix is $-289$. Thus, we would need $|\mathbf{A}|^4 = -289$. But no real number raised to the fourth power is negative.

**(b)** The determinant of the given matrix is zero, making the given matrix singular. Hence it cannot be the inverse of any matrix.

**12.** **B** similar to **A** implies there is a matrix **P** such that $\mathbf{B} = \mathbf{P}^{-1}\mathbf{AP}$.

**(b)** $|\mathbf{B}^T| = |\mathbf{B}| = |\mathbf{P}^{-1}\mathbf{AP}| = |\mathbf{P}^{-1}||\mathbf{A}||\mathbf{P}| = \frac{1}{|\mathbf{P}|}|\mathbf{A}||\mathbf{P}| = |\mathbf{A}| = |\mathbf{A}^T|$.

**(e)** $\mathbf{B} + \mathbf{I}_n = \mathbf{P}^{-1}\mathbf{AP} + \mathbf{I}_n = \mathbf{P}^{-1}\mathbf{AP} + \mathbf{P}^{-1}\mathbf{I}_n\mathbf{P} = \mathbf{P}^{-1}(\mathbf{A} + \mathbf{I}_n)\mathbf{P}$.

**14.** **(b)** $p_{\mathbf{A}}(x) = x^3 + x^2 - 21x - 45 = (x+3)^2(x-5)$; eigenvalues: $\lambda_1 = 5, \lambda_2 = -3$; eigenspaces: $E_5 = \{a[-1, 4, 4] \mid a \in \mathbb{R}\}$, $E_{-3} = \{a[-2, 1, 0] + b[2, 0, 1] \mid a, b \in \mathbb{R}\}$; $\mathbf{P} = \begin{bmatrix} -1 & -2 & 2 \\ 4 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$; $\mathbf{D} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$

**15.** **(b)** $p_{\mathbf{A}}(x) = x^4 + 6x^3 + 9x^2 = x^2(x+3)^2$. Even though the eigenvalue $\lambda = -3$ has algebraic multiplicity 2, only 1 fundamental eigenvector is produced for $\lambda$ because $(-3\mathbf{I}_4 - \mathbf{A})$ has rank 3. In fact, we get only 3 fundamental eigenvectors overall, which is insufficient by Step 4 of the Diagonalization Method.

**16.** $\mathbf{A}^{13} = \begin{bmatrix} -9565941 & 9565942 & 4782976 \\ -12754588 & 12754589 & 6377300 \\ 3188648 & -3188648 & -1594325 \end{bmatrix}$

**17.** **(a)** $\lambda_1 = 2, \lambda_2 = -1, \lambda_3 = 3$

**(b)** $E_2 = \{a[1, -2, 1, 1] \mid a \in \mathbb{R}\}$, $E_{-1} = \{a[1, 0, 0, 1] + b[3, 7, -3, 2] \mid a, b \in \mathbb{R}\}$, $E_3 = \{a[2, 8, -4, 3] \mid a \in \mathbb{R}\}$

**(c)** $|\mathbf{A}| = 6$

**18.** **(a)** F **(d)** F **(g)** T **(j)** T **(m)** T **(p)** F **(s)** T **(v)** T **(y)** F
**(b)** F **(e)** T **(h)** T **(k)** F **(n)** F **(q)** F **(t)** F **(w)** T **(z)** F
**(c)** F **(f)** T **(i)** F **(l)** F **(o)** F **(r)** T **(u)** F **(x)** T

## Section 4.1 (p. 147–148)

**5.** The set of singular 2 × 2 matrices is not closed under addition. For example, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are both singular, but their sum $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}_2$ is nonsingular.

**8.** Properties (2), (3), and (6) are not satisfied, and property (4) makes no sense without property (3). The following is a counterexample for property (2): $3 \oplus (4 \oplus 5) = 3 \oplus 18 = 42$, but $(3 \oplus 4) \oplus 5 = 14 \oplus 5 = 38$.

**20.** **(a)** F **(b)** F **(c)** T **(d)** T **(e)** F **(f)** T **(g)** T

## Section 4.2 (p. 153–155)

**1.** **(a)** Not a subspace; no zero vector
**(c)** Subspace
**(e)** Not a subspace; no zero vector
**(g)** Not a subspace; not closed under addition
**(j)** Not a subspace; not closed under addition
**(l)** Not a subspace; not closed under scalar multiplication

**2.** Only starred parts are listed:
Subspaces: (a), (c), (e), (g)
Part (h) is not a subspace because it is not closed under addition.

**3.** Only starred parts are listed:
Subspaces: (a), (b), (g)
Part (e) is not a subspace because it does not contain the zero polynomial.

**12.** **(e)** No; if $|\mathbf{A}| \neq 0$ and $c = 0$, then $|c\mathbf{A}| = 0$.

**15.** $S = \{\mathbf{0}\}$, the trivial subspace of $\mathbb{R}^n$.

**22.** **(a)** F **(b)** T **(c)** F **(d)** T **(e)** T **(f)** F **(g)** T **(h)** T

## Section 4.3 (p. 161–163)

**1.** **(a)** $\{[a, b, -a+b] \mid a, b \in \mathbb{R}\}$
**(c)** $\{[a, b, -b] \mid a, b \in \mathbb{R}\}$
**(e)** $\{[a, b, c, -2a+b+c] \mid a, b, c \in \mathbb{R}\}$

**2.** **(a)** $\{ax^3 + bx^2 + cx - (a+b+c) \mid a, b, c \in \mathbb{R}\}$ **(c)** $\{ax^3 - ax + b \mid a, b \in \mathbb{R}\}$

**3.** **(a)** $\left\{ \begin{bmatrix} a & b \\ c & -a-b-c \end{bmatrix} \,\middle|\, a, b, c \in \mathbb{R} \right\}$ **(c)** $\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \,\middle|\, a, b, c, d \in \mathbb{R} \right\} = \mathcal{M}_{22}$

**4.** **(a)** $[a+b, a+c, b+c, c] = a[1,1,0,0] + b[1,0,1,0] + c[0,1,1,1]$. The set of vectors of this form is the row space of $\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$.

**(b)** $\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} \end{bmatrix}$.

**(c)** Row space of $\mathbf{B} = \{a[1, 0, 0, -\frac{1}{2}] + b[0, 1, 0, \frac{1}{2}] + c[0, 0, 1, \frac{1}{2}] \mid a, b, c \in \mathbb{R}\}$
$= \{[a, b, c, -\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}c] \mid a, b, c \in \mathbb{R}\}$

**11.** One answer is: $-1(x^3 - 2x^2 + x - 3) + 2(2x^3 - 3x^2 + 2x + 5) - 1(4x^2 + x - 3) + 0(4x^3 - 7x^2 + 4x - 1)$.

**15.** **(a)** $S = \{[-3, 2, 0], [4, 0, 5]\}$

**20.** See Theorem 1.15.

**25.** **(b)** $S_1 = \{[1,0,0], [0,1,0]\}$, $S_2 = \{[0,1,0], [0,0,1]\}$ **(c)** $S_1 = \{[1,0,0], [0,1,0]\}$, $S_2 = \{[1,0,0], [1,1,0]\}$

**26.** **(c)** $S_1 = \{x^5\}$, $S_2 = \{x^4\}$

**30.** **(a)** F **(b)** T **(c)** F **(d)** F **(e)** F **(f)** T

## Section 4.4 (p. 172–174)

**1.** Linearly independent: (a), (b)
Linearly dependent: (c), (d), (e)

**2.** Answers given for starred parts only:
Linearly independent: (b)
Linearly dependent: (a), (e)

**3.** Answers given for starred parts only:
Linearly independent: (a)
Linearly dependent: (c)

**4.** Answers given for starred parts only:
Linearly independent: (a), (e)
Linearly dependent: (c)

**7. (b)** Two possibilities: [0, 1, 0], [0, 0, 1]

**(c)** Any linear combination of [1, 1, 0] and [−2, 0, 1] works, other than [1, 1, 0] and [−2, 0, 1] themselves. One possibility is $\mathbf{u} = [1, 1, 0] + [-2, 0, 1] = [-1, 1, 1]$.

**11. (a)** One answer is $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$.

**(c)** One answer is $\{1, x, x^2, x^3\}$.

**(e)** One answer is $\left\{\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}\right\}$.

(Notice that each matrix is symmetric.)

**13. (b)** Let $S$ be the given set of three vectors. To prove that $\mathbf{v} = [0, 0, -6, 0]$ is redundant, we need to show that $\text{span}(S - \{\mathbf{v}\}) = \text{span}(S)$. We apply the Simplified Span Method to both $S - \{\mathbf{v}\}$ and $S$.

For $\text{span}(S - \{\mathbf{v}\})$:

$$\begin{bmatrix}1&1&0&0\\1&1&1&0\end{bmatrix} \text{ row reduces to } \begin{bmatrix}1&1&0&0\\0&0&1&0\end{bmatrix}.$$

For $\text{span}(S)$:

$$\begin{bmatrix}1&1&0&0\\1&1&1&0\\0&0&-6&0\end{bmatrix} \text{ row reduces to } \begin{bmatrix}1&1&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}.$$

Since the reduced row echelon form matrices are the same, except for the extra row of zeroes, the two spans are equal, and $\mathbf{v} = [0, 0, -6, 0]$ is a redundant vector. (More simply, $[0, 0, -6, 0]$ is redundant since $[0, 0, -6, 0] = 6[1, 1, 0, 0] - 6[1, 1, 1, 0]$.)

**17. (b)** Let $\mathbf{A} = \mathbf{O}_{nm}$ and $S$ be any nontrivial linearly independent subset of $\mathbb{R}^m$.

**21. (b)** $S$ is linearly dependent if every vector $\mathbf{v}$ in $\text{span}(S)$ can be expressed in more than one way as a linear combination of vectors in $S$ (ignoring zero coefficients).

**25. (a)** F **(b)** T **(c)** T **(d)** F **(e)** T **(f)** T **(g)** F **(h)** T **(i)** T

## Section 4.5 (p. 180–182)

**4. (a)** Not a basis (linearly independent but does not span)
**(c)** Basis
**(e)** Not a basis (linearly dependent but spans)

**5. (a)** $\mathcal{W}$ is nonempty, since $\mathbf{0} \in \mathcal{W}$. Let $\mathbf{X}_1, \mathbf{X}_2 \in \mathcal{W}$, $c \in \mathbb{R}$. Then $\mathbf{A}(\mathbf{X}_1 + \mathbf{X}_2) = \mathbf{A}\mathbf{X}_1 + \mathbf{A}\mathbf{X}_2 = \mathbf{0} + \mathbf{0} = \mathbf{0}$. Also, $\mathbf{A}(c\mathbf{X}_1) = c(\mathbf{A}\mathbf{X}_1) = c\mathbf{0} = \mathbf{0}$. Hence $\mathcal{W}$ is closed under addition and scalar multiplication, and so is a subspace by Theorem 4.2.

**(b)** $\{[-1, -2, 5, 0, 0], [-2, 1, 0, 5, 0], [-1, 1, 0, 0, 1]\}$

**(c)** $\dim(\mathcal{W}) = 3$; $\text{rank}(\mathbf{A}) = 2$; $3 + 2 = 5$

**9. (b)** 5

**(c)** $\{(x-2)(x-3), x(x-2)(x-3), x^2(x-2)(x-3), x^3(x-2)(x-3)\}$

**(d)** 4

**10. (b)** When the Simplified Span Method is applied to both $S$ and $B$, we obtain the same simplified spanning set $\left\{\left[1, 0, -\frac{3}{5}, \frac{2}{5}\right], \left[0, 1, \frac{2}{5}, -\frac{3}{5}\right]\right\}$. Hence, span($S$) and span($B$) are actually the same subspace of $\mathbb{R}^4$. Therefore, $B$ spans span($S$), and it is linearly independent (by part (a)). So, $B$ is a basis for span($S$). Because the basis $B$ for span($S$) contains two vectors, $\dim(\text{span}(S)) = 2$.

**(c)** No; $\dim(\text{span}(S)) = 2 \neq 4 = \dim(\mathbb{R}^4)$

**12. (a)** Let $\mathcal{V} = \mathbb{R}^3$, and let $S = \{[1, 0, 0], [2, 0, 0], [3, 0, 0]\}$. **(b)** Let $\mathcal{V} = \mathbb{R}^3$, and let $T = \{[1, 0, 0], [2, 0, 0], [3, 0, 0]\}$.

**24. (a)** T **(b)** F **(c)** F **(d)** F **(e)** F **(f)** T **(g)** F **(h)** F **(i)** T

## Section 4.6 (p. 186–188)

**1. (a)** $\{[1, 0, 0, 2, -2], [0, 1, 0, 0, 1], [0, 0, 1, -1, 0]\}$ **(d)** $\left\{[1, 0, 0, -2, -\frac{13}{4}], [0, 1, 0, 3, \frac{9}{2}], [0, 0, 1, 0, -\frac{1}{4}]\right\}$

**2.** $\{x^3 - 3x, x^2 - x, 1\}$

**3.** $\left\{\begin{bmatrix} 1 & 0 \\ \frac{4}{3} & \frac{1}{3} \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$

**4. (a)** One answer is $\{[1, 3, -2], [2, 1, 4], [0, 1, -1]\}$.

**(c)** One answer is $\{[3, -2, 2], [1, 2, -1], [3, -2, 7]\}$.

**5. (a)** One answer is $\{x^3 - 8x^2 + 1, 3x^3 - 2x^2 + x, 4x^3 + 2x - 10, x^3 - 20x^2 - x + 12\}$.

**6. (a)** $\{[3, 1, -2], [6, 2, -3]\}$ **(d)** One answer is $\{[1, -3, 0], [0, 1, 1]\}$.

**7. (a)** One answer is $\{x^3, x^2, x\}$. **(c)** One answer is $\{x^3 + x^2, x, 1\}$.

**8. (a)** One answer is $\{\boldsymbol{\Psi}_{ij} \mid 1 \le i, j \le 3\}$, where $\boldsymbol{\Psi}_{ij}$ is the $3 \times 3$ matrix with $(i, j)$ entry $= 1$ and all other entries 0.

**(c)** One answer is $\left\{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\}$.

**10. (a)** One answer is $\{[1, -3, 0, 1, 4], [2, 2, 1, -3, 1], [1, 0, 0, 0, 0], [0, 1, 0, 0, 0], [0, 0, 1, 0, 0]\}$.

**(c)** One answer is $\{[1, 0, -1, 0, 0], [0, 1, -1, 1, 0], [2, 3, -8, -1, 0], [1, 0, 0, 0, 0], [0, 0, 0, 0, 1]\}$.

**11. (a)** One answer is $\{x^3 - x^2, x^4 - 3x^3 + 5x^2 - x, x^4, x^3, 1\}$.

**(c)** One answer is $\{x^4 - x^3 + x^2 - x + 1, x^3 - x^2 + x - 1, x^2 - x + 1, x^2, x\}$.

**12. (a)** One answer is $\left\{\begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}\right\}$.

**(c)** One answer is $\left\{\begin{bmatrix} 3 & -1 \\ 2 & -6 \\ 5 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ -4 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 6 & 2 \\ -2 & -9 \\ 10 & 2 \end{bmatrix}, \begin{bmatrix} 3 & -4 \\ 8 & -9 \\ 5 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}\right\}$.

**13. (b)** 8 **(d)** 3

**14. (b)** $(n^2 - n)/2$

**17. (b)** No; consider the subspace $\mathcal{W}$ of $\mathbb{R}^3$ given by $\mathcal{W} = \{[a, 0, 0] \mid a \in \mathbb{R}\}$. No subset of $B = \{[1, 1, 0], [1, -1, 0], [0, 0, 1]\}$ (a basis for $\mathbb{R}^3$) is a basis for $\mathcal{W}$.

**(c)** Yes; consider $\mathcal{Y} = \text{span}(B')$.

**18. (b)** In $\mathbb{R}^3$, consider $\mathcal{W} = \{[a, b, 0] \mid a, b \in \mathbb{R}\}$. We could let $\mathcal{W}' = \{[0, 0, c] \mid c \in \mathbb{R}\}$ or $\mathcal{W}' = \{[0, c, c] \mid c \in \mathbb{R}\}$.

**22. (a)** T **(b)** T **(c)** F **(d)** T **(e)** F **(f)** F **(g)** F

## Section 4.7 (p. 198–201)

**1. (a)** $[\mathbf{v}]_B = [7, -1, -5]$ **(g)** $[\mathbf{v}]_B = [-1, 4, -2]$

**(c)** $[\mathbf{v}]_B = [-2, 4, -5]$ **(h)** $[\mathbf{v}]_B = [2, -3, 1]$

**(e)** $[\mathbf{v}]_B = [4, -5, 3]$ **(j)** $[\mathbf{v}]_B = [5, -2]$

**2.** **(a)** $\begin{bmatrix} -102 & 20 & 3 \\ 67 & -13 & -2 \\ 36 & -7 & -1 \end{bmatrix}$

**(c)** $\begin{bmatrix} 20 & -30 & -69 \\ 24 & -24 & -80 \\ -9 & 11 & 31 \end{bmatrix}$

**(d)** $\begin{bmatrix} -1 & -4 & 2 & -9 \\ 4 & 5 & 1 & 3 \\ 0 & 2 & -3 & 1 \\ -4 & -13 & 13 & -15 \end{bmatrix}$

**(f)** $\begin{bmatrix} 6 & 1 & 2 \\ 1 & 1 & 2 \\ -1 & -1 & -3 \end{bmatrix}$

**4.** **(a)** $\mathbf{P} = \begin{bmatrix} 13 & 31 \\ -18 & -43 \end{bmatrix}$; $\mathbf{Q} = \begin{bmatrix} -11 & -8 \\ 29 & 21 \end{bmatrix}$; $\mathbf{T} = \begin{bmatrix} 1 & 3 \\ -1 & -4 \end{bmatrix}$

**(c)** $\mathbf{P} = \begin{bmatrix} 2 & 8 & 13 \\ -6 & -25 & -43 \\ 11 & 45 & 76 \end{bmatrix}$; $\mathbf{Q} = \begin{bmatrix} -24 & -2 & 1 \\ 30 & 3 & -1 \\ 139 & 13 & -5 \end{bmatrix}$; $\mathbf{T} = \begin{bmatrix} -25 & -97 & -150 \\ 31 & 120 & 185 \\ 145 & 562 & 868 \end{bmatrix}$

**5.** **(a)** $C = ([1,-4,0,-2,0], [0,0,1,4,0], [0,0,0,0,1])$; $\mathbf{P} = \begin{bmatrix} 1 & 6 & 3 \\ 1 & 5 & 3 \\ 1 & 3 & 2 \end{bmatrix}$;

$\mathbf{Q} = \mathbf{P}^{-1} = \begin{bmatrix} 1 & -3 & 3 \\ 1 & -1 & 0 \\ -2 & 3 & -1 \end{bmatrix}$; $[\mathbf{v}]_B = [17, 4, -13]$; $[\mathbf{v}]_C = [2, -2, 3]$

**(c)** $C = ([1,0,0,0], [0,1,0,0], [0,0,1,0], [0,0,0,1])$; $\mathbf{P} = \begin{bmatrix} 3 & 6 & -4 & -2 \\ -1 & 7 & -3 & 0 \\ 4 & -3 & 3 & 1 \\ 6 & -2 & 4 & 2 \end{bmatrix}$;

$\mathbf{Q} = \mathbf{P}^{-1} = \begin{bmatrix} 1 & -4 & -12 & 7 \\ -2 & 9 & 27 & -\frac{31}{2} \\ -5 & 22 & 67 & -\frac{77}{2} \\ 5 & -23 & -71 & 41 \end{bmatrix}$; $[\mathbf{v}]_B = [2, 1, -3, 7]$; $[\mathbf{v}]_C = [10, 14, 3, 12]$

**7.** **(a)** Transition matrix to $C_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

Transition matrix to $C_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Transition matrix to $C_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Transition matrix to $C_4 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Transition matrix to $C_5 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

**10.** $C = ([-142, 64, 167], [-53, 24, 63], [-246, 111, 290])$

**11.** **(b)** $\mathbf{D}[\mathbf{v}]_B = [\mathbf{Av}]_B = [2, -2, 3]$.

**18.** **(a)** F **(b)** T **(c)** T **(d)** F **(e)** F **(f)** T **(g)** F

## Chapter 4 Review Exercises (p. 201–204)

**2.** Zero vector $= [-4, 5]$; Additive inverse of $[x, y]$ is $[-x-8, -y+10]$

**3.** (a), (d), and (f) are not subspaces; (c) is a subspace

**4.** **(a)** $\text{span}(S) = \{[a, b, c, 5a-3b+c] \mid a, b, c \in \mathbb{R}\} \neq \mathbb{R}^4$

**(b)** Basis $= \{[1,0,0,5], [0,1,0,-3], [0,0,1,1]\}$; $\dim(\text{span}(S)) = 3$

**7.** **(a)** $S$ is linearly independent

**(b)** $S$ itself is a basis for span($S$). $S$ spans $\mathbb{R}^3$.

**(c)** No, by Theorem 4.9

**8.** **(a)** $S$ is linearly dependent; $x^3 - 2x^2 - x + 2 = 3(-5x^3 + 2x^2 + 5x - 2) + 8(2x^3 - x^2 - 2x + 1)$

**(b)** The subset $\{-5x^3 + 2x^2 + 5x - 2, 2x^3 - x^2 - 2x + 1, -2x^3 + 2x^2 + 3x - 5\}$ of $S$ is a basis for span($S$). $S$ does not span $\mathcal{P}_3$.

**(c)** Yes, there is. See part (b) of Exercise 21 in Section 4.4.

**12.** **(a)** The matrix whose rows are the given vectors row reduces to $\mathbf{I}_4$, so the Simplified Span Method shows that the set spans $\mathbb{R}^4$. Since the set has 4 vectors and $\dim(\mathbb{R}^4) = 4$, part (1) of Theorem 4.12 shows that it is a basis.

**13.** **(a)** $\mathcal{W}$ nonempty: $\mathbf{0} \in \mathcal{W}$ because $\mathbf{A0} = \mathbf{0}$. Closure under addition: If $\mathbf{X}_1, \mathbf{X}_2 \in \mathcal{W}$, $\mathbf{A}(\mathbf{X}_1 + \mathbf{X}_2) = \mathbf{AX}_1 + \mathbf{AX}_2 = \mathbf{0} + \mathbf{0} = \mathbf{0}$. Closure under scalar multiplication: If $\mathbf{X} \in \mathcal{W}$, $\mathbf{A}(c\mathbf{X}) = c(\mathbf{AX}) = c\mathbf{0} = \mathbf{0}$.

**(b)** Basis $= \{[3,1,0,0], [-2,0,1,1]\}$

**(c)** $\dim(\mathcal{W}) = 2$; $\text{rank}(\mathbf{A}) = 2$; $2 + 2 = 4$

**14.** **(a)** First, use direct computation to check that every polynomial in $B$ is in $\mathcal{V}$. Next, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ row reduces to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, and so $B$ is linearly independent by the Independence Test Method. Finally, since the polynomial $x \notin \mathcal{V}$, $\dim(\mathcal{V}) < \dim(\mathcal{P}_3) = 4$ by Theorem 4.13. But, part (2) of Theorem 4.12 shows that $|B| \leq \dim(\mathcal{V})$. Hence, $3 = |B| \leq \dim(\mathcal{V}) < \dim(\mathcal{P}_3) = 4$, and so $\dim(\mathcal{V}) = 3$. Part (2) of Theorem 4.12 then implies that $B$ is a basis for $\mathcal{V}$.

**(b)** $C = \{1, x^3 - 3x^2 + 3x\}$ is a basis for $\mathcal{W}$; $\dim(\mathcal{W}) = 2$

**15.** $T = \{[2,-3,0,1], [4,3,0,4], [1,0,2,1]\}$

**17.** $\{[2,1,-1,2], [1,-2,2,-4], [0,1,0,0], [0,0,1,0]\}$

**20.** **(a)** $[\mathbf{v}]_B = [-3,-1,-2]$ **(c)** $[\mathbf{v}]_B = [-3,5,-1]$

**21.** **(a)** $[\mathbf{v}]_B = [27,-62,6]$; $\mathbf{P} = \begin{bmatrix} 4 & 2 & 5 \\ -1 & 0 & 1 \\ 3 & 1 & -2 \end{bmatrix}$; $[\mathbf{v}]_C = [14,-21,7]$

**(b)** $[\mathbf{v}]_B = [-4,-1,3]$; $\mathbf{P} = \begin{bmatrix} 4 & 1 & -2 \\ 1 & 2 & 0 \\ -1 & 1 & 1 \end{bmatrix}$; $[\mathbf{v}]_C = [-23,-6,6]$

**23.** **(c)** $\begin{bmatrix} 4 & -3 & 3 \\ 4 & 2 & 0 \\ 13 & 0 & 4 \end{bmatrix}$

**26.**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| **(a)** T | **(d)** T | **(g)** T | **(j)** T | **(m)** T | **(p)** F | **(s)** T | **(v)** T | **(y)** T |
| **(b)** T | **(e)** F | **(h)** F | **(k)** F | **(n)** T | **(q)** T | **(t)** T | **(w)** F | **(z)** F |
| **(c)** F | **(f)** T | **(i)** F | **(l)** F | **(o)** F | **(r)** F | **(u)** T | **(x)** T | |

## Section 5.1 (p. 212–215)

**1.** Only starred parts are listed:
Linear transformations: (a), (d), (h), (k)
Linear operators: (a), (d)

**10. (c)** $\begin{bmatrix} \sin\theta & 0 & \cos\theta \\ 0 & 1 & 0 \\ \cos\theta & 0 & -\sin\theta \end{bmatrix}$

**26.** $L(\mathbf{i}) = \frac{7}{5}\mathbf{i} - \frac{11}{5}\mathbf{j}$; $L(\mathbf{j}) = -\frac{2}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$

**30. (b)** Consider the zero linear transformation.

**38. (a)** F **(b)** T **(c)** F **(d)** F **(e)** T **(f)** F **(g)** T **(h)** T

## Section 5.2 (p. 223–227)

**2. (a)** $\begin{bmatrix} -6 & 4 & -1 \\ -2 & 3 & -5 \\ 3 & -1 & 7 \end{bmatrix}$

**(c)** $\begin{bmatrix} 4 & -1 & 3 & 3 \\ 1 & 3 & -1 & 5 \\ -2 & -7 & 5 & -1 \end{bmatrix}$

**3. (a)** $\begin{bmatrix} -47 & 128 & -288 \\ -18 & 51 & -104 \end{bmatrix}$

**(c)** $\begin{bmatrix} 22 & 14 \\ 62 & 39 \\ 68 & 43 \end{bmatrix}$

**(e)** $\begin{bmatrix} 5 & 6 & 0 \\ -11 & -26 & -6 \\ -14 & -19 & -1 \\ 6 & 3 & -2 \\ -1 & 1 & 1 \\ 11 & 13 & 0 \end{bmatrix}$

**4. (a)** $\begin{bmatrix} -202 & -32 & -43 \\ -146 & -23 & -31 \\ 83 & 14 & 18 \end{bmatrix}$

**(b)** $\begin{bmatrix} 21 & 7 & 21 & 16 \\ -51 & -13 & -51 & -38 \end{bmatrix}$

**6. (a)** $\begin{bmatrix} 67 & -123 \\ 37 & -68 \end{bmatrix}$

**(b)** $\begin{bmatrix} -7 & 2 & 10 \\ 5 & -2 & -9 \\ -6 & 1 & 8 \end{bmatrix}$

**7. (a)** $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$; $12x^2 - 10x + 6$

**8. (a)** $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

**9. (b)** $\frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 0 & 1 & -1 \end{bmatrix}$

**10.** $\begin{bmatrix} -12 & 12 & -2 \\ -4 & 6 & -2 \\ -10 & -3 & 7 \end{bmatrix}$

**13. (a)** $\mathbf{I}_n$
**(c)** $c\mathbf{I}_n$
**(e)** The $n \times n$ matrix whose columns are $\mathbf{e}_n, \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{n-1}$, respectively

**18.** **(a)** $p_{\mathbf{A}_{BB}}(x) = x^3 - 2x^2 + x = x(x-1)^2$

**(b)** Basis for $E_1 = ([2,1,0],[2,0,1])$; basis for $E_0 = ([-1,2,2])$

**(c)** One answer is: $\mathbf{P} = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

**26.** **(a)** T **(b)** T **(c)** F **(d)** F **(e)** T **(f)** F **(g)** T **(h)** T **(i)** T **(j)** F

## Section 5.3 (p. 232–235)

**1.** **(a)** Yes, because $L([10,-5,1,1]) = [0,0,0]$

**(b)** No, because $\ker(L) \subseteq \mathbb{R}^4$, and $[4,2,-5] \in \mathbb{R}^3$

**(e)** No, because the system

$$\begin{cases} x_1 + x_2 - x_3 - 4x_4 = 2 \\ 3x_1 + 5x_2 + x_3 - 6x_4 = -1 \\ -x_1 \qquad + 3x_3 + 7x_4 = 4 \end{cases}$$

has no solutions

**2.** **(a)** No, since $L(x^3 - 5x^2 + 3x - 6) = 19x^3 - 21x^2 + 8 \neq 0$

**(d)** Yes, because, for example, $L(x^3 + x^2 - x + 1) = -4x^3 + x^2 - 1$

**3.** **(a)** $\dim(\ker(L)) = 1$; basis for $\ker(L) = \{[-2,3,1]\}$; $\dim(\text{range}(L)) = 2$; basis for $\text{range}(L) = \{[1,-2,3],[-1,3,-3]\}$

**(d)** $\dim(\ker(L)) = 2$; basis for $\ker(L) = \{[1,-3,1,0],[-1,2,0,1]\}$; $\dim(\text{range}(L)) = 2$; basis for $\text{range}(L) = \{[-14,-4,-6,3,4],[-8,-1,2,-7,2]\}$

**4.** **(a)** $\dim(\ker(L)) = 2$; basis for $\ker(L) = \{[1,0,0],[0,0,1]\}$; $\dim(\text{range}(L)) = 1$; basis for $\text{range}(L) = \{[0,1]\}$

**(d)** $\dim(\ker(L)) = 2$; basis for $\ker(L) = \{x^4, x^3\}$; $\dim(\text{range}(L)) = 3$; basis for $\text{range}(L) = \{x^2, x, 1\}$

**(f)** $\dim(\ker(L)) = 1$; basis for $\ker(L) = \{[0,1,1]\}$; $\dim(\text{range}(L)) = 2$; basis for $\text{range}(L) = \{[1,0,1],[0,0,-1]\}$ (A simpler basis for $\text{range}(L) = \{[1,0,0],[0,0,1]\}$.)

**(g)** $\dim(\ker(L)) = 0$; basis for $\ker(L) = \{\}$ (empty set); $\dim(\text{range}(L)) = 4$; basis for $\text{range}(L) =$ standard basis for $\mathcal{M}_{22}$

**(i)** $\dim(\ker(L)) = 1$; basis for $\ker(L) = \{x^2 - 2x + 1\}$; $\dim(\text{range}(L)) = 2$; basis for $\text{range}(L) = \{[1,2],[1,1]\}$ (A simpler basis for $\text{range}(L) =$ standard basis for $\mathbb{R}^2$.)

**6.** $\ker(L) = \left\{ \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & -a-e \end{bmatrix} \middle|\ a,b,c,d,e,f,g,h \in \mathbb{R} \right\}$; $\dim(\ker(L)) = 8$; $\text{range}(L) = \mathbb{R}$; $\dim(\text{range}(L)) = 1$.

**8.** $\ker(L) = \{\mathbf{0}\}$; $\text{range}(L) = \{ax^4 + bx^3 + cx^2\}$; $\dim(\ker(L)) = 0$; $\dim(\text{range}(L)) = 3$

**10.** When $1 \le k \le n$, $\ker(L) =$ all polynomials of degree less than $k$, $\dim(\ker(L)) = k$, $\text{range}(L) = \mathcal{P}_{n-k}$, and $\dim(\text{range}(L)) = n - k + 1$. When $k > n$, $\ker(L) = \mathcal{P}_n$, $\dim(\ker(L)) = n + 1$, $\text{range}(L) = \{\mathbf{0}\}$, and $\dim(\text{range}(L)) = 0$.

**12.** $\ker(L) = \{[0,0,\ldots,0]\}$; $\text{range}(L) = \mathbb{R}^n$ (Note: Every vector $\mathbf{X}$ is in the range since $L(\mathbf{A}^{-1}\mathbf{X}) = \mathbf{A}(\mathbf{A}^{-1}\mathbf{X}) = \mathbf{X}$.)

**16.** Consider $L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. Then, $\ker(L) = \text{range}(L) = \{[a,a] \mid a \in \mathbb{R}\}$.

**20.** **(a)** F **(b)** F **(c)** T **(d)** F **(e)** T **(f)** F **(g)** F **(h)** F

## Section 5.4 (p. 238–240)

**1.** **(a)** Not one-to-one, because $L([1,0,0]) = L([0,0,0]) = [0,0,0,0]$; not onto, because $[0,0,0,1]$ is not in $\text{range}(L)$

**(c)** One-to-one, because $L([x,y,z]) = [0,0,0]$ implies that $[2x, x+y+z, -y] = [0,0,0]$, which gives $x = y = z = 0$; onto, because every vector $[a,b,c]$ can be expressed as $[2x, x+y+z, -y]$, where $x = \frac{a}{2}$, $y = -c$, and $z = b - \frac{a}{2} + c$

**(e)** One-to-one, because $L(ax^2 + bx + c) = 0$ implies that $a + b = b + c = a + c = 0$, which gives $a = b = c = 0$; onto, because every polynomial $Ax^2 + Bx + C$ can be expressed as $(a+b)x^2 + (b+c)x + (a+c)$, where $a = (A - B + C)/2$, $b = (A + B - C)/2$, and $c = (-A + B + C)/2$

**(g)** Not one-to-one, because $L\left(\begin{bmatrix}0 & 1 & 0\\1 & 0 & -1\end{bmatrix}\right) = L\left(\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\end{bmatrix}\right) = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$; onto, because every $2 \times 2$ matrix $\begin{bmatrix}A & B\\C & D\end{bmatrix}$ can be expressed as $\begin{bmatrix}a & -c\\2e & d+f\end{bmatrix}$, where $a = A$, $c = -B$, $e = C/2$, $d = D$, and $f = 0$

**(h)** One-to-one, because $L(ax^2 + bx + c) = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$ implies that $a + c = b - c = -3a = 0$, which gives $a = b = c = 0$; not onto, because $\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}$ is not in range($L$)

**2.** **(a)** One-to-one; onto; the matrix row reduces to $\mathbf{I}_2$, which means that $\dim(\ker(L)) = 0$ and $\dim(\text{range}(L)) = 2$.

**(c)** Not one-to-one; not onto; the matrix row reduces to $\begin{bmatrix}1 & 0 & -\frac{2}{5}\\0 & 1 & -\frac{6}{5}\\0 & 0 & 0\end{bmatrix}$, which means that $\dim(\ker(L)) = 1$ and $\dim(\text{range}(L)) = 2$.

**3.** **(a)** One-to-one; onto; the matrix row reduces to $\mathbf{I}_3$, which means that $\dim(\ker(L)) = 0$ and $\dim(\text{range}(L)) = 3$.

**(c)** Not one-to-one; not onto; the matrix row reduces to $\begin{bmatrix}1 & 0 & -\frac{10}{11} & \frac{19}{11}\\0 & 1 & \frac{3}{11} & -\frac{9}{11}\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix}$, which means that $\dim(\ker(L)) = 2$ and $\dim(\text{range}(L)) = 2$.

**7.** **(a)** No, by Corollary 5.13, because $\dim(\mathbb{R}^6) = \dim(\mathcal{P}_5)$.

**(b)** No, by Corollary 5.13, because $\dim(\mathcal{M}_{22}) = \dim(\mathcal{P}_3)$.

**12.** **(a)** F **(b)** F **(c)** T **(d)** T **(e)** T **(f)** T **(g)** T **(h)** F **(i)** F

## Section 5.5 (p. 246–248)

**1.** In each part, let $\mathbf{A}$ represent the given matrix for $L_1$ and let $\mathbf{B}$ represent the given matrix for $L_2$. By Theorem 5.16, both $L_1$ and $L_2$ are isomorphisms if and only if $\mathbf{A}$ and $\mathbf{B}$ are nonsingular. In each part, we state $|\mathbf{A}|$ and $|\mathbf{B}|$ to show that $\mathbf{A}$ and $\mathbf{B}$ are nonsingular.

**(a)** $|\mathbf{A}| = 1$, $|\mathbf{B}| = 3$, $L_1^{-1}\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = \begin{bmatrix}0 & 0 & 1\\0 & -1 & 0\\1 & -2 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$,

$L_2^{-1}\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = \begin{bmatrix}1 & 0 & 0\\0 & 0 & -\frac{1}{3}\\2 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$,

$(L_2 \circ L_1)\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = \begin{bmatrix}0 & -2 & 1\\1 & 4 & -2\\0 & 3 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$,

$(L_2 \circ L_1)^{-1}\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = (L_1^{-1} \circ L_2^{-1})\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = \begin{bmatrix}2 & 1 & 0\\0 & 0 & \frac{1}{3}\\1 & 0 & \frac{2}{3}\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$

**(c)** $|\mathbf{A}| = -1$, $|\mathbf{B}| = 1$, $L_1^{-1}\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = \begin{bmatrix}2 & -4 & -1\\7 & -13 & -3\\5 & -10 & -3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$,

$$L_2^{-1}\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)=\begin{bmatrix}1&0&-1\\3&1&-3\\-1&-2&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix},$$

$$(L_2\circ L_1)\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)=\begin{bmatrix}29&-6&-4\\21&-5&-2\\38&-8&-5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix},$$

$$(L_2\circ L_1)^{-1}\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)=(L_1^{-1}\circ L_2^{-1})\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)=\begin{bmatrix}-9&-2&8\\-29&-7&26\\-22&-4&19\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

**5.** **(a)** $\begin{bmatrix}0&1\\1&0\end{bmatrix}$

**21.** **(a)** T **(b)** T **(c)** F **(d)** F **(e)** F **(f)** T **(g)** T **(h)** T

## Section 5.6 (p. 257–260)

**1.** **(a)** $\lambda_1=2$; basis for $E_2=([1,0])$; algebraic multiplicity of $\lambda_1$ is 2; geometric multiplicity of $\lambda_1$ is 1

**(c)** $\lambda_1=-1$; basis for $E_{-1}=([-1,2,3])$; algebraic multiplicity of $\lambda_1$ is 2; geometric multiplicity of $\lambda_1$ is 1; $\lambda_2=0$; basis for $E_0=([-1,1,3])$; algebraic multiplicity for $\lambda_2$ is 1; geometric multiplicity for $\lambda_2$ is 1

**(d)** $\lambda_1=2$; basis for $E_2=([5,4,0],[3,0,2])$; algebraic multiplicity of $\lambda_1$ is 2; geometric multiplicity of $\lambda_1$ is 2; $\lambda_2=3$; basis for $E_3=([0,-1,1])$; algebraic multiplicity of $\lambda_2$ is 1; geometric multiplicity of $\lambda_2$ is 1

**2.** **(b)** $C=(x^2,x,1)$; $\mathbf{A}=\begin{bmatrix}2&0&0\\-2&1&0\\0&-1&0\end{bmatrix}$; $B=(x^2-2x+1,-x+1,1)$; $\mathbf{D}=\begin{bmatrix}\mathbf{2}&0&0\\0&\mathbf{1}&0\\0&0&\mathbf{0}\end{bmatrix}$; $\mathbf{P}=\begin{bmatrix}1&0&0\\-2&-1&0\\1&1&1\end{bmatrix}$

**(d)** $C=(x^2,x,1)$; $\mathbf{A}=\begin{bmatrix}-1&0&0\\-12&-4&0\\18&0&-5\end{bmatrix}$; $B=(2x^2-8x+9,x,1)$; $\mathbf{D}=\begin{bmatrix}\mathbf{-1}&0&0\\0&\mathbf{-4}&0\\0&0&\mathbf{-5}\end{bmatrix}$; $\mathbf{P}=\begin{bmatrix}2&0&0\\-8&1&0\\9&0&1\end{bmatrix}$

**(e)** $C=(\mathbf{i},\mathbf{j})$; $\mathbf{A}=\frac{1}{2}\begin{bmatrix}1&-\sqrt{3}\\\sqrt{3}&1\end{bmatrix}$; no eigenvalues; not diagonalizable

**(h)** $C=\left(\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\right)$; $\mathbf{A}=\begin{bmatrix}-4&0&3&0\\0&-4&0&3\\-10&0&7&0\\0&-10&0&7\end{bmatrix}$;

$B=\left(\begin{bmatrix}3&0\\5&0\end{bmatrix},\begin{bmatrix}0&3\\0&5\end{bmatrix},\begin{bmatrix}1&0\\2&0\end{bmatrix},\begin{bmatrix}0&1\\0&2\end{bmatrix}\right)$; $\mathbf{D}=\begin{bmatrix}\mathbf{1}&0&0&0\\0&\mathbf{1}&0&0\\0&0&\mathbf{2}&0\\0&0&0&\mathbf{2}\end{bmatrix}$; $\mathbf{P}=\begin{bmatrix}3&0&1&0\\0&3&0&1\\5&0&2&0\\0&5&0&2\end{bmatrix}$

**4.** **(a)** The only eigenvalue is $\lambda=1$; $E_1=\{1\}$

**7.** **(a)** $\begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix}$; eigenvalue $\lambda=1$; basis for $E_1=\{[1,0,0]\}$; $\lambda$ has algebraic multiplicity 3 and geometric multiplicity 1

**(c)** $\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$; eigenvalues $\lambda_1=1,\lambda_2=0$; basis for $E_1=\{[1,0,0],[0,1,0]\}$; $\lambda_1$ has algebraic and geometric multiplicity 2

**23.** **(a)** F **(b)** T **(c)** T **(d)** F **(e)** T **(f)** T **(g)** T **(h)** F **(i)** F **(j)** T

## Chapter 5 Review Exercises (p. 260–264)

**1. (a)** Not a linear transformation

**3.** $f(\mathbf{A}_1) + f(\mathbf{A}_2) = \mathbf{CA}_1\mathbf{B}^{-1} + \mathbf{CA}_2\mathbf{B}^{-1} = \mathbf{C}(\mathbf{A}_1\mathbf{B}^{-1} + \mathbf{A}_2\mathbf{B}^{-1}) = \mathbf{C}(\mathbf{A}_1 + \mathbf{A}_2)\mathbf{B}^{-1} = f(\mathbf{A}_1 + \mathbf{A}_2)$; $f(k\mathbf{A}) = \mathbf{C}(k\mathbf{A})\mathbf{B}^{-1} = k\mathbf{CAB}^{-1} = kf(\mathbf{A})$.

**4.** $L([6,2,-7]) = [20,10,44]$; $L([x,y,z]) = \begin{bmatrix} -3 & 5 & -4 \\ 2 & -1 & 0 \\ 4 & 3 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

**5. (b)** Use Theorem 5.2 and part (2) of Theorem 5.3.

**6. (a)** $\mathbf{A}_{BC} = \begin{bmatrix} 29 & 32 & -2 \\ 43 & 42 & -6 \end{bmatrix}$

**7. (b)** $\mathbf{A}_{DE} = \begin{bmatrix} 115 & -45 & 59 \\ 374 & -146 & 190 \\ -46 & 15 & -25 \\ -271 & 108 & -137 \end{bmatrix}$

**9. (a)** $p_{\mathbf{A}_{BB}}(x) = x^3 - x^2 - x + 1 = (x+1)(x-1)^2$

**10. (a)** Basis for $\ker(L) = \{[2,-3,1,0], [-3,4,0,1]\}$;
basis for $\text{range}(L) = \{[3,2,2,1], [1,1,3,4]\}$

**12. (a)** First show that $\ker(L_1) \subseteq \ker(L_2 \circ L_1)$. Conclude that $\dim(\ker(L_1)) \le \dim(\ker(L_2 \circ L_1))$.

**(b)** Let $L_1([x,y]) = [x,0]$ and $L_2([x,y]) = [0,y]$.

**15. (a)** $\dim(\ker(L)) = 0$, $\dim(\text{range}(L)) = 3$. $L$ is both one-to-one and onto.

**18. (a)** The matrices for $L_1$ and $L_2$, respectively, have determinants 5 and 2. Apply Theorem 5.16.

**(b)** Matrix for $L_2 \circ L_1 = \begin{bmatrix} 81 & 71 & -15 & 18 \\ 107 & 77 & -31 & 19 \\ 69 & 45 & -23 & 11 \\ -29 & -36 & -1 & -9 \end{bmatrix}$; for $L_1^{-1}$: $\frac{1}{5}\begin{bmatrix} 2 & -10 & 19 & 11 \\ 0 & 5 & -10 & -5 \\ 3 & -15 & 26 & 14 \\ -4 & 15 & -23 & -17 \end{bmatrix}$;

for $L_2^{-1}$: $\frac{1}{2}\begin{bmatrix} -8 & 26 & -30 & 2 \\ 10 & -35 & 41 & -4 \\ 10 & -30 & 34 & -2 \\ -14 & 49 & -57 & 6 \end{bmatrix}$.

**21. (a)** $L(ax^4 + bx^3 + cx^2) = 4ax^3 + (3b + 12a)x^2 + (2c + 6b)x + 2c$. Clearly $\ker(L) = \{\mathbf{0}\}$. Apply part (1) of Theorem 5.12.

**22.** In parts (a) and (c), let $\mathbf{A}$ represent the given matrix.

**(a) (i)** $p_{\mathbf{A}}(x) = x^3 - 3x^2 - x + 3 = (x-1)(x+1)(x-3)$; eigenvalues for $L$: $\lambda_1 = 1$, $\lambda_2 = -1$, and $\lambda_3 = 3$; basis for $E_1$: $\{[-1,3,4]\}$; basis for $E_{-1}$: $\{[-1,4,5]\}$; basis for $E_3$: $\{[-6,20,27]\}$

**(ii)** All algebraic and geometric multiplicities equal 1; $L$ is diagonalizable.

**(iii)** $B = ([-1,3,4], [-1,4,5], [-6,20,27])$; $\mathbf{D} = \begin{bmatrix} \mathbf{1} & 0 & 0 \\ 0 & \mathbf{-1} & 0 \\ 0 & 0 & \mathbf{3} \end{bmatrix}$; $\mathbf{P} = \begin{bmatrix} -1 & -1 & -6 \\ 3 & 4 & 20 \\ 4 & 5 & 27 \end{bmatrix}$

(Note that $\mathbf{P}^{-1} = \begin{bmatrix} -8 & 3 & -4 \\ 1 & 3 & -2 \\ 1 & -1 & 1 \end{bmatrix}$.)

**(c) (i)** $p_{\mathbf{A}}(x) = x^3 - 5x^2 + 3x + 9 = (x+1)(x-3)^2$; eigenvalues for $L$: $\lambda_1 = -1$, and $\lambda_2 = 3$; Basis for $E_{-1}$: $\{[1,3,3]\}$; Basis for $E_3$: $\{[1,5,0], [3,0,25]\}$

**(ii)** For $\lambda_1 = -1$: algebraic multiplicity = geometric multiplicity = 1; for $\lambda_2 = 3$: algebraic multiplicity = geometric multiplicity = 2; $L$ is diagonalizable.

**(iii)** $B = ([1,3,3], [1,5,0], [3,0,25])$; $\mathbf{D} = \begin{bmatrix} \mathbf{-1} & 0 & 0 \\ 0 & \mathbf{3} & 0 \\ 0 & 0 & \mathbf{3} \end{bmatrix}$; $\mathbf{P} = \begin{bmatrix} 1 & 1 & 3 \\ 3 & 5 & 0 \\ 3 & 0 & 25 \end{bmatrix}$

(Note that $\mathbf{P}^{-1} = \frac{1}{5}\begin{bmatrix} 125 & -25 & -15 \\ -75 & 16 & 9 \\ -15 & 3 & 2 \end{bmatrix}$.)

**29.** **(a)** T **(b)** F **(c)** T **(d)** T **(e)** T **(f)** F **(g)** F **(h)** T **(i)** T **(j)** F **(k)** F **(l)** T **(m)** F **(n)** T **(o)** T **(p)** F **(q)** F **(r)** T **(s)** F **(t)** T **(u)** T **(v)** F **(w)** T **(x)** T **(y)** T **(z)** T

## Section 6.1 (p. 272–274)

**1.** **(a)** Orthogonal, not orthonormal
**(c)** Neither
**(f)** Orthogonal, not orthonormal

**2.** **(a)** Orthogonal
**(c)** Not orthogonal: columns not normalized
**(e)** Orthogonal

**3.** **(a)** $[\mathbf{v}]_B = \left[\frac{2\sqrt{3}+3}{2}, \frac{3\sqrt{3}-2}{2}\right]$
**(c)** $[\mathbf{v}]_B = [3, \frac{13\sqrt{3}}{3}, \frac{5\sqrt{6}}{3}, 4\sqrt{2}]$

**4.** **(a)** $\{[5, -1, 2], [5, -3, -14]\}$
**(c)** $\{[2, 1, 0, -1], [-1, 1, 3, -1], [5, -7, 5, 3]\}$

**5.** **(a)** $\{[2, 2, -3], [13, -4, 6], [0, 3, 2]\}$
**(c)** $\{[1, -3, 1], [2, 5, 13], [4, 1, -1]\}$
**(e)** $\{[2, 1, -2, 1], [3, -1, 2, -1], [0, 5, 2, -1], [0, 0, 1, 2]\}$

**7.** **(a)** $[-1, 3, 3]$
**(c)** $[5, 1, 1]$

**8.** **(b)** No

**16.** **(b)** $\begin{bmatrix} \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{3} & \frac{\sqrt{6}}{6} \\ \frac{\sqrt{30}}{6} & -\frac{\sqrt{30}}{15} & -\frac{\sqrt{30}}{30} \\ 0 & \frac{\sqrt{5}}{5} & -\frac{2\sqrt{5}}{5} \end{bmatrix}$

**22.** **(a)** F **(b)** T **(c)** T **(d)** F **(e)** T **(f)** T **(g)** T **(h)** F **(i)** T **(j)** T

## Section 6.2 (p. 282–284)

**1.** **(a)** $\mathcal{W}^\perp = \text{span}(\{[2, 3]\})$
**(c)** $\mathcal{W}^\perp = \text{span}(\{[2, 3, 7]\})$
**(e)** $\mathcal{W}^\perp = \text{span}(\{[-2, 5, -1]\})$
**(f)** $\mathcal{W}^\perp = \text{span}(\{[7, 1, -2, -3], [0, 4, -1, 2]\})$

**2.** **(a)** $\mathbf{w}_1 = \mathbf{proj}_{\mathcal{W}}\mathbf{v} = \left[-\frac{33}{35}, \frac{111}{35}, \frac{12}{7}\right]; \mathbf{w}_2 = \left[-\frac{2}{35}, -\frac{6}{35}, \frac{2}{7}\right]$
**(b)** $\mathbf{w}_1 = \mathbf{proj}_{\mathcal{W}}\mathbf{v} = \left[-\frac{17}{9}, -\frac{10}{9}, \frac{14}{9}\right]; \mathbf{w}_2 = \left[\frac{26}{9}, -\frac{26}{9}, \frac{13}{9}\right]$

**4.** **(a)** $\frac{3\sqrt{129}}{43}$
**(d)** $\frac{8\sqrt{17}}{17}$

**5.** **(a)** Orthogonal projection onto $3x + y + z = 0$
**(d)** Neither

**6.** $\frac{1}{9}\begin{bmatrix} 5 & 2 & -4 \\ 2 & 8 & 2 \\ -4 & 2 & 5 \end{bmatrix}$

**9.** **(a)** $x^3 - 2x^2 + x$
**(c)** $x^3 - x^2 - x + 1$

**10.** **(a)** $\frac{1}{59}\begin{bmatrix} 50 & -21 & -3 \\ -21 & 10 & -7 \\ -3 & -7 & 58 \end{bmatrix}$
**(c)** $\frac{1}{9}\begin{bmatrix} 2 & 2 & 3 & -1 \\ 2 & 8 & 0 & 2 \\ 3 & 0 & 6 & -3 \\ -1 & 2 & -3 & 2 \end{bmatrix}$

**25.** **(a)** T **(b)** T **(c)** F **(d)** F **(e)** T **(f)** F **(g)** T **(h)** T **(i)** T **(j)** F **(k)** T **(l)** T

## Section 6.3 (p. 292–295)

**1.** **(a)** Symmetric, because the matrix for $L$ with respect to the standard basis is symmetric
**(d)** Symmetric, since $L$ is orthogonally diagonalizable
**(e)** Not symmetric, since $L$ is not diagonalizable, and hence not orthogonally diagonalizable
**(g)** Symmetric, because the matrix with respect to the standard basis is symmetric

**2.** **(a)** $\frac{1}{25}\begin{bmatrix} -7 & 24 \\ 24 & 7 \end{bmatrix}$

**(d)** $\frac{1}{169}\begin{bmatrix} -119 & -72 & -96 & 0 \\ -72 & 119 & 0 & 96 \\ -96 & 0 & 119 & -72 \\ 0 & 96 & -72 & -119 \end{bmatrix}$

**3.** **(a)** $B = \left(\frac{1}{13}[5,12], \frac{1}{13}[-12,5]\right)$; $\mathbf{P} = \frac{1}{13}\begin{bmatrix} 5 & -12 \\ 12 & 5 \end{bmatrix}$; $\mathbf{D} = \begin{bmatrix} \mathbf{0} & 0 \\ 0 & \mathbf{169} \end{bmatrix}$. The Spectral Theorem is verified by direct computation. Details appear in the Student Solutions Manual.

**(c)** $B = \left(\frac{1}{\sqrt{2}}[-1,1,0], \frac{1}{3\sqrt{2}}[1,1,4], \frac{1}{3}[-2,-2,1]\right)$ (other bases are possible, since $E_1$ is two-dimensional), $\mathbf{P} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & -\frac{2}{3} \\ \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & -\frac{2}{3} \\ 0 & \frac{4}{3\sqrt{2}} & \frac{1}{3} \end{bmatrix}$, $\mathbf{D} = \begin{bmatrix} \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{3} \end{bmatrix}$. The Spectral Theorem is verified by direct computation. Details appear in the Student Solutions Manual.

**(e)** $B = \left(\frac{1}{\sqrt{14}}[3,2,1,0], \frac{1}{\sqrt{14}}[-2,3,0,1], \frac{1}{\sqrt{14}}[1,0,-3,2], \frac{1}{\sqrt{14}}[0,-1,2,3]\right)$;
$\mathbf{P} = \frac{1}{\sqrt{14}}\begin{bmatrix} 3 & -2 & 1 & 0 \\ 2 & 3 & 0 & -1 \\ 1 & 0 & -3 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$; $\mathbf{D} = \begin{bmatrix} \mathbf{2} & 0 & 0 & 0 \\ 0 & \mathbf{2} & 0 & 0 \\ 0 & 0 & \mathbf{-3} & 0 \\ 0 & 0 & 0 & \mathbf{5} \end{bmatrix}$. The Spectral Theorem is verified by direct computation. Details appear in the Student Solutions Manual.

**(g)** $B = \left(\frac{1}{\sqrt{5}}[1,2,0], \frac{1}{\sqrt{6}}[-2,1,1], \frac{1}{\sqrt{30}}[2,-1,5]\right)$ (other bases are possible, since $E_{15}$ is two-dimensional);
$\mathbf{P} = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{6}} & \frac{2}{\sqrt{30}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{30}} \\ 0 & \frac{1}{\sqrt{6}} & \frac{5}{\sqrt{30}} \end{bmatrix}$; $\mathbf{D} = \begin{bmatrix} \mathbf{15} & 0 & 0 \\ 0 & \mathbf{15} & 0 \\ 0 & 0 & \mathbf{-15} \end{bmatrix}$. The Spectral Theorem is verified by direct computation. Details appear in the Student Solutions Manual.

**4.** **(a)** $C = \left(\frac{1}{19}[-10,15,6], \frac{1}{19}[15,6,10]\right)$; $\mathbf{A} = \begin{bmatrix} -2 & 2 \\ 2 & 1 \end{bmatrix}$; $B = \left(\frac{1}{19\sqrt{5}}[20,27,26], \frac{1}{19\sqrt{5}}[35,-24,-2]\right)$;
$\mathbf{P} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$; $\mathbf{D} = \begin{bmatrix} \mathbf{2} & 0 \\ 0 & \mathbf{-3} \end{bmatrix}$. The Spectral Theorem is verified by direct computation. Details appear in the Student Solutions Manual.

**5.** **(a)** $\frac{1}{25}\begin{bmatrix} 23 & -36 \\ -36 & 2 \end{bmatrix}$ **(c)** $\frac{1}{3}\begin{bmatrix} 11 & 4 & -4 \\ 4 & 17 & -8 \\ -4 & -8 & 17 \end{bmatrix}$

**6.** The matrix $\mathbf{A}$ in Examples 1, 4, and 5 of Section 3.4 is diagonalizable, but not symmetric, and hence, not orthogonally diagonalizable.

**7.** $\frac{1}{2}\begin{bmatrix} a+c+\sqrt{(a-c)^2+4b^2} & 0 \\ 0 & a+c-\sqrt{(a-c)^2+4b^2} \end{bmatrix}$

**8.** **(b)** $L$ must be the zero linear operator. Since $L$ is diagonalizable, the eigenspace for 0 must be all of $\mathcal{V}$.

**13.** **(e)** $\theta \approx 278°$. This is a counterclockwise rotation about the vector $[-1,3,3]$, looking down from the point $(-1,3,3)$ toward the plane through the origin spanned by $[6,1,1]$ and $[0,1,-1]$.

**16.** **(a)** T **(b)** F **(c)** T **(d)** T **(e)** T **(f)** T **(g)** F

## Chapter 6 Review Exercises (p. 295–297)

**1. (a)** $[\mathbf{v}]_B = [-1, 4, 2]$

**2. (a)** $\{[1, -1, -1, 1], [1, 1, 1, 1]\}$

**3.** $\{[6, 3, -6], [3, 6, 6], [2, -2, 1]\}$

**6. (a)** $\mathbf{w}_1 = \mathbf{proj}_{\mathcal{W}}\mathbf{v} = [0, -9, 18]$; $\mathbf{w}_2 = \mathbf{proj}_{\mathcal{W}^\perp}\mathbf{v} = [2, 16, 8]$

**7. (a)** Distance $\approx 10.141294$

**10.** $\frac{1}{7}\begin{bmatrix} 3 & 6 & -2 \\ 6 & -2 & 3 \\ -2 & 3 & 6 \end{bmatrix}$

**11. (b)** $p_L(x) = x^2(x-1) = x^3 - x^2$

**12. (a)** Not a symmetric operator, since the matrix for $L$ with respect to the standard basis is $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, which is not symmetric.

**13. (a)** $B = \left(\frac{1}{\sqrt{6}}[-1, -2, 1], \frac{1}{\sqrt{30}}[1, 2, 5], \frac{1}{\sqrt{5}}[-2, 1, 0]\right)$; $\mathbf{P} = \begin{bmatrix} -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{30}} & -\frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{6}} & \frac{2}{\sqrt{30}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{6}} & \frac{5}{\sqrt{30}} & 0 \end{bmatrix}$; $\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. The Spectral Theorem is verified by direct computation. Details appear in the Student Solutions Manual.

**16.** **(a)** T **(b)** T **(c)** T **(d)** T **(e)** F **(f)** F **(g)** T **(h)** T **(i)** T **(j)** T **(k)** T **(l)** T **(m)** F **(n)** T **(o)** T **(p)** T **(q)** F **(r)** F **(s)** T **(t)** T **(u)** F **(v)** T **(w)** F

## Section 7.1 (p. 304–305)

**1. (a)** $[1+4i, 1+i, 6-i]$
**(b)** $[-12-32i, -7+30i, 53-29i]$
**(d)** $[-24-12i, -28-8i, -32i]$
**(e)** $1+28i$

**3. (a)** $\begin{bmatrix} 11+4i & -4-2i \\ 2-4i & 12 \end{bmatrix}$

**(c)** $\begin{bmatrix} 1-i & 0 & 10i \\ 2i & 3-i & 0 \\ 6-4i & 5 & 7+3i \end{bmatrix}$

**(d)** $\begin{bmatrix} -3-15i & -3 & 9i \\ 9-6i & 0 & 3+12i \end{bmatrix}$

**(f)** $\begin{bmatrix} 1+40i & -4-14i \\ 13-50i & 23+21i \end{bmatrix}$

**(i)** $\begin{bmatrix} 4+36i & -5+39i \\ 1-7i & -6-4i \\ 5+40i & -7-5i \end{bmatrix}$

**5. (a)** Skew-Hermitian
**(b)** Neither
**(c)** Hermitian
**(d)** Skew-Hermitian
**(e)** Hermitian

**11.** **(a)** F **(b)** F **(c)** T **(d)** F **(e)** F **(f)** T

## Section 7.2 (p. 308–309)

**1. (a)** $w = \frac{1}{5} + \frac{13}{5}i$, $z = \frac{28}{5} - \frac{3}{5}i$
**(c)** $x = (2+5i) - (4-3i)c$, $y = (5+2i) + ic$, $z = c$
**(e)** Solution set $= \{\,\}$

**2. (b)** $|\mathbf{A}| = -15-23i$; $\mathbf{A}$ is nonsingular; $|\mathbf{A}^*| = -15+23i = \overline{|\mathbf{A}|}$

**3. (a)** Eigenvalues: $\lambda_1 = i$; $\lambda_2 = -1$, with respective eigenvectors $[1+i, 2]$ and $[7+6i, 17]$. Hence, $E_i = \{c[1+i, 2] \mid c \in \mathbb{C}\}$, and $E_{-1} = \{c[7+6i, 17] \mid c \in \mathbb{C}\}$.

(c) Eigenvalues: $\lambda_1 = i$ and $\lambda_2 = -2$, with eigenvectors $[1, 1, 0]$ and $[(-3-2i), 0, 2]$ for $\lambda_1$, and eigenvector $[-1, i, 1]$ for $\lambda_2$. Hence, $E_i = \{c[1, 1, 0] + d[(-3-2i), 0, 2] \mid c, d \in \mathbb{C}\}$ and $E_{-2} = \{c[-1, i, 1] \mid c \in \mathbb{C}\}$.

**4.** (a) The $2 \times 2$ matrix **A** is diagonalizable since two eigenvectors were found in the diagonalization process;
$\mathbf{P} = \begin{bmatrix} 1+i & 7+6i \\ 2 & 17 \end{bmatrix}$; $\mathbf{D} = \begin{bmatrix} i & 0 \\ 0 & -1 \end{bmatrix}$

**6.** (a) T (b) F (c) F (d) F

## Section 7.3 (p. 311)

**2.** (b) Not linearly independent, dim = 1 (d) Not linearly independent, dim = 2

**3.** (b) Linearly independent, dim = 2 (d) Linearly independent, dim = 3

**4.** (b) $[i, 1+i, -1]$

**5.** Ordered basis $= ([1, 0], [i, 0], [0, 1], [0, i])$; matrix $= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$.

**8.** $\begin{bmatrix} -3+i & -\frac{2}{5} - \frac{11}{5}i \\ \frac{1}{2} - \frac{3}{2}i & -i \\ -\frac{1}{2} + \frac{7}{2}i & -\frac{8}{5} - \frac{4}{5}i \end{bmatrix}$

**9.** (a) T (b) F (c) T (d) F

## Section 7.4 (p. 315–316)

**1.** (a) Not orthogonal (c) Orthogonal

**3.** (a) $\{[1+i, i, 1], [2, -1-i, -1+i], [0, 1, i]\}$

(b) $\begin{bmatrix} \frac{1+i}{2} & \frac{i}{2} & \frac{1}{2} \\ \frac{2}{\sqrt{8}} & \frac{-1-i}{\sqrt{8}} & \frac{-1+i}{\sqrt{8}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix}$

**10.** (b) $\mathbf{P} = \begin{bmatrix} \frac{-1+i}{\sqrt{6}} & \frac{1-i}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$; the corresponding diagonal matrix is $\begin{bmatrix} 9+6i & 0 \\ 0 & -3-12i \end{bmatrix}$

**13.** Eigenvalues: $-4$, $2+\sqrt{6}$ and $2-\sqrt{6}$

**15.** (a) F (b) T (c) T (d) T (e) F

## Section 7.5 (p. 324–327)

**1.** (b) $\langle \mathbf{x}, \mathbf{y} \rangle = -183$; $\|\mathbf{x}\| = \sqrt{314}$

**3.** (b) $\langle \mathbf{f}, \mathbf{g} \rangle = \frac{1}{2}(e^{\pi} + 1)$; $\|\mathbf{f}\| = \sqrt{\frac{1}{2}\left(e^{2\pi} - 1\right)}$

**9.** (a) $\sqrt{\frac{\pi^3}{3} - \frac{3\pi}{2}}$

**10.** (b) 0.586 radians, or 33.6°

**14.** (a) Orthogonal (c) Not orthogonal

**19.** Using $\mathbf{w}_1 = t^2 - t + 1$, $\mathbf{w}_2 = 1$, and $\mathbf{w}_3 = t$ yields the orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, with $\mathbf{v}_1 = t^2 - t + 1$, $\mathbf{v}_2 = -20t^2 + 20t + 13$, and $\mathbf{v}_3 = 15t^2 + 4t - 5$.

**23.** $\mathcal{W}^{\perp} = \text{span}(\{t^3 - t^2, t + 1\})$

**26.** $\mathbf{w}_1 = \frac{1}{2\pi}(\sin t - \cos t)$, $\mathbf{w}_2 = \frac{1}{k}e^t - \frac{1}{2\pi}\sin t + \frac{1}{2\pi}\cos t$

**29. (b)** $\ker(L) = \mathcal{W}^{\perp}$; $\text{range}(L) = \mathcal{W}$

**30. (a)** F **(b)** F **(c)** F **(d)** T **(e)** F

## Chapter 7 Review Exercises (p. 327–329)

**1. (a)** 0

**(b)** $(1+2i)(\mathbf{v}\cdot\mathbf{z}) = ((1+2i)\mathbf{v})\cdot\mathbf{z} = -21+43i$, $\mathbf{v}\cdot((1+2i)\mathbf{z}) = 47-9i$

**4. (a)** $w = 4+3i$, $z = -2i$

**(d)** $\{[(2+i)-(3-i)c,\ (7+i)-ic,\ c] \mid c\in\mathbb{C}\} = \{[(2-3c)+(1+c)i, 7+(1-c)i, c] \mid c\in\mathbb{C}\}$

**6. (a)** $p_{\mathbf{A}}(x) = x^3 - x^2 + x - 1 = (x^2+1)(x-1) = (x-i)(x+i)(x-1)$; $\mathbf{D} = \begin{bmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 1 \end{bmatrix}$;

$$\mathbf{P} = \begin{bmatrix} -2-i & -2+i & 0 \\ 1-i & 1+i & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

**7. (a)** One possibility: Consider $L\colon \mathbb{C}\to\mathbb{C}$ given by $L(\mathbf{z}) = \overline{\mathbf{z}}$. Note that $L(\mathbf{v}+\mathbf{w}) = \overline{(\mathbf{v}+\mathbf{w})} = \overline{\mathbf{v}}+\overline{\mathbf{w}} = L(\mathbf{v})+L(\mathbf{w})$. But $L$ is not a linear operator on $\mathbb{C}$ because $L(i) = -i$, but $iL(1) = i(1) = i$, so the rule "$L(c\mathbf{v}) = cL(\mathbf{v})$" is not satisfied.

**(b)** The example given in part (a) *is* a linear operator on $\mathbb{C}$, thought of as a *real* vector space. In that case we may use only real scalars, and so, if $\mathbf{v} = a+bi$, then $L(c\mathbf{v}) = L(ca+cbi) = ca - cbi = c(a-bi) = cL(\mathbf{v})$.

**8. (a)** $B = \{[1, i, 1, -i], [4+5i, 7-4i, i, 1], [10, -2+6i, -8-i, -1+8i], [0, 0, 1, i]\}$

**9. (a)** $p_{\mathbf{A}}(x) = x^2 - 5x + 4 = (x-1)(x-4)$; $\mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$; $\mathbf{P} = \begin{bmatrix} \frac{1}{\sqrt{6}}(-1-i) & \frac{1}{\sqrt{3}}(1+i) \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

**10.** Show that $\mathbf{A}$ is normal, and then apply Theorem 7.9.

**13.** Distance $= \sqrt{\frac{8}{105}} \approx 0.276$

**14.** $\{[1, 0, 0], [4, 3, 0], [5, 4, 2]\}$

**16.**

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| (a) T | (d) F | (g) F | (j) T | (m) F | (p) T | (s) T | (v) T |
| (b) F | (e) F | (h) T | (k) T | (n) T | (q) T | (t) T | (w) F |
| (c) T | (f) T | (i) F | (l) T | (o) T | (r) T | (u) T | |

## Section 8.1 (p. 336–340)

**1.** Symmetric: (a), (b), (c), (d), (g)

**(a)** Matrix for $G_1$: $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

**(b)** Matrix for $G_2$: $\begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix}$

**(c)** Matrix for $G_3$: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

**(d)** Matrix for $G_4$: $\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$

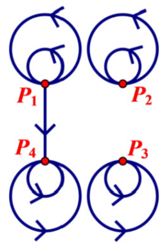
**(e)** Matrix for $D_1$: $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$

(f) Matrix for $D_2$: $\begin{bmatrix} 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

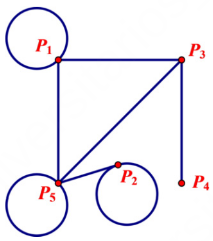
(g) Matrix for $D_3$: $\begin{bmatrix} 0 & 2 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 2 & 0 \end{bmatrix}$

(h) Matrix for $D_4$: $\begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

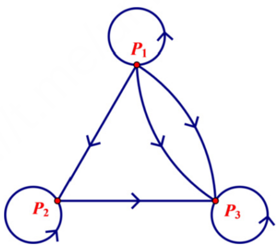
**2.** **C** can be the adjacency matrix for a digraph (only).

Digraph for **C** =


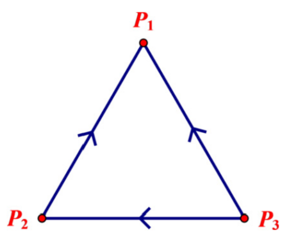

**F** can be the adjacency matrix for either a graph or digraph.

Graph for **F** =


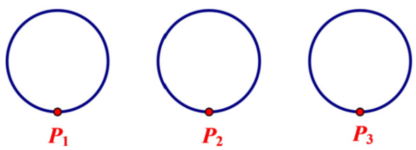

**G** can be the adjacency matrix for a digraph (only).

Digraph for **G** =


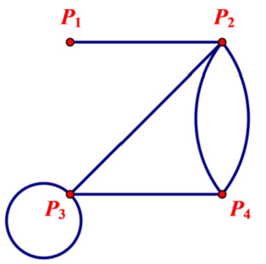

**H** can be the adjacency matrix for a digraph (only).

Digraph for **H** =



**I** can be the adjacency matrix for a graph or digraph.

Graph for **I** =


**J** can be the adjacency matrix for a graph or digraph.

Graph for $\mathbf{J} =$ 


**K** can be the adjacency matrix for a graph or digraph.

Graph for $\mathbf{K} =$ 


**4. (a)** 15 **(c)** $74 = 1 + 2 + 15 + 56$ **(e)** Length 2

**5. (a)** 4 **(c)** $7 = 1 + 2 + 4$ **(e)** No such path exists.

**6. (a)** 8 **(c)** $92 = 0 + 6 + 12 + 74$

**7. (a)** 3 **(c)** $19 = 1 + 2 + 4 + 12$

**8. (a)** If the vertex is the $i$th vertex, then the $i$th row and $i$th column entries of the adjacency matrix all equal 0, except possibly for the $(i, i)$ entry.

**9. (a)** The trace equals the total number of loops in the graph or digraph.

**10. (a)** $\mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 = \begin{bmatrix} 26 & 0 & 20 & 28 \\ 0 & 3 & 0 & 0 \\ 20 & 0 & 14 & 24 \\ 28 & 0 & 24 & 20 \end{bmatrix}$; disconnected **(b)** $\mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 = \begin{bmatrix} 6 & 2 & 5 & 1 \\ 2 & 2 & 4 & 3 \\ 5 & 4 & 3 & 1 \\ 1 & 3 & 1 & 1 \end{bmatrix}$; connected

**11. (a)** Since $G_2$ has the same edges as $G_1$, along with one new loop at each vertex, $G_2$ has the same number of edges connecting any two distinct vertices as $G_1$. Thus, the entries off the main diagonal of the adjacency matrices for the two graphs are the same. But $G_2$ has one more loop at each vertex than $G_1$ has. Hence, the entries on the main diagonal of the adjacency matrix for $G_2$ are all 1 larger than the entries of **A**, so the adjacency matrix for $G_2$ is found by adding $\mathbf{I}_n$ to **A**. This does not change any entries off the main diagonal, but adds 1 to every entry on the main diagonal.

**(b)** Connectivity only involves the existence of a path between *distinct* vertices. If there is a path from $P_i$ to $P_j$ in $G_1$, for $i \neq j$, the same path connects these two vertices in $G_2$. If there is a path from $P_i$ to $P_j$ in $G_2$, then a similar path can be found from $P_i$ to $P_j$ in $G_1$ merely by deleting any loops that might appear in the path from $P_i$ to $P_j$ in $G_2$. Thus, $P_i$ is connected by a path to $P_j$ in $G_1$ if and only if $P_i$ is connected by a path to $P_j$ in $G_2$. Hence, $G_1$ is connected if and only if $G_2$ is connected.

**(c)** Suppose there is a path of length $m$ from $P_i$ to $P_j$ in $G_1$, where $m \leq k$. Then, we can find a path of length $k$ from $P_i$ to $P_j$ in $G_2$ by first following the known path of length $m$ from $P_i$ to $P_j$ that exists in $G_1$ (and hence in $G_2$), and then going $(k - m)$ times around the newly added loop at $P_j$.

**13. (a)** Digraph $D_2$ in Fig. 8.2 is strongly connected; digraph in Fig. 8.7 is not strongly connected

**(d)** Digraph for **A** is not weakly connected; digraph for **B** is weakly connected.

**14. (b)** Yes, it is a dominance digraph because no tie games are possible and because each team plays every other team. Thus, if $P_i$ and $P_j$ are two given teams, either $P_i$ defeats $P_j$ or vice versa.

**16. (a)** T **(b)** F **(c)** T **(d)** F **(e)** T **(f)** T **(g)** T **(h)** F **(i)** F **(j)** F **(k)** F **(l)** F **(m)** T

## Section 8.2 (p. 342)

**1. (a)** $I_1 = 8, I_2 = 5, I_3 = 3$ **(c)** $I_1 = 12, I_2 = 5, I_3 = 3, I_4 = 2, I_5 = 2, I_6 = 7$

**2. (a)** T **(b)** T

## Section 8.3 (p. 347–348)

**1.** **(a)** $y = -0.8x - 3.3,\ y = -7.3$ when $x = 5$ **(c)** $y = -1.5x + 3.8,\ y = -3.7$ when $x = 5$

**2.** **(a)** $y = 0.375x^2 + 0.35x + 3.60$ **(c)** $y = -0.042x^2 + 0.633x + 0.266$

**3.** **(a)** $y = \frac{1}{4}x^3 + \frac{25}{28}x^2 + \frac{25}{14}x + \frac{37}{35}$

**4.** **(a)** $y = 4.4286x^2 - 2.0571$ **(e)** $y = 0x^3 - 0.3954x^2 + 0.9706$
**(c)** $y = -0.1014x^2 + 0.9633x - 0.8534$

**5.** **(a)** $y = 0.2x + 2.74$; the angle reaches 4.34° in the 8th month
**(c)** $y = 0.007143x^2 + 0.1286x + 2.8614$; the tower will be leaning at 5.43° in the 12th month

**7.** The least-squares polynomial is $y = \frac{4}{5}x^2 - \frac{2}{5}x + 2$, which is the *exact* quadratic through the three given points.

**9.** **(a)** $x_1 = \frac{230}{39},\ x_2 = \frac{155}{39};\ \begin{cases} 4x_1 - 3x_2 = 11\frac{2}{3}, \text{ which is almost } 12 \\ 2x_1 + 5x_2 = 31\frac{2}{3}, \text{ which is almost } 32 \\ 3x_1 + \ \ x_2 = 21\frac{2}{3}, \text{ which is close to } 21 \end{cases}$

**10.** **(a)** T **(b)** F **(c)** F **(d)** F

## Section 8.4 (p. 354–356)

**1.** **A** is not a stochastic matrix, since **A** is not square. (However, **A** is a stochastic vector.) **A** is not regular, since **A** is not a stochastic matrix.
**B** is not stochastic, since the entries of column 2 do not sum to 1; **B** is not regular, since **B** is not stochastic.
**C** is stochastic; **C** is regular, since **C** is stochastic and has all nonzero entries.
**D** is stochastic; **D** is not regular, since every positive power of **D** is a matrix whose rows are the rows of **D** rearranged in some order, and hence every such power contains zero entries.
**E** is not stochastic, since the entries of column 1 do not sum to 1; **E** is not regular, since **E** is not stochastic.
**F** is stochastic; **F** is not regular, since every positive power of **F** has all second row entries zero.
**G** is not stochastic, since **G** is not square; **G** is not regular, since **G** is not stochastic.
**H** is stochastic; **H** is regular, since **H** is stochastic and $\mathbf{H}^2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$, which has all nonzero entries.

**2.** **(a)** $\mathbf{p}_1 = \left[\frac{5}{18}, \frac{13}{18}\right],\ \mathbf{p}_2 = \left[\frac{67}{216}, \frac{149}{216}\right]$ **(c)** $\mathbf{p}_1 = \left[\frac{17}{48}, \frac{1}{3}, \frac{5}{16}\right],\ \mathbf{p}_2 = \left[\frac{205}{576}, \frac{49}{144}, \frac{175}{576}\right]$

**3.** **(a)** $\left[\frac{2}{5}, \frac{3}{5}\right]$

**5.** **(a)** [0.34, 0.175, 0.34, 0.145] in the next election; [0.3555, 0.1875, 0.2875, 0.1695] in the election after that
**(b)** The steady-state vector is [0.36, 0.20, 0.24, 0.20]. After a century, the votes would be 36% for Party A and 24% for Party C.

**6.** **(a)** $\mathbf{M} = \begin{bmatrix} \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & \frac{1}{5} & 0 \\ \frac{1}{8} & \frac{1}{2} & 0 & 0 & \frac{1}{5} \\ \frac{1}{8} & 0 & \frac{1}{2} & \frac{1}{10} & \frac{1}{10} \\ \frac{1}{4} & 0 & \frac{1}{6} & \frac{1}{2} & \frac{1}{5} \\ 0 & \frac{1}{3} & \frac{1}{6} & \frac{1}{5} & \frac{1}{2} \end{bmatrix}$
**(b)** $\mathbf{M}^2$ has all nonzero entries
**(c)** $\frac{29}{120}$, since the probability vector after 2 time intervals is $\left[\frac{1}{5}, \frac{13}{240}, \frac{73}{240}, \frac{29}{120}, \frac{1}{5}\right]$
**(d)** $\left[\frac{1}{5}, \frac{3}{20}, \frac{3}{20}, \frac{1}{4}, \frac{1}{4}\right]$; Over time, the rat frequents rooms B and C the least, and rooms D and E the most.

**11.** **(a)** F **(b)** T **(c)** T **(d)** T **(e)** F

## Section 8.5 (p. 358–359)

**1. (a)** −24 −46 −15 −30 10 16 39 62 26 42 51 84 24 37 −11 −23

**2. (a)** WHO IS BURIED IN GRANT(')S TOMB –

**3. (a)** T **(b)** T **(c)** F

## Section 8.6 (p. 366–369)

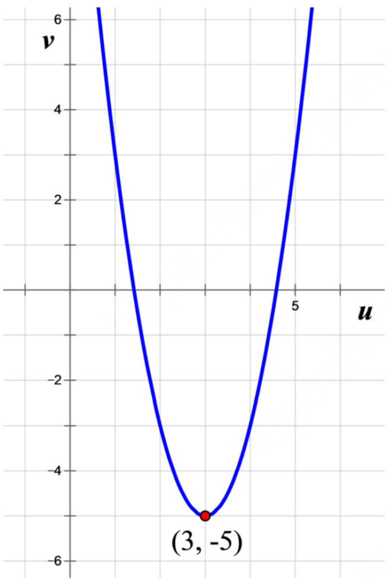
**1. (a)** $a_n = 3^{n-1} + 2^n$; $a_1 = 3$, $a_2 = 7$, $a_3 = 17$, $a_4 = 43$, $a_5 = 113$

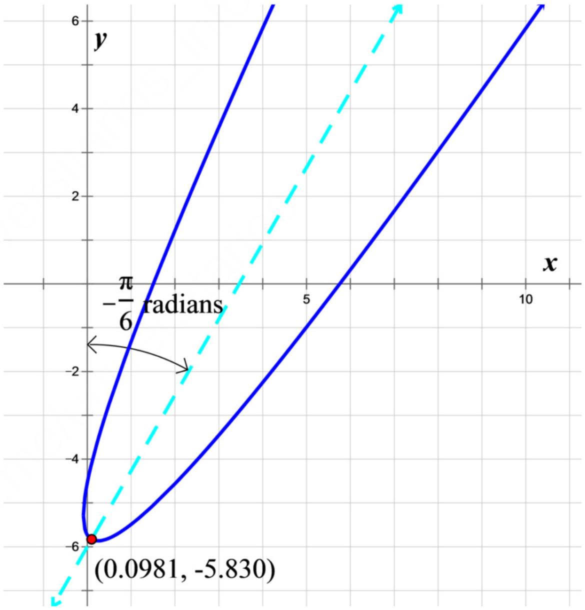
**(d)** $a_n = \left(\frac{1}{20}\right) 22 + 30(-3)^{n-1} - 12(-4)^{n-1}$; $a_1 = 2$, $a_2 = -1$, $a_3 = 5$, $a_4 = -1$, $a_5 = -31$

**10. (a)** F **(b)** T **(c)** F **(d)** T **(e)** T **(f)** T **(g)** T

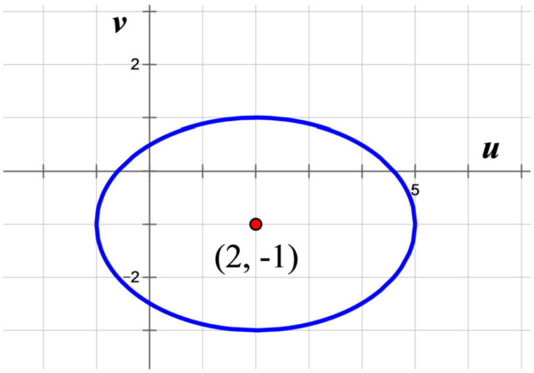
## Section 8.7 (p. 373)

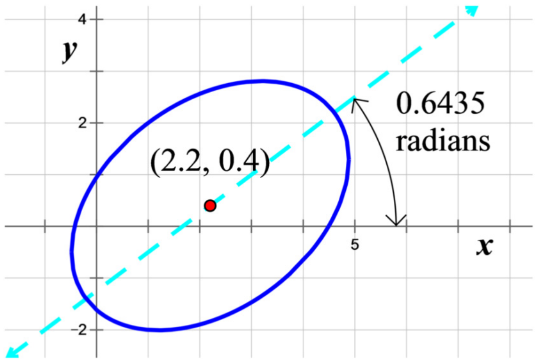
**1. (c)** $\theta = \frac{1}{2}\arctan(-\sqrt{3}) = -\frac{\pi}{6}$; $\mathbf{P} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$; equation in $uv$-coordinates: $v = 2u^2 - 12u + 13$, or, $(v+5) = 2(u-3)^2$; vertex in $uv$-coordinates: $(3, -5)$; vertex in $xy$-coordinates: $(0.0981, -5.830)$ (see accompanying figures)



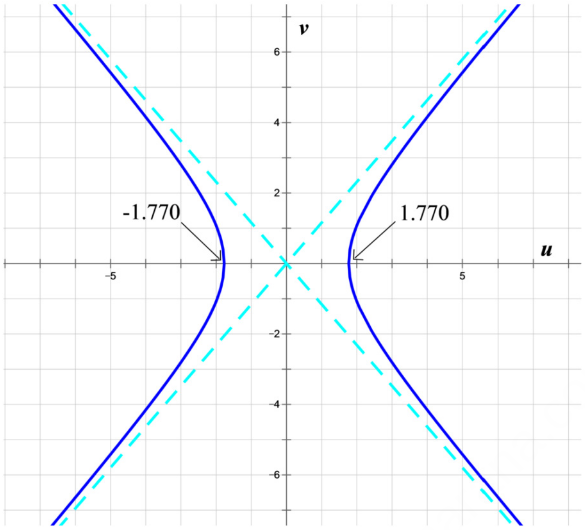



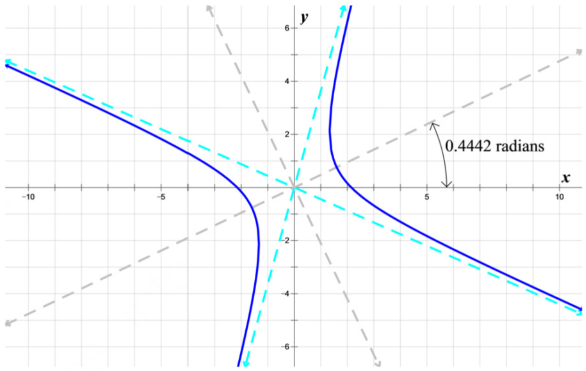

**(d)** $\theta \approx 0.6435$ radians (about 36°52′); $\mathbf{P} = \frac{1}{5}\begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}$; equation in $uv$-coordinates: $\frac{(u-2)^2}{9} + \frac{(v+1)^2}{4} = 1$; center in $uv$-coordinates $= (2, -1)$; center in $xy$-coordinates $= \left(\frac{11}{5}, \frac{2}{5}\right)$ (see accompanying figures)



**(f)** All answers rounded to 4 significant digits: $\theta \approx 0.4442$ radians (about 25°27′); $\mathbf{P} = \begin{bmatrix} 0.9029 & -0.4298 \\ 0.4298 & 0.9029 \end{bmatrix}$; equation in $uv$-coordinates: $\frac{u^2}{(1.770)^2} - \frac{v^2}{(2.050)^2} = 1$; center in $uv$-coordinates: $(0, 0)$; center in $xy$-coordinates $= (0, 0)$ (see accompanying figures)



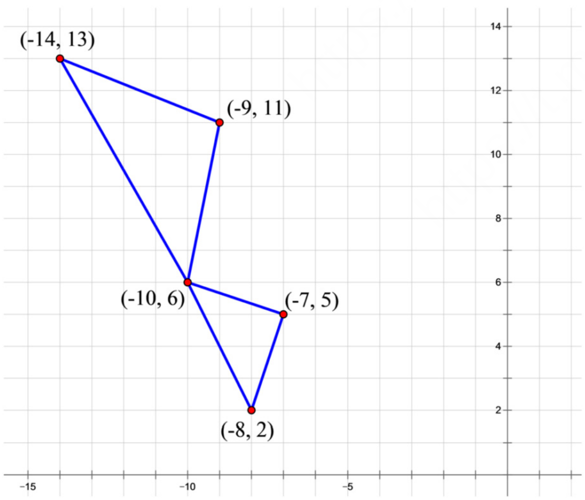



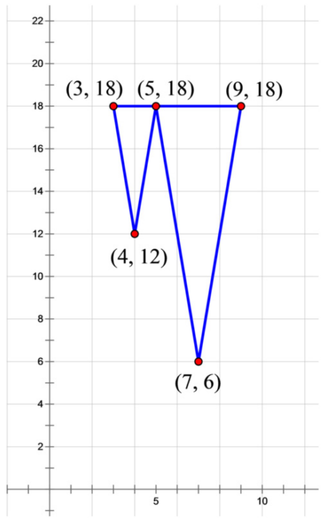

**2. (a)** T **(b)** F **(c)** T **(d)** F

## Section 8.8 (p. 381–385)

**1. (a)** (9, 1), (9, 5), (12, 1), (12, 5), (14, 3) **(c)** (−2, 5), (0, 9), (−5, 7), (−2, 10), (−5, 10)

**2. (b)** (−8, 2), (−7, 5), (−10, 6), (−9, 11), (−14, 13) (see accompanying figure) **(d)** (3, 18), (4, 12), (5, 18), (7, 6), (9, 18) (see accompanying figure)



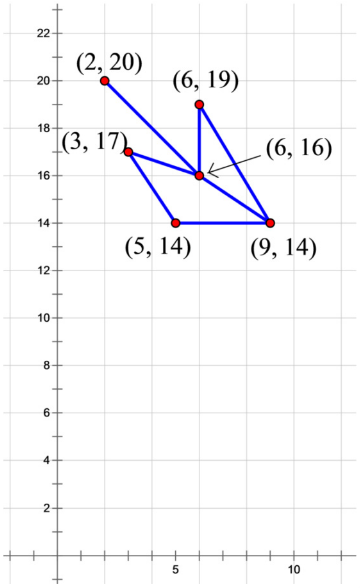




**3. (a)** (3, −4), (3, −10), (7, −6), (9, −9), (10, −3) **(c)** (−2, 6), (0, 8), (−8, 17), (−10, 22), (−16, 25)

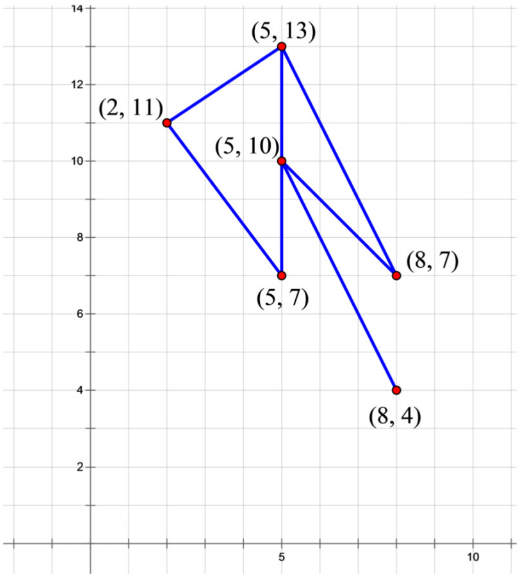
**4. (a)** (14, 9), (10, 6), (11, 11), (8, 9), (6, 8), (11, 14) **(c)** (2, 4), (2, 6), (8, 5), (8, 6), (8, 6), (14, 4)

**5. (b)** (0, 5), (1, 7), (0, 11), (−5, 8), (−4, 10)

**6. (a)** (2, 20), (3, 17), (5, 14), (6, 19), (6, 16), (9, 14) (see accompanying figure)




**(c)** (8, 4), (5, 7), (2, 11), (8, 7), (5, 10), (5, 13) (see accompanying figure)




**11. (b)** Consider the reflection about the $y$-axis and a counterclockwise rotation of 90° about the origin. Starting from the point $(1, 0)$, performing the rotation and then the reflection yields $(0, 1)$. However, performing the reflection followed by the rotation produces $(0, -1)$. Hence, the two transformations do not commute.

**13. (a)** F **(b)** F **(c)** F **(d)** T **(e)** T **(f)** F

## Section 8.9 (p. 390–391)

**1. (a)** $b_1 e^{t}\begin{bmatrix}7\\3\end{bmatrix} + b_2 e^{-t}\begin{bmatrix}2\\1\end{bmatrix}$

**(c)** $b_1\begin{bmatrix}0\\-1\\1\end{bmatrix} + b_2 e^{t}\begin{bmatrix}1\\-1\\1\end{bmatrix} + b_3 e^{3t}\begin{bmatrix}2\\0\\1\end{bmatrix}$

**(d)** $b_1 e^{t}\begin{bmatrix}6\\-1\\2\end{bmatrix} + b_2 e^{t}\begin{bmatrix}1\\-1\\0\end{bmatrix} + b_3 e^{4t}\begin{bmatrix}1\\1\\1\end{bmatrix}$ (There are other possible answers. For example, the first two vectors in the sum could be any basis for the two-dimensional eigenspace corresponding to the eigenvalue 1.)

**2. (a)** $y = b_1 e^{2t} + b_2 e^{-3t}$ **(c)** $y = b_1 e^{2t} + b_2 e^{-2t} + b_3 e^{\left(\sqrt{2}\right)t} + b_4 e^{-\left(\sqrt{2}\right)t}$

**4. (b)** $\mathbf{F}(t) = 2e^{5t}\begin{bmatrix}1\\0\\1\end{bmatrix} + e^{t}\begin{bmatrix}1\\-2\\0\end{bmatrix} - 2e^{-t}\begin{bmatrix}1\\1\\1\end{bmatrix}$

**7. (a)** T **(b)** T **(c)** T **(d)** F

## Section 8.10 (p. 395–396)

**1. (a)** Unique least-squares solution: $\mathbf{v} = \left[\frac{23}{30}, \frac{11}{10}\right]$; $||\mathbf{A}\mathbf{v} - \mathbf{b}|| = \frac{\sqrt{6}}{6} \approx 0.408$; $||\mathbf{A}\mathbf{z} - \mathbf{b}|| = 1$

**(c)** Infinite number of least-squares solutions, all of the following form: $\left[7c + \frac{17}{3}, -13c - \frac{23}{3}, c\right]$. Two particular least-squares solutions are $\left[\frac{17}{3}, -\frac{23}{3}, 0\right]$ and $\left[8, -12, \frac{1}{3}\right]$. Also, with $\mathbf{v}$ as either of these vectors, $||\mathbf{A}\mathbf{v} - \mathbf{b}|| = \frac{\sqrt{6}}{3} \approx 0.816$; $||\mathbf{A}\mathbf{z} - \mathbf{b}|| = 3$.

**2. (a)** Infinite number of least-squares solutions, all of the form $\left[-\frac{4}{7}c + \frac{19}{42}, \frac{8}{7}c - \frac{5}{21}, c\right]$, with $\frac{5}{24} \le c \le \frac{19}{24}$.

**3. (b)** $\mathbf{v} \approx [0.46, -0.36, 0.90]$; $(\lambda'\mathbf{I} - \mathbf{C})\mathbf{v} \approx [0.03, -0.04, 0.07]$.

**7. (a)** F **(b)** T **(c)** T **(d)** T **(e)** T

## Section 8.11 (p. 400)

**1. (a)** $\mathbf{C} = \begin{bmatrix} 8 & 12 \\ 0 & -9 \end{bmatrix}$; $\mathbf{A} = \begin{bmatrix} 8 & 6 \\ 6 & -9 \end{bmatrix}$

**(c)** $\mathbf{C} = \begin{bmatrix} 5 & 4 & -3 \\ 0 & -2 & 5 \\ 0 & 0 & 0 \end{bmatrix}$; $\mathbf{A} = \begin{bmatrix} 5 & 2 & -\frac{3}{2} \\ 2 & -2 & \frac{5}{2} \\ -\frac{3}{2} & \frac{5}{2} & 0 \end{bmatrix}$

**2. (a)** $\mathbf{A} = \begin{bmatrix} 43 & -24 \\ -24 & 57 \end{bmatrix}$; $\mathbf{P} = \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}$; $\mathbf{D} = \begin{bmatrix} 75 & 0 \\ 0 & 25 \end{bmatrix}$;

$B = \left(\frac{1}{5}[-3, 4], \frac{1}{5}[4, 3]\right)$; $[\mathbf{x}]_B = [-7, -4]$; $Q(\mathbf{x}) = 4075$

**(c)** $\mathbf{A} = \begin{bmatrix} 18 & 48 & -30 \\ 48 & -68 & 18 \\ -30 & 18 & 1 \end{bmatrix}$; $\mathbf{P} = \frac{1}{7}\begin{bmatrix} 2 & -6 & 3 \\ 3 & -2 & -6 \\ 6 & 3 & 2 \end{bmatrix}$; $\mathbf{D} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 49 & 0 \\ 0 & 0 & -98 \end{bmatrix}$;

$B = \left(\frac{1}{7}[2, 3, 6], \frac{1}{7}[-6, -2, 3], \frac{1}{7}[3, -6, 2]\right)$; $[\mathbf{x}]_B = [5, 0, 6]$; $Q(\mathbf{x}) = -3528$

**4.** Yes. If $Q(\mathbf{x}) = \sum_{1 \le i \le j \le n} a_{ij}x_ix_j$, then $Q(\mathbf{x}) = \mathbf{x}^T\mathbf{C}_1\mathbf{x}$ and $\mathbf{C}_1$ upper triangular imply that the $(i, j)$ entry for $\mathbf{C}_1$ is 0 if $i > j$ and $a_{ij}$ if $i \le j$. A similar argument describes $\mathbf{C}_2$. Thus $\mathbf{C}_1 = \mathbf{C}_2$.

**6. (a)** T **(b)** F **(c)** F **(d)** T **(e)** T

## Section 9.1 (p. 408–409)

**1. (a)** Solution to first system: (602, 1500); solution to second system: (302, 750). The system is ill-conditioned because a very small change in the coefficient of $y$ leads to a very large change in the solution.

**2.** Answers to this problem may differ significantly from the following depending on where rounding is done in the algorithm.

**(a)** Without partial pivoting: (3210, 0.765); with partial pivoting: (3230, 0.767). (Actual solution is (3214, 0.765).)

**(c)** Without partial pivoting: (2.26, 1.01, −2.11); with partial pivoting: (277, −327, 595). (Actual solution is (267, −315, 573).)

**3.** Answers to this problem may differ significantly from the following depending on where rounding is done in the algorithm.

**(a)** Without partial pivoting: (3214, 0.7651); with partial pivoting: (3213, 0.7648). (Actual solution is (3214, 0.765).)

**(c)** Without partial pivoting: (−2.380, 8.801, −16.30); with partial pivoting: (267.8, −315.9, 574.6). (Actual solution is (267, −315, 573).)

**4. (a)**

| | $x_1$ | $x_2$ |
|---|---|---|
| Initial Values | 0.000 | 0.000 |
| After 1 Step | 5.200 | −6.000 |
| After 2 Steps | 6.400 | −8.229 |
| After 3 Steps | 6.846 | −8.743 |
| After 4 Steps | 6.949 | −8.934 |
| After 5 Steps | 6.987 | −8.978 |
| After 6 Steps | 6.996 | −8.994 |
| After 7 Steps | 6.999 | −8.998 |
| After 8 Steps | 7.000 | −9.000 |
| After 9 Steps | 7.000 | −9.000 |

**(c)**

| | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|
| Initial Values | 0.000 | 0.000 | 0.000 |
| After 1 Step | −8.857 | 4.500 | −4.333 |
| After 2 Steps | −10.738 | 3.746 | −8.036 |
| After 3 Steps | −11.688 | 4.050 | −8.537 |
| After 4 Steps | −11.875 | 3.975 | −8.904 |
| After 5 Steps | −11.969 | 4.005 | −8.954 |
| After 6 Steps | −11.988 | 3.998 | −8.991 |
| After 7 Steps | −11.997 | 4.001 | −8.996 |
| After 8 Steps | −11.999 | 4.000 | −8.999 |
| After 9 Steps | −12.000 | 4.000 | −9.000 |
| After 10 Steps | −12.000 | 4.000 | −9.000 |

**5. (a)**

| | $x_1$ | $x_2$ |
|---|---|---|
| Initial Values | 0.000 | 0.000 |
| After 1 Step | 5.200 | −8.229 |
| After 2 Steps | 6.846 | −8.934 |
| After 3 Steps | 6.987 | −8.994 |
| After 4 Steps | 6.999 | −9.000 |
| After 5 Steps | 7.000 | −9.000 |
| After 6 Steps | 7.000 | −9.000 |

**(c)**

| | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|
| Initial Values | 0.000 | 0.000 | 0.000 |
| After 1 Step | −8.857 | 3.024 | −7.790 |
| After 2 Steps | −11.515 | 3.879 | −8.818 |
| After 3 Steps | −11.931 | 3.981 | −8.974 |
| After 4 Steps | −11.990 | 3.997 | −8.996 |
| After 5 Steps | −11.998 | 4.000 | −8.999 |
| After 6 Steps | −12.000 | 4.000 | −9.000 |
| After 7 Steps | −12.000 | 4.000 | −9.000 |

**6.** Strictly diagonally dominant: (a), (c)

**7. (a)** Put the third equation first, and move the other two down to get the following:

| | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|
| Initial Values | 0.000 | 0.000 | 0.000 |
| After 1 Step | 3.125 | −0.481 | 1.461 |
| After 2 Steps | 2.517 | −0.500 | 1.499 |
| After 3 Steps | 2.500 | −0.500 | 1.500 |
| After 4 Steps | 2.500 | −0.500 | 1.500 |

**(c)** Put the second equation first, the fourth equation second, the first equation third, and the third equation fourth to get the following:

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|---|---|---|---|---|
| Initial Values | 0.000 | 0.000 | 0.000 | 0.000 |
| After 1 Step | 5.444 | −5.379 | 9.226 | −10.447 |
| After 2 Steps | 8.826 | −8.435 | 10.808 | −11.698 |
| After 3 Steps | 9.820 | −8.920 | 10.961 | −11.954 |
| After 4 Steps | 9.973 | −8.986 | 10.994 | −11.993 |
| After 5 Steps | 9.995 | −8.998 | 10.999 | −11.999 |
| After 6 Steps | 9.999 | −9.000 | 11.000 | −12.000 |
| After 7 Steps | 10.000 | −9.000 | 11.000 | −12.000 |
| After 8 Steps | 10.000 | −9.000 | 11.000 | −12.000 |

**8.** Jacobi method yields the following:

| | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|
| Initial Values | 0.0 | 0.0 | 0.0 |
| After 1 Step | 16.0 | −13.0 | 12.0 |
| After 2 Steps | −37.0 | 59.0 | −87.0 |
| After 3 Steps | 224.0 | −61.0 | 212.0 |
| After 4 Steps | −77.0 | 907.0 | −1495.0 |
| After 5 Steps | 3056.0 | 2515.0 | −356.0 |
| After 6 Steps | 12235.0 | 19035.0 | −23895.0 |

Gauss-Seidel method yields the following:

| | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|
| Initial Values | 0.0 | 0.0 | 0.0 |
| After 1 Step | 16.0 | 83.0 | −183.0 |
| After 2 Steps | 248.0 | 1841.0 | −3565.0 |
| After 3 Steps | 5656.0 | 41053.0 | −80633.0 |
| After 4 Steps | 124648.0 | 909141.0 | −1781665.0 |

The actual solution is: (2, −3, 1).

**10. (a)** T **(b)** F **(c)** F **(d)** T **(e)** F **(f)** F

## Section 9.2 (p. 415)

**1. (a)** $\mathbf{LDU} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

**(c)** $\mathbf{LDU} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 4 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$

**(e)** $\mathbf{LDU} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{4}{3} & 1 & 0 & 0 \\ -2 & -\frac{3}{2} & 1 & 0 \\ \frac{2}{3} & -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & -\frac{2}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 & \frac{5}{2} & -\frac{11}{2} \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

**3. (a)** $\mathbf{KU} = \begin{bmatrix} -1 & 0 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix}$; the solution is $\{(4, -1)\}$.

**(c)** $\mathbf{KU} = \begin{bmatrix} -1 & 0 & 0 \\ 4 & 3 & 0 \\ -2 & 5 & -2 \end{bmatrix} \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}$; the solution is $\{(2, -3, 1)\}$.

**4. (a)** F **(b)** T **(c)** F **(d)** F

## Section 9.3 (p. 419–420)

**1. (a)** After 9 iterations, eigenvector = [0.60, 0.80], eigenvalue = 50

**(c)** After 7 iterations, eigenvector = [0.41, 0.41, 0.82], eigenvalue = 3.0

**(e)** After 15 iterations, eigenvector = [0.346, 0.852, 0.185, 0.346], eigenvalue = 5.405

**3. (b)** Let $\lambda_1, \ldots \lambda_n$ be the eigenvalues of $\mathbf{A}$ with $|\lambda_1| > |\lambda_j|$, for $2 \le j \le n$. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be as given in the exercise. Suppose the initial vector in the Power Method is $\mathbf{u}_0 = a_{01}\mathbf{v}_1 + \cdots + a_{0n}\mathbf{v}_n$ and the $i$th iteration yields $\mathbf{u}_i = a_{i1}\mathbf{v}_1 + \cdots + a_{in}\mathbf{v}_n$. A proof by induction shows that $\mathbf{u}_i = k_i\mathbf{A}^i\mathbf{u}_0$ for some nonzero constant $k_i$. Therefore, $\mathbf{u}_i = k_i a_{01}\mathbf{A}^i\mathbf{v}_1 + k_i a_{02}\mathbf{A}^i\mathbf{v}_2 + \cdots + k_i a_{0n}\mathbf{A}^i\mathbf{v}_n = k_i a_{01}\lambda_1^i\mathbf{v}_1 + k_i a_{02}\lambda_2^i\mathbf{v}_2 + \cdots + k_i a_{0n}\lambda_n^i\mathbf{v}_n$. Hence, $a_{ij} = k_i a_{0j}\lambda_j^i$. Thus, for $2 \le j \le n$, $\lambda_j \ne 0$, and $a_{0j} \ne 0$, we have

$$\frac{|a_{i1}|}{|a_{ij}|} = \frac{\left|k_i a_{01}\lambda_1^i\right|}{\left|k_i a_{0j}\lambda_j^i\right|} = \left|\frac{\lambda_1}{\lambda_j}\right|^i \frac{|a_{01}|}{|a_{0j}|}.$$

**4. (a)** F **(b)** T **(c)** T **(d)** F

## Section 9.4 (p. 425)

**1. (a)** $\mathbf{Q} = \frac{1}{3}\begin{bmatrix} 2 & 1 & -2 \\ -2 & 2 & -1 \\ 1 & 2 & 2 \end{bmatrix}$; $\mathbf{R} = \begin{bmatrix} 3 & 6 & 3 \\ 0 & 6 & -9 \\ 0 & 0 & 3 \end{bmatrix}$

**(c)** $\mathbf{Q} = \begin{bmatrix} \frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{6}}{3} & \frac{\sqrt{3}}{3} & 0 \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3} & -\frac{\sqrt{2}}{2} \end{bmatrix}$; $\mathbf{R} = \begin{bmatrix} \sqrt{6} & 3\sqrt{6} & -\frac{2\sqrt{6}}{3} \\ 0 & 2\sqrt{3} & -\frac{10\sqrt{3}}{3} \\ 0 & 0 & \sqrt{2} \end{bmatrix}$

**2. (a)** $x \approx 5.562$, $y \approx -2.142$ **(c)** $x \approx -0.565$, $y \approx 0.602$, $z \approx 0.611$

**5. (a)** T **(b)** T **(c)** T **(d)** F **(e)** T

## Section 9.5 (p. 437–439)

**1.** For each part, one possible answer is given.

**(a)** $\mathbf{U} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $\mathbf{\Sigma} = \begin{bmatrix} 2\sqrt{10} & 0 \\ 0 & \sqrt{10} \end{bmatrix}$, $\mathbf{V} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$

**(c)** $\mathbf{U} = \frac{1}{\sqrt{10}}\begin{bmatrix} -3 & 1 \\ 1 & 3 \end{bmatrix}$, $\mathbf{\Sigma} = \begin{bmatrix} 9\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{bmatrix}$, $\mathbf{V} = \frac{1}{3}\begin{bmatrix} -1 & -2 & 2 \\ -2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$

**(f)** $\mathbf{U} = \frac{1}{7}\begin{bmatrix} 6 & 2 & 3 \\ 3 & -6 & -2 \\ 2 & 3 & -6 \end{bmatrix}$, $\mathbf{\Sigma} = \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}$, $\mathbf{V} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

**2. (a)** $\mathbf{A}^+ = \frac{1}{2250}\begin{bmatrix} 104 & 70 & 122 \\ -158 & 110 & 31 \end{bmatrix}$, $\mathbf{v} = \frac{1}{2250}\begin{bmatrix} 5618 \\ 3364 \end{bmatrix}$, $\mathbf{A}^T\mathbf{A}\mathbf{v} = \mathbf{A}^T\mathbf{b} = \frac{1}{15}\begin{bmatrix} 6823 \\ 3874 \end{bmatrix}$

**(c)** $\mathbf{A}^+ = \frac{1}{84}\begin{bmatrix} 36 & 24 & 12 & 0 \\ 12 & 36 & -24 & 0 \\ -31 & -23 & 41 & 49 \end{bmatrix}$, $\mathbf{v} = \frac{1}{14}\begin{bmatrix} 44 \\ -18 \\ 71 \end{bmatrix}$, $\mathbf{A}^T\mathbf{A}\mathbf{v} = \mathbf{A}^T\mathbf{b} = \frac{1}{7}\begin{bmatrix} 127 \\ -30 \\ 60 \end{bmatrix}$

**3. (a)** $\mathbf{A} = 2\sqrt{10}\left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)\left(\frac{1}{\sqrt{5}}\begin{bmatrix} 2 & 1 \end{bmatrix}\right) + \sqrt{10}\left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)\left(\frac{1}{\sqrt{5}}\begin{bmatrix} -1 & 2 \end{bmatrix}\right)$

**(c)** $\mathbf{A} = 2\sqrt{2}\left(\frac{1}{7}\begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}\right)\left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \end{bmatrix}\right) + \sqrt{2}\left(\frac{1}{7}\begin{bmatrix} 2 \\ -6 \\ 3 \end{bmatrix}\right)\left(\frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \end{bmatrix}\right)$

**14. (a)** $\mathbf{A} = \begin{bmatrix} 40 & -5 & 15 & -15 & 5 & -30 \\ 1.8 & 3 & 1.2 & 1.2 & -0.6 & 1.8 \\ 50 & 5 & 45 & -45 & -5 & -60 \\ -2.4 & -1.5 & 0.9 & 0.9 & 3.3 & -2.4 \\ 42.5 & -2.5 & 60 & -60 & 2.5 & -37.5 \end{bmatrix}$

**(b)** $\mathbf{A}_1 = \begin{bmatrix} 25 & 0 & 25 & -25 & 0 & -25 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 50 & 0 & 50 & -50 & 0 & -50 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 50 & 0 & 50 & -50 & 0 & -50 \end{bmatrix}$,

$\mathbf{A}_2 = \begin{bmatrix} 35 & 0 & 15 & -15 & 0 & -35 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 55 & 0 & 45 & -45 & 0 & -55 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 40 & 0 & 60 & -60 & 0 & -40 \end{bmatrix}$,

$\mathbf{A}_3 = \begin{bmatrix} 40 & -5 & 15 & -15 & 5 & -30 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 50 & 5 & 45 & -45 & -5 & -60 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 42.5 & -2.5 & 60 & -60 & 2.5 & -37.5 \end{bmatrix}$,

$\mathbf{A}_4 = \begin{bmatrix} 40 & -5 & 15 & -15 & 5 & -30 \\ 1.8 & 1.8 & 0 & 0 & -1.8 & 1.8 \\ 50 & 5 & 45 & -45 & -5 & -60 \\ -2.4 & -2.4 & 0 & 0 & 2.4 & -2.4 \\ 42.5 & -2.5 & 60 & -60 & 2.5 & -37.5 \end{bmatrix}$

**(c)** $N(\mathbf{A}) \approx 153.85$; $N(\mathbf{A} - \mathbf{A}_1)/N(\mathbf{A}) \approx 0.2223$; $N(\mathbf{A} - \mathbf{A}_2)/N(\mathbf{A}) \approx 0.1068$; $N(\mathbf{A} - \mathbf{A}_3)/N(\mathbf{A}) \approx 0.0436$; $N(\mathbf{A} - \mathbf{A}_4)/N(\mathbf{A}) \approx 0.0195$

**(d)** The method described in the text for the compression of digital images takes the matrix describing the image and alters it by zeroing out some of the lower singular values. This exercise illustrates how the matrices $\mathbf{A}_i$ that use only the first $i$ singular values for a matrix $\mathbf{A}$ get closer to approximating $\mathbf{A}$ as $i$ increases.

**16.** **(a)** F **(c)** F **(e)** F **(g)** F **(i)** F **(k)** T
**(b)** T **(d)** F **(f)** T **(h)** T **(j)** T

## Appendix B (p. 452–453)

**1.** **(a)** Not a function; undefined for $x < 1$
**(c)** Not a function; two values assigned to each $x \neq 1$
**(e)** Not a function ($k$ is undefined at $\theta = \frac{\pi}{2}$)
**(f)** Function; range = all prime numbers; image of 2 is 2; pre-image of 2 = $\{0, 1, 2\}$

**2.** **(a)** $\{-15, -10, -5, 5, 10, 15\}$ **(c)** $\{\ldots, -8, -6, -4, -2, 0, 2, 4, 6, 8, \ldots\}$

**3.** $(g \circ f)(x) = \frac{1}{4}\sqrt{75x^2 - 30x + 35}$; $(f \circ g)(x) = \frac{1}{4}(5\sqrt{3x^2+2} - 1)$

**4.** $(g \circ f)\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -8 & 24 \\ 2 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$; $(f \circ g)\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -12 & 8 \\ -4 & 12 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

**8.** $f$ is not one-to-one because $f(x^2+1) = f(x^2+2) = 2x$; $f$ is not onto because there is no pre-image for $x^n$. The pre-image of $\mathcal{P}_2$ is $\mathcal{P}_3$.

**10.** $f$ is one-to-one because if $f(\mathbf{A}_1) = f(\mathbf{A}_2)$, then $\mathbf{B}(f(\mathbf{A}_1))\mathbf{B}^{-1} = \mathbf{B}(f(\mathbf{A}_2))\mathbf{B}^{-1} \Longrightarrow \mathbf{B}\mathbf{B}^{-1}\mathbf{A}_1\mathbf{B}\mathbf{B}^{-1} = \mathbf{B}\mathbf{B}^{-1}\mathbf{A}_2\mathbf{B}\mathbf{B}^{-1} \Longrightarrow \mathbf{A}_1 = \mathbf{A}_2$. $f$ is onto because, for any $\mathbf{C} \in \mathcal{M}_{nn}$, $f(\mathbf{BCB}^{-1}) = \mathbf{B}^{-1}(\mathbf{BCB}^{-1})\mathbf{B} = \mathbf{C}$. Finally, $f^{-1}(\mathbf{A}) = \mathbf{BAB}^{-1}$.

**12.** **(a)** F **(b)** T **(c)** F **(d)** F **(e)** F **(f)** F **(g)** F **(h)** F

## Appendix C (p. 457)

**1.** **(a)** $11 - i$
**(c)** $20 - 12i$
**(e)** $9 + 19i$
**(g)** $-17 - 19i$
**(i)** $9 + 2i$
**(k)** $16 + 22i$
**(m)** $\sqrt{53}$

**2.** **(a)** $\frac{3}{20} + \frac{1}{20}i$ **(c)** $-\frac{4}{17} - \frac{1}{17}i$

**5.** **(a)** F **(b)** F **(c)** T **(d)** T **(e)** F

## Appendix D (p. 462–463)

**1.** **(a)** (III): $\langle 2\rangle \leftrightarrow \langle 3\rangle$; inverse operation is (III): $\langle 2\rangle \leftrightarrow \langle 3\rangle$. The matrix is its own inverse.

**(b)** (I): $\langle 2\rangle \leftarrow -2\,\langle 2\rangle$; inverse operation is (I): $\langle 2\rangle \leftarrow -\frac{1}{2}\,\langle 2\rangle$. The inverse matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

**(e)** (II): $\langle 3\rangle \leftarrow -2\,\langle 4\rangle + \langle 3\rangle$; inverse operation is (II): $\langle 3\rangle \leftarrow 2\,\langle 4\rangle + \langle 3\rangle$. The inverse matrix is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

**2.** **(a)** $\begin{bmatrix} 4 & 9 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{bmatrix}\begin{bmatrix} 1 & \frac{9}{4} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

**(d)** The product of the following matrices in the order listed:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{5}{3} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & \frac{2}{3} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{6} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

**10. (a)** T **(b)** F **(c)** F **(d)** T **(e)** T

# Index

## D

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## Equivalent Conditions for Singular and Nonsingular Matrices

Let **A** be an $n \times n$ matrix. Any pair of statements in the same column are equivalent.

| | |
|---|---|
| **A** is singular ($\mathbf{A}^{-1}$ does not exist). | **A** is nonsingular ($\mathbf{A}^{-1}$ exists). |
| Rank$(\mathbf{A}) \neq n$. | Rank$(\mathbf{A}) = n$. |
| $\lvert\mathbf{A}\rvert = 0$. | $\lvert\mathbf{A}\rvert \neq 0$. |
| **A** is *not* row equivalent to $\mathbf{I}_n$. | **A** is row equivalent to $\mathbf{I}_n$. |
| $\mathbf{AX} = \mathbf{O}$ has a *nontrivial* solution for **X**. | $\mathbf{AX} = \mathbf{O}$ has only the trivial solution for **X**. |
| $\mathbf{AX} = \mathbf{B}$ does *not* have a unique solution (no solutions or infinitely many solutions). | $\mathbf{AX} = \mathbf{B}$ has a *unique* solution for **X** (namely, $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$). |
| The rows of **A** do *not* form a basis for $\mathbb{R}^n$. | The rows of **A** form a basis for $\mathbb{R}^n$. |
| The columns of **A** do *not* form a basis for $\mathbb{R}^n$. | The columns of **A** form a basis for $\mathbb{R}^n$. |
| The linear operator $L\colon \mathbb{R}^n \to \mathbb{R}^n$ given by $L(\mathbf{X}) = \mathbf{AX}$ is *not* an isomorphism. | The linear operator $L\colon \mathbb{R}^n \to \mathbb{R}^n$ given by $L(\mathbf{X}) = \mathbf{AX}$ is an isomorphism. |

## Diagonalization Method

To diagonalize (if possible) an $n \times n$ matrix **A**:

**Step 1:** Calculate $p_{\mathbf{A}}(x) = |x\mathbf{I}_n - \mathbf{A}|$.
**Step 2:** Find all real roots of $p_{\mathbf{A}}(x)$ (that is, all real solutions to $p_{\mathbf{A}}(x) = 0$).
These are the eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k$ for **A**.
**Step 3:** For each eigenvalue $\lambda_m$ in turn:
Row reduce the augmented matrix $[\lambda_m \mathbf{I}_n - \mathbf{A} \mid \mathbf{0}]$ to obtain the fundamental solutions of the homogeneous system $(\lambda_m \mathbf{I}_n - \mathbf{A})\mathbf{X} = \mathbf{0}$. (These are found by setting each independent variable in turn equal to 1 while setting all other independent variables equal to 0.) These fundamental solutions are the fundamental eigenvectors for $\lambda_m$.
**Step 4:** If after repeating Step 3 for each eigenvalue, you have less than $n$ fundamental eigenvectors overall for **A**, then **A** cannot be diagonalized. Stop.
**Step 5:** Otherwise, form a matrix **P** whose columns are these $n$ fundamental eigenvectors.
**Step 6:** Verify that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$ is a diagonal matrix whose $d_{ii}$ entry is the eigenvalue for the fundamental eigenvector forming the $i$th column of **P**.
Also note that $\mathbf{A} = \mathbf{PDP}^{-1}$.

## Simplified Span Method (Simplifying Span($S$))

Suppose that $S$ is a finite subset of $\mathbb{R}^n$ containing $k$ vectors, with $k \geq 2$.
To find a simplified form for span($S$), perform the following steps:

**Step 1:** Form a $k \times n$ matrix **A** by using the vectors in $S$ as the rows of **A**.
(Thus, span($S$) is the row space of **A**.)
**Step 2:** Let **C** be the reduced row echelon form matrix for **A**.
**Step 3:** Then, a simplified form for span($S$) is given by the set of all linear combinations of the *nonzero* rows of **C**.

## Independence Test Method (Testing for Linear Independence of $S$)

Let $S$ be a finite nonempty set of vectors in $\mathbb{R}^n$.
To determine whether $S$ is linearly independent, perform the following steps:

**Step 1:** Create the matrix **A** whose *columns* are the vectors in $S$.
**Step 2:** Find **B**, the reduced row echelon form of **A**.
**Step 3:** If there is a pivot in every column of **B**, then $S$ is linearly independent.
Otherwise, $S$ is linearly dependent.

## Equivalent Conditions for Linearly Independent and Linearly Dependent Sets

Let $S$ be a (possibly infinite) set of vectors in a vector space. Any pair of statements in the same column are equivalent.

| $S$ is linearly independent | $S$ is linearly dependent |
|---|---|
| If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subseteq S$ and $a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = \mathbf{0}$, then $a_1 = a_2 = \cdots = a_n = 0$. (The zero vector requires zero coefficients.) | There is a subset $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of $S$ such that $a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = \mathbf{0}$, for some scalars $a_1, a_2, \ldots, a_n$, with some $a_i \neq 0$. (The zero vector does not require all coefficients to be 0.) |
| *No* vector in $S$ is a finite linear combination of *other* vectors in $S$. | *Some* vector in $S$ is a finite linear combination of *other* vectors in $S$. |
| For every $\mathbf{v} \in S$, we have $\mathbf{v} \notin \text{span}(S - \{\mathbf{v}\})$. | There is a $\mathbf{v} \in S$ such that $\mathbf{v} \in \text{span}(S - \{\mathbf{v}\})$. |
| For every $\mathbf{v} \in S$, $\text{span}(S - \{\mathbf{v}\})$ does *not* contain all the vectors of $\text{span}(S)$. | There is some $\mathbf{v} \in S$ such that $\text{span}(S - \{\mathbf{v}\}) = \text{span}(S)$. |
| If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, then $\mathbf{v}_1 \neq \mathbf{0}$, and, for each $k \geq 2$, $\mathbf{v}_k \notin \text{span}(\{\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}\})$. (*Each* $\mathbf{v}_k$ is *not* a linear combination of the *previous* vectors in $S$.) | If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, then $\mathbf{v}_1 = \mathbf{0}$, or, for some $k \geq 2$, $\mathbf{v}_k = a_1\mathbf{v}_1 + \cdots + a_{k-1}\mathbf{v}_{k-1}$. (*Some* $\mathbf{v}_k$ is a linear combination of the *previous* vectors in $S$.) |
| *Every* finite subset of $S$ is linearly independent. | *Some* finite subset of $S$ is linearly dependent. |
| *Every* vector in $\text{span}(S)$ can be *uniquely* expressed as a linear combination of the vectors in $S$. | *Some* vector in $\text{span}(S)$ can be expressed in more than one way as a linear combination of vectors in $S$. |

## Coordinatization Method (Coordinatizing v with Respect to an Ordered Basis $B$)

Let $\mathcal{V}$ be a nontrivial subspace of $\mathbb{R}^n$, let $B = (\mathbf{v}_1, \ldots, \mathbf{v}_k)$ be an ordered basis for $\mathcal{V}$, and let $\mathbf{v} \in \mathbb{R}^n$. To calculate $[\mathbf{v}]_B$, if it exists, perform the following:

**Step 1:** Form an augmented matrix $[\mathbf{A} \mid \mathbf{v}]$ by using the vectors in $B$ as the *columns* of $\mathbf{A}$, in *order*, and using $\mathbf{v}$ as a column on the right.

**Step 2:** Row reduce $[\mathbf{A} \mid \mathbf{v}]$ to obtain the reduced row echelon form $[\mathbf{C} \mid \mathbf{w}]$.

**Step 3:** If there is a row of $[\mathbf{C} \mid \mathbf{w}]$ that contains all zeroes on the left and has a nonzero entry on the right, then $\mathbf{v} \notin \text{span}(B) = \mathcal{V}$, and coordinatization is not possible. Stop.

**Step 4:** Otherwise, $\mathbf{v} \in \text{span}(B) = \mathcal{V}$. Eliminate all rows consisting entirely of zeroes in $[\mathbf{C} \mid \mathbf{w}]$ to obtain $[\mathbf{I}_k \mid \mathbf{y}]$. Then, $[\mathbf{v}]_B = \mathbf{y}$, the last column of $[\mathbf{I}_k \mid \mathbf{y}]$.

## Transition Matrix Method (Calculating a Transition Matrix from $B$ to $C$)

To find the transition matrix $\mathbf{P}$ from $B$ to $C$ where $B$ and $C$ are ordered bases for a nontrivial $k$-dimensional subspace of $\mathbb{R}^n$, use row reduction on

$$\left[\begin{array}{cccc|cccc} \text{1st} & \text{2nd} & & k\text{th} & \text{1st} & \text{2nd} & & k\text{th} \\ \text{vector} & \text{vector} & \cdots & \text{vector} & \text{vector} & \text{vector} & \cdots & \text{vector} \\ \text{in} & \text{in} & & \text{in} & \text{in} & \text{in} & & \text{in} \\ C & C & & C & B & B & & B \end{array}\right]$$

to produce

$$\left[\begin{array}{c|c} \mathbf{I}_k & \mathbf{P} \\ \hline \text{rows of} & \text{zeroes} \end{array}\right].$$

## Kernel Method (Finding a Basis for the Kernel of $L$)

Let $L\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation given by $L(\mathbf{X}) = \mathbf{AX}$ for some $m \times n$ matrix $\mathbf{A}$. To find a basis for $\ker(L)$, perform the following steps:

**Step 1:** Find $\mathbf{B}$, the reduced row echelon form of $\mathbf{A}$.

**Step 2:** Solve for the fundamental solutions $\mathbf{v}_1, \ldots, \mathbf{v}_k$ of the homogeneous system $\mathbf{BX} = \mathbf{O}$. (The $i$th such solution, $\mathbf{v}_i$, is found by setting the $i$th independent variable equal to 1 and setting all other independent variables equal to 0.)

**Step 3:** The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a basis for $\ker(L)$.

## Range Method (Finding a Basis for the Range of $L$)

Let $L\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation given by $L(\mathbf{X}) = \mathbf{AX}$, for some $m \times n$ matrix $\mathbf{A}$. To find a basis for range($L$), perform the following steps:

**Step 1:** Find $\mathbf{B}$, the reduced row echelon form of $\mathbf{A}$.

**Step 2:** Form the set of those columns of $\mathbf{A}$ whose corresponding columns in $\mathbf{B}$ have nonzero pivots. This set is a basis for range($L$).

## Equivalence Conditions for One-to-One, Onto, and Isomorphism

Let $L\colon \mathcal{V} \to \mathcal{W}$ be a linear transformation between finite dimensional vector spaces, and let $B$ be a basis for $\mathcal{V}$.

| | |
|---|---|
| $L$ is *one-to-one* | |
| $\iff$ | $\ker(L) = \{\mathbf{0}_{\mathcal{V}}\}$ |
| $\iff$ | $\dim(\ker(L)) = 0$ |
| $\iff$ | the image of every linearly independent set in $\mathcal{V}$ is linearly independent in $\mathcal{W}$ |
| $L$ is *onto* | |
| $\iff$ | $\text{range}(L) = \mathcal{W}$ |
| $\iff$ | $\dim(\text{range}(L)) = \dim(\mathcal{W})$ |
| $\iff$ | the image of *every* spanning set for $\mathcal{V}$ is a spanning set for $\mathcal{W}$ |
| $\iff$ | the image of *some* spanning set for $\mathcal{V}$ is a spanning set for $\mathcal{W}$ |
| $L$ is an *isomorphism* | |
| $\iff$ | $L$ is both one-to-one and onto |
| $\iff$ | $L$ is invertible (that is, $L^{-1}\colon \mathcal{W} \to \mathcal{V}$ exists) |
| $\iff$ | the matrix for $L$ (with respect to *every* pair of ordered bases for $\mathcal{V}$ and $\mathcal{W}$) is nonsingular |
| $\iff$ | the matrix for $L$ (with respect to *some* pair of ordered bases for $\mathcal{V}$ and $\mathcal{W}$) is nonsingular |
| $\iff$ | the images of vectors in $B$ are distinct and $L(B)$ is a basis for $\mathcal{W}$ |
| $\iff$ | $L$ is one-to-one and $\dim(\mathcal{V}) = \dim(\mathcal{W})$ |
| $\iff$ | $L$ is onto and $\dim(\mathcal{V}) = \dim(\mathcal{W})$ |

## Dimension Theorem

If $L\colon \mathcal{V} \to \mathcal{W}$ is a linear transformation and $\mathcal{V}$ is finite dimensional, then range($L$) is finite dimensional, and $\dim(\ker(L)) + \dim(\text{range}(L)) = \dim(\mathcal{V})$.

## Gram-Schmidt Process

Let $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ be a linearly independent subset of $\mathbb{R}^n$. Create an orthogonal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ for span($\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$) as follows:

Let $\mathbf{v}_1 = \mathbf{w}_1$.

Let $\mathbf{v}_2 = \mathbf{w}_2 - \left(\dfrac{\mathbf{w}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1$.

Let $\mathbf{v}_3 = \mathbf{w}_3 - \left(\dfrac{\mathbf{w}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\dfrac{\mathbf{w}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2$.

$\vdots$

Let $\mathbf{v}_k = \mathbf{w}_k - \left(\dfrac{\mathbf{w}_k \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\dfrac{\mathbf{w}_k \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 - \cdots - \left(\dfrac{\mathbf{w}_k \cdot \mathbf{v}_{k-1}}{\mathbf{v}_{k-1} \cdot \mathbf{v}_{k-1}}\right)\mathbf{v}_{k-1}$.